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FLAT-SYMMETRICAL SOLUTIONS IN A SO(4)-INVARIANT SELF-GRAVITATING σ-MODEL

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Exact solutions of four-dimensional self-gravitating compact and noncompact σ-models with an internal SO(4) invariance, in spaces of flat symmetry, are obtained. It is shown that the compactness of the chiral field may not influence the spacetime metric.

The theory of two-dimensional chiral models (σ-models), born as a nonlinear theory of strong interactions [1], has by now developed as a nonlinear field-theoretical model in which the interaction is introduced geometrically [2]. Such an approach is used in some unified models of electromagnetic and weak interactions [3], or when studying metastable states of a two-dimensional isotropic ferromagnetic [4]. Besides, two-dimensional chiral models have properties similar to those of four-dimensional non-Abelian gauge theories, which at the moment are thought to be most promising candidates for the description of strong interactions [2]. The transition to four-dimensional σ-models involves some difficulties on both classical and quantum levels [5]. On the classical level, a four-dimensional generalization of the nonlinear chiral model with O(4) invariance, interacting with the metric tensor field satisfying the Einstein equations, was suggested and developed in [5, 6]. The case of arbitrary internal symmetry for self-gravitating σ-models, and the connection between spacetime and chiral field symmetries, was studied by Ivanov [7]. In the same paper examples are given of exact solutions for a σ-model with an internal SO(3) invariance, in spaces with flat symmetry. In the present paper the self-gravitating compact and noncompact σ-models with a SO(4) internal invariance are considered in spaces with flat symmetry. Exact solutions are found.

The Lagrangian of a self-gravitating four-dimensional σ-model, whose chiral (scalar) fields φ^C are defined on the Riemannian spacetime $M(V_4, g_{ij})$ and take values in the Riemannian space (chiral field) $\Phi(V_N, g_{AB})$, has the form [7]:

$$\Lambda = \sqrt{-g}\,(R\varkappa^{-1} + k g_{AB}\varphi_i{}^A \varphi_\kappa{}^B g^{i\kappa})/2. \tag{1}$$

Here $g = \det g_{ik}$, g_{ik} is the metric of M; k is the interaction constant of the gravitational and chiral fields; g_{AB} is the metric of Φ; $\varphi_i{}^A \equiv \partial_i \varphi^A$; A = 1, 2, ..., N. From the Lagrangian (1) we can obtain the system of Einstein equations

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$$R_{i\kappa} = \varkappa k g_{AB}\varphi_i^A \varphi_\kappa^B \tag{2}$$

with the energy-momentum tensor (EMT) of the multiplet of N scalar fields

$$T_{i\kappa} = k g_{AB}\varphi_i^A\varphi_\kappa^B - k g_{i\kappa} g_{AB}\varphi_n^A\varphi_m^B g^{mn}/2 \tag{3}$$

and the equations for chiral or scalar fields:

$$\partial_i(\sqrt{-g}\, g_{CB}\varphi_\kappa^B g^{i\kappa})/\sqrt{-g} - g^{i\kappa}\varphi_i^A\varphi_\kappa^B \partial g_{AB}/2\partial\varphi^C. \tag{4}$$

We search for solutions of the equations of the self-gravitating σ-model (2) and (4) in the class of metrics with flat symmetry

$$dS^2 = A(dx^2+dy^2) - 4e^F d\zeta d\eta = A(dx^2+dy^2) - e^F(dz^2-dt^2), \tag{5}$$

where $2\zeta = z + t$, $2\eta = z - t$, $A = A(\zeta, \eta)$, $F = F(\zeta, \eta)$.

