note that the value of $\chi(z)$ is independent of φ, so that the estimates of $\chi(z)$ presented below are valid for arbitrary neutrino mixing angle. Numerical estimates of $\chi(z)$ are presented in Fig. 3 for the values $\omega = 0.1$ MeV and $\omega = 0.5$ MeV. At the value $z = 4$ ($\delta m_\nu^2 = 10^{-2}$ eV^2, L = 400 m) the lowest value of $\chi(z) \approx -0.6$ occurs for $\omega = 0.1$ MeV, above which there is a slow increase until a maximum is reached at z = 15 (L = 1500 m). With further increase in z the quantity $\chi(z)$ approaches a constant value of $\sim$0.5. From the analysis performed it is evident that neutrino oscillation has a marked effect on the energy spectrum of photons produced by the process of Eq. (1).

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HAMILTON FORMULATION OF A THEORY WITH HIGH DERIVATIVES

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A method of "Hamiltonization" of a singular theory with high derivatives is described. In the nonsingular case the result agrees with the known Ostrogradskii formulation. It is shown that the Lagrange equations of motion reduce to normal form in the nonsingular theory.

I. It is known that the addition of terms containing high derivatives to the standard Lagrange field theory improves the convergence of the appropriate Feynmann diagrams in a number of cases [1]. Here, gravitation with the Lagrangian R^2 and the Yang–Mills theory with high derivatives can be mentioned.* At this time, gauge field theories naturally attract the greatest attention, which also indicates an interest in gauge theories with high derivatives.

In this connection, the problem of quantization of such theories arises, particularly canonical quantization. As is known, canonical quantization actually reduces to the problem of "Hamiltonization" of the appropriate classical theory.

For a theory with high derivatives, Ostrogradskii [2] first considered "Hamiltonization." However, his method is not directly applicable to singular theories among which are gauge theories in particular. This was noted even in the original paper [2]. Nevertheless, in a number of particular cases of singular theories with high derivatives, different modifications of the Ostrogradskii method afforded an opportunity to construct a Hamilton formalism [3-5].

A generalization of the Ostrogradskii method is proposed in this paper which will permit an arbitrary theory with high derivatives to be reduced to a form allowing application of the Dirac method [6], i.e., actually to "Hamiltonize" it.

The examination is made in an example of a theory with a finite number of degrees of freedom. Transformation of the results to field theory is trivial and not presented in the text.

*It must be noted that the introduction of high derivatives creates definite difficulties, for instance, with the appearance of the indefinite metric.

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II. Let us consider the classical system with N degrees of freedom in the case when the Lagrange function L depends on the generalized coordinates q_i^l, i = s, ..., N and their derivatives above the first order.

The action S is expressed in terms of the Lagrange function in the usual way:

$$S = \int_{t_1}^{t_2} L(q_i^1, q_i^{1(1)}, \ldots, q_i^{1(n_i)})\, dt, \quad f^{(\kappa)} = \frac{d^\kappa f}{dt^\kappa}. \tag{1}$$

The Lagrange equations are obtained from the appropriate action principle:

$$\frac{\delta S}{\delta q_i^1} \equiv \sum_{l=0}^{n_i} (-1)^l \left(\frac{\partial L}{\partial q_i^{1(l)}}\right)^{(l)} = 0. \tag{2}$$

In the following it is convenient to go over to the problem on the conditional extremum of the action:

$$S' = \int L(q_i^1, q_i^2, \ldots, q_i^{n_i}, v_i)\, dt \tag{3}$$

under the additional conditions

$$q_i^{s_i} = \dot{q}_i^{s_i - 1}, \quad \dot{q}_i^{n_i} = v_i, \quad s_i = 2, \ldots, n_i. \tag{4}$$

This problem is equivalent to the problem for an absolute extremum of the functional S* [7]

$$S^* = \int [p_i^{s_i - 1}(\dot{q}_i^{s_i - 1} - q_i^{s_i}) + p_i^{n_i}(\dot{q}_i^{n_i} - v_i) + L(q, v)]\, dt, \tag{5}$$

where all the q, p, v are independent functions of the time subject to variation. The appropriate equations of motion have the form

$$\left.\begin{aligned} \dot{p}_i^1 &= \frac{\partial L}{\partial q_i^1}, \\ \dot{p}_i^{s_i} &= \frac{\partial L}{\partial q_i^{s_i}} - p_i^{s_i - 1}, \quad s_i = 2, \ldots, n_i, \end{aligned}\right\} \tag{6}$$

$$\dot{q}_i^{s_i - 1} = q_i^{s_i}, \quad \dot{q}_i^{n_i} = v_i, \tag{7}$$

$$p_i^{n_i} - \frac{\partial L}{\partial v_i} = 0. \tag{8}$$

The action S* in (5) and the equations (6)-(8) are called an expanded Lagrange system. If $q_i^{s_i}$, $p_i^{r_i}$, v_i, $r_i = 1, \ldots, n_i$ are eliminated in the expanded system (as is always possible), we return to the action (1) and the Lagrange equations (2). Only v_i must be eliminated to go over to the Hamilton formulation.

III. 1) Let the following condition be satisfied

$$\det \frac{\partial^2 L(q_i^1 \ldots q_i^{1(n_i)})}{\partial q_i^{1(n_i)} \partial q_j^{1(n_j)}} \neq 0 \tag{9}$$

(the matrix $\frac{\partial^2 L}{\partial q_i^{(n_i)} \partial q_j^{(n_j)}}$ is called the Hessian). In this case v_i can be expressed in terms of q and p^n from (8)

$$v_i = v_i(q, p^n). \tag{10}$$

Let us substitute $v_i(q, p^n)$ into (6) and (7) in place of v_i. The equations of motion (6) and (7) can afterwards be written in the canonical form:

$$\dot{p}_i^{r_i} = \{p_i^{r_i}, H\},\ \dot{q}_i^{r_i} = \{q_i^{r_i}, H\}, \tag{11}$$

where the Hamilton function H(q, p) has the form

$$H \equiv \sum_{i=1}^{N}\Big(\sum_{s_i=2}^{n_i} p_i^{s_i-1} q_i^{s_i} + p_i^{n_i} v_i(q, p^n)\Big) - L(q, v(q, p^n)); \tag{12}$$

and $\{A, B\} = \sum_{i=1}^{N}\sum_{r_i=1}^{n_i}\Big(\frac{\partial A}{\partial q_i^{r_i}}\frac{\partial B}{\partial p_i^{r_i}} - \frac{\partial A}{\partial p_i^{r_i}}\frac{\partial B}{\partial q_i^{r_i}}\Big)$ is the Poisson bracket. In this case (9), we say that the theory is nonsingular. Ostrogradskii [2] first obtained such a formulation of a nonsingular theory.

2) If the Hessian is degenerate

$$\operatorname{rank}\frac{\partial^2 L(q, v)}{\partial v_i \partial v_j} = R,\ N - R = m > 0, \tag{13}$$

then the relationship (8) in the form (10) cannot be solved. In this case we call the theory singular. It can be considered that the minor of maximal rank in the Hessian is in the upper left corner.* We shall sometimes use the following notation

$$\begin{aligned} p_\alpha^{n_\alpha} &= P_\alpha,\ v_\alpha = V_\alpha,\ \alpha = 1,\dots, R;\\ p_{R+a}^{n_{R+a}} &= \pi_a,\ v_{R+a} = \lambda_a,\ a = 1,\dots, m. \end{aligned} \tag{14}$$

From the first R equations in (8) we express V in terms of q, P, λ

$$V_\alpha = V_\alpha(q, P, \lambda). \tag{15}$$

We substitute V in the form (15) into (6) and (7) and the remaining m equations in (8). Then it turns out that these latter have the following structure

$$\Phi_a^{(1)} \equiv \pi_a - f_a(q, P) = 0, \tag{16}$$

where

$$f_a(q, P) = \frac{\partial L(q, v)}{\partial \lambda_a}\Big|_{V=V(q,P,\lambda)}.$$