The linear element of the Riemannian space $\Phi(V_3, g_{AB})$ invariant under the transformations of the group SO(4) has the following forms:

a) for the compact model:

$$d\sigma^2 = dX^2 + \sin^2 X\,(d\Theta^2 + \sin^2\Theta\, d\Psi^2), \tag{6a}$$

where $X = \varphi^1$, $\Theta = \varphi^2$, $\Psi = \varphi^3$;

b) for the noncompact model:

$$d\sigma^2 = dX^2 + \text{sh}^2 X\,(d\Theta^2 + \sin^2\Theta\, d\Psi^2). \tag{6b}$$

By virtue of the Einstein equations (2) $A_{\zeta\eta} = 0$, and using admissible coordinate transformations, we come to the two possibilities:

$$A = z, \tag{A}$$

$$A = t. \tag{B}$$

The system of equations (2) and (4) for the metrics (5) and (6a) takes the form:

$$(2A^2)^{-1} + F_\zeta\cdot A^{-1} = k\varkappa\,[X_\zeta^2 + \sin^2 X\,(\Theta_\zeta^2 + \sin^2\Theta\cdot\Psi_\zeta^2)]; \tag{7.1}$$

$$(2A^2)^{-1} + eF_\eta\cdot A^{-1} = k\varkappa\,[X_\eta^2 + \sin^2 X\,(\Theta_\eta^2 + \sin^2\Theta\Psi_\eta^2)]; \tag{7.2}$$

$$-F_{\zeta\eta} + e\,(2A^2)^{-1} = k\varkappa\,[X_\zeta X_\eta + \sin^2 X\,(\Theta_\zeta\Theta_\eta + \sin^2\Theta\cdot\Psi_\zeta\cdot\Psi_\eta)]; \tag{7.3}$$

$$X_\eta + eX_\zeta + 2AX_{\zeta\eta} - A\sin 2X\,(\Theta_\zeta\Theta_\eta + \sin^2\Theta\cdot\Psi_\zeta\Psi_\eta) = 0; \tag{7.4}$$

$$\Theta_\eta + e\Theta_\zeta + 2A\Theta_{\zeta\eta} + 2A\,\text{ctg}\,X\,(X_\zeta\Theta_\eta + X_\eta\Theta_\zeta) - A\sin 2\Theta\cdot\Psi_\zeta\Psi_\eta = 0; \tag{7.5}$$

$$\Psi_\eta + e\Psi_\zeta + 2A\Psi_{\zeta\eta} + 2A\,\text{ctg}\,X\,(X_\zeta\Psi_\eta + X_\eta\Psi_\zeta) + 2A\,\text{ctg}\,\Theta\,(\Theta_\zeta\Psi_\eta + \Theta_\eta\Psi_\zeta) = 0. \tag{7.6}$$

Here X, Θ, and Ψ are functions of the advanced and retarded times ζ and η; the lower index denotes the derivative with respect to the corresponding argument.

In the case (A), $A = z < 0$, $e = 1$; in the case (B), $A = t < 0$, $e = -1$. For the noncompact model (6b), one has to replace $\sin X$ by $\sinh X$ and $\cot X$ by $\coth X$ in Eqs. (7). The equations of scalar fields (7.4)-(7.6) have to be substituted by the integrability conditions for the Einstein equations (7.1)-(7.3). Equations (7.1)-(7.6) are invariant with respect to the substitution $\varphi^A \to -\varphi^A$ and $\zeta \to \eta$.

SOLUTION OF THE COMPACT MODEL

Without going into all the details of the computation, let us give examples of exact solutions.

1A. In the case of stationary scalar fields we have the static solution:

$$dS^2 = z\,(dx^2+dy^2) - D_0\,(-z)^{-1/2+k\varkappa\beta^2}\cdot 4d\zeta d\eta, \tag{8}$$

where β^2 is the integration constant, and D_0 is a positive constant.