The functions $\Phi_a^{(1)}(q, p^n)$ will be called the primary couplings. Let us introduce the function H

$$H \equiv p_i^{s_i-1} q_i^{s_i} + P_\alpha V_\alpha(q, P, \lambda) + \lambda_a f_a(q, P) - L(q^1,\dots, q^n, V(q, P, \lambda)\lambda). \tag{17}$$

It is easy to see that H is independent of λ:

$$\frac{\partial H}{\partial \lambda} = P\frac{\partial V}{\partial \lambda} + f(q, P) - \frac{\partial L}{\partial V}\frac{\partial V}{\partial \lambda} - \frac{\partial L}{\partial \lambda} = 0. \tag{18}$$

Let us also introduce the function $H^{(1)}(q, p, \lambda)$:

$$H^{(1)}(q, p, \lambda) \equiv p_i^{s_i-1} q_i^{s_i} + P_\alpha V_\alpha(q, P, \lambda) + \pi_a\lambda_a - L(q, V, \lambda). \tag{19}$$

It is evident that

$$H^{(1)} = H + \lambda_a \Phi_a^{(1)}. \tag{20}$$

By using $H^{(1)}$ the equations of motion can be rewritten as follows:

*This does not limit the generality since the minor of the maximal rank can always be relocated in the upper left corner in a symmetric matrix by the simultaneous renumbering of the rows and columns.

$$\dot{p}_i^{r_i} = \{p_i^{r_i}, H^{(1)}\}, \; \dot{q}_i^{r_i} = \{q_i^{r_i}, H^{(1)}\},$$
$$\Phi_a^{(1)}(q, p^n) = 0, \tag{21}$$

which have the form of the equations of motion of Hamiltonian mechanics with couplings [6], which permits application of the Dirac method and assures the possibility of canonical quantization.

Let us clarify the meaning of the function H(q, p). As already noted, the momenta $p_i^{r_i}$, $r_i = 1, \ldots, n_i$ and v_i can always be eliminated from the expanded Lagrange system (6)-(8). The following expressions are obtained for the momenta

$$\sum_{l=r_i}^{n_i} (-1)^{l-r_i} \left(\frac{\partial L}{\partial q_i^{1(l)}}\right)^{(l-r_i)} = p_i^{r_i}. \tag{22}$$

If the momenta $p_i^{r_i}$ are replaced by their expressions (22) in H(q, p), then it turns out that H agrees with the energy*

$$\varepsilon = \sum_{i=1}^{N} \sum_{r_i=1}^{n_i} q_i^{1(r_i)} \sum_{l=r_i}^{n_i} (-1)^{l-r_i} \left(\frac{\partial L}{\partial q_i^{1(l)}}\right)^{(l-r_i)} - L. \tag{23}$$

IV. Let us examine the question of the possibility of reducing the system of Lagrange equations (3) to normal form.† For simplicity we limit ourselves to the case

$$L(x, x^{(1)}, \ldots, x^{(n)}, y, y^{(1)}, \ldots, y^{(m)}), \; x = (x_1 \ldots x_M), \; y = (y_1 \ldots y_M). \tag{24}$$

It is considered here that n and m are the real order to which the derivatives of x and y (respectively) enter into L, i.e., that

$$\frac{\partial L}{\partial x^{(n)}} \neq 0, \; \frac{\partial L}{\partial y^{(m)}} \neq 0.$$

1) If m = n, then evidently the nondegeneracy of the Hessian is sufficient for the Lagrange equation to be reduced to the form

$$\left.\begin{aligned} x^{(2n)} &= X(x \ldots x^{(2n-1)}, y \ldots y^{(2n-1)}) \\ y^{(2n)} &= Y(y \ldots y^{(2n-1)}, x \ldots x^{(2n-1)}) \end{aligned}\right\}. \tag{25}$$

If the Hessian is degenerate, then the Lagrange equations in the variables x, y are not reduced to the form (25).