The scalar field X(z) has the solutions:

$$\underset{1}{X} = \pm \arccos\,[\beta^{-1}\sqrt{\alpha^2+\beta^2}\sin(\pm\beta u)] + 2\pi m,\ m = 0;\ \pm 1;\ \pm 2;\ldots; \tag{9.1}$$

$$\underset{2}{X} = \pm \arccos\,[\beta^{-1}\sqrt{\beta^2-\alpha^2}\sin(\pm\beta u)] + 2\pi m,\ \beta^2 > \alpha^2. \tag{9.2}$$

Here u = ln(−z); α^2 is the integration constant. A particular solution for the fields Θ and Ψ (for each X_a, where a = 1, 2 numbers the solutions) has the form:

$$\begin{gathered}\underset{a}{\Theta} = \pm \arccos\,(\sqrt{1-\underset{a}{b}^{-2}}\cos\underset{a}{b}\,\underset{a}{\xi}\,\underset{a}{B}^{-2}) + 2\pi m,\ \underset{a}{b}^2 > 1;\\ \underset{a}{\Psi} = \underset{a}{B}\,\mathrm{arctg}\,(\underset{a}{b}\,\mathrm{tg}\,\underset{a}{b}\,\underset{a}{\xi}\,\underset{a}{B}^{-2}) + \underset{a}{\Psi}_0 \quad \text{(no summation)};\\ \underset{1}{\xi} = B(2\alpha)^{-1}\ln|(\alpha\beta^{-1}\,\mathrm{tg}\,\beta u + 1)/(\alpha\beta^{-1}\,\mathrm{tg}\,\beta u - 1)|;\\ \underset{2}{\xi} = B\alpha^{-1}\,\mathrm{arctg}\,(\alpha\beta^{-1}\,\mathrm{tg}\,\beta u).\end{gathered} \tag{10}$$

Here $\underset{a}{b}$, $\underset{a}{B}$, Ψ_0 are integration constants.

The nonvanishing components of the EMT (3) are

$$T_{11} = T_{22} = -(2D_0)^{-1}(-z)^{-1/2-k\varkappa\beta^2};\ T_{33} = T_{44} = k\beta^2(-z)^{-2}/2. \tag{11}$$

2A. In the case when the scalar fields depend only on time t, we obtain the stationary metric:

$$dS^2 = z(dx^2+dy^2) - D_0(-z)^{-1/2}\exp\{k\varkappa\beta^2 z^2/2\}\cdot 4d\zeta d\eta. \tag{12}$$

The scalar fields are defined by the expressions (9.1), (9.2), and (10) after a formal substitution u → t. The nonvanishing components of the EMT (3) are:

$$T_{11} = T_{22} = k\beta^2(2D_0)^{-1}(-z)^{3/2}\exp\{-k\varkappa\beta^2 z^2/2\};\ T_{33} = T_{44} = k\beta^2/2. \tag{13}$$

3A. The wave solutions, when X = X(ζ, η), Θ = Θ(ζ), Ψ = Ψ(ζ) are:

$$dS^2 = z(dx^2+dy^2) - D_0(z^2-t^2)^{k\varkappa/4}\cdot(-z)^{k\varkappa/4-1/2}\cdot\ln f(\zeta)\cdot(dz^2-dt^2), \tag{14}$$

where $\ln f(\zeta) = k\varkappa\int\zeta(\Theta_\zeta^2 + \sin^2\Theta\cdot\Psi_\zeta^2)\,d\zeta$,

$$X = (-1)^m \arcsin(\pm\sqrt{\zeta\cdot z^{-1}}) + \pi m,\ m = 0;\ \pm 1;\ \pm 2;\ldots \tag{15}$$

The following components of the EMT do not vanish:

$$T_{11} = T_{22} = k\beta^2\cdot 2D_0(z^2-t^2)^{-k\varkappa/4}\cdot(z)^{-k\varkappa/4+1/2}\cdot f^{-1};\ T_{33} = T_{44} = k\beta^2/2. \tag{16}$$

4A. The wave solution similar to 3A: By virtue of the invariance of the system (7) under the substitution ζ → η, one has to make this substitution in Eqs. (14)-(16).

1B. In the case of fields depending only on the time t, we obtain an analog of the solutions 1A with the substitution of z by t in the expressions (9)-(11).

2B. In the case of fields depending only on z, substitute z by t in the solutions of the case 2A.

3B. Substitute z by t in the expressions of the case 3A.

4B. Substitute z by t in the expressions of the case 4A.

THE SOLUTIONS OF THE NONCOMPACT MODEL

For the noncompact model we can obtain the analogs of all of the above solutions, the gravitational fields having the same form, while the key expressions for determining the scalar fields (similar to Eqs. (9) and (10)) are:

$$\underset{1}{X} = \mathrm{Arch}\ [\beta^{-1}\sqrt{\alpha^2+\beta^2}\,\mathrm{ch}\,(\pm\beta u)];$$

$$\underset{2}{X} = \begin{cases} \mathrm{Arch}\,[\beta^{-1}\cdot\sqrt{\alpha^2-\beta^2}\,\mathrm{sh}\,(\pm\beta u)], & \alpha^2 > \beta^2; \\ \mathrm{Arch}\exp(\pm\beta u), & \alpha^2 = \beta^2; \\ \mathrm{Arch}\,[\beta^{-1}\sqrt{\beta^2-\alpha^2}\,\mathrm{ch}\,(\pm\beta u)], & \beta^2 > \alpha^2; \end{cases} \tag{17}$$

$$\underset{a}{\Theta} = \pm\arccos\,(\sqrt{1-\underset{a}{b}^{-2}}\cos\underset{a}{b}\,\xi\,\underset{a}{B}^{-2}) + 2\pi m,\ \ \underset{a}{b}^2 > 1;$$

$$\underset{a}{\Psi} = B\cdot\mathrm{arctg}\ (\underset{a}{b}\,\mathrm{tg}\,\underset{a}{b}\xi\underset{a}{B}^{-2}) + \underset{a}{\Psi_0};$$

$$\underset{1}{\xi} = B\alpha^{-1}\,\mathrm{arctg}\ [\alpha\beta^{-1}\,\mathrm{cth}\,(\pm\beta u)];$$

$$\underset{2}{\xi} = \begin{cases} B\alpha^{-1}\,\mathrm{Arth}\,[\alpha\beta^{-1}\,\mathrm{th}\ (\pm\beta u)], & \alpha^2 > \beta^2; \\ B\beta^{-1}\,[\ln\sqrt{\exp(\pm 2\beta u)-1} - (\pm\beta u)], & \alpha^2 = \beta^2; \\ B\alpha^{-1}\,\mathrm{Arth}\,[\alpha\beta^{-1}\,\mathrm{cth}\,(\pm\beta u)], & \alpha^2 < \beta^2. \end{cases} \tag{18}$$

DISCUSSION OF THE RESULTS

The above examples of exact solutions follow from the immediate integration of the equations of the self-gravitating σ-model, and cannot be obtained by the harmonic-maps method [5], because all the solutions are measurelike (with an infinite action integral). None of the solutions are totally geodesic [5], this condition imposing strict limitations on the function $F: F = F(\eta)$, which are not satisfied in our case. Comparing the results of the compact and noncompact models, we can see that the compactness of the chiral field Φ may not affect the metric of spacetime.

Note that the wave solutions 3A and 4A (3B and 4B) can be generalized to the case of SO(N) symmetry, if we assume that all fields φ^C $(C \geqslant 2)$ are totally dependent on ζ (on η).). The spacetime metric will have the form (14), the function $f(\zeta)$ being of the form

$$\ln f(\zeta) = k\varkappa \int \zeta\,(\Theta_\zeta^2 + \sin^2\Theta\,(\Psi_\zeta^2 + \sin^2\Psi\,(\ldots + \sin^2\varphi^{N-1}\,[\varphi_\zeta^N]^2)\ldots)\,d\zeta. \tag{19}$$

The field X can be determined from Eq. (15), and the components of the EMT — from Eq. (16), taking into account Eq. (19).

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