2) Let $n < m$. The Lagrange equations have the following structure

$$\frac{\delta S}{\delta x} \equiv (-1)^n \left(\frac{\partial^2 L}{\partial x^{(n)} \partial x^{(n)}} x^{(2n)} + \frac{\partial^2 L}{\partial y^{(m)} \partial x^{(n)}} y^{(n+m)}\right) + F(x, \ldots, x^{(2n-1)}, y, \ldots, y^{(n+m-1)}); \tag{26}$$

$$\frac{\delta S}{\delta y} \equiv (-1)^m \left(\frac{\partial^2 L}{\partial x^{(n)} \partial y^{(m)}} x^{(n+m)} + \frac{\partial^2 L}{\partial y^{(m)} \partial y^{(m)}} y^{(2m)}\right) + G(x, \ldots, x^{(n+m-1)}, y, \ldots, y^{(2m-1)}). \tag{27}$$

Let us note that the high-order derivatives $x^{(n+m)}$ and $y^{(2m)}$ can enter only into (27). Therefore, it is impossible to solve the Lagrange equations (26) and (27) for $x^{(n+m)}$, $y^{(2m)}$ simultaneously. Nevertheless, when the Hessian is nondegenerate:

*The expression (23) for the energy is the derivative of the action (2) with respect to t_2 at the extremals. Expressions (22) for the momenta $p_i^{r_i}$ are the derivatives of the action (2) with respect to $q_i^{1(r_i-1)}(t_2)$.

†That form of the differential equations $q_i^{(d_i)} = Q_i(q_j, \ldots, q_j^{(d_j-1)})$ is called normal. The Cauchy problem has a unique solution for normal equations.

$$\det\begin{pmatrix} \dfrac{\partial^2 L}{\partial x^{(n)}\partial x^{(n)}} & \dfrac{\partial^2 L}{\partial y^{(m)}\partial x^{(n)}} \\ \dfrac{\partial^2 L}{\partial x^{(n)}\partial y^{(m)}} & \dfrac{\partial^2 L}{\partial y^{(m)}\partial y^{(m)}} \end{pmatrix} \neq 0, \tag{28}$$

the Lagrange equations are reduced to normal form.

Let g denote the minor of maximal rank N in the matrix

$$\left(\frac{\partial^2 L}{\partial x^{(n)}\partial x^{(n)}} \quad \frac{\partial^2 L}{\partial x^{(n)}\partial y^{(m)}}\right).$$

We partition the coordinates x, y as follows

$$x_i = X_i, \quad \text{if the column} \quad \frac{\partial^2 L}{\partial x^{(n)}\partial x_i^{(n)}} \subset g;$$

$$x_i = x_i, \quad \text{if the column} \quad \frac{\partial^2 L}{\partial x^{(n)}\partial x_i^{(n)}} \not\subset g;$$

$$y_j = Y_j, \quad \text{if the column} \quad \frac{\partial^2 L}{\partial x^{(n)}\partial y_j^{(m)}} \subset g;$$

$$y_j = u_j, \quad \text{if the column} \quad \frac{\partial^2 L}{\partial x^{(n)}\partial y_j^{(m)}} \not\subset g.$$

CONCLUSION

The Lagrange equations in the nonsingular theory (28) reduce to the following normal form:

$$\begin{aligned} X^{(2n)} &= F\,(X\ldots X^{(2n-1)},\ x\ldots x^{(2n)},\ Y\ldots Y^{(n+m-1)},\ u\ldots u^{(n+m)}); \\ Y^{(n+m)} &= G\,(X\ldots X^{(2n-1)},\ x\ldots x^{(2n)},\ Y\ldots Y^{(n+m-1)},\ u\ldots u^{(n+m)}); \\ x^{(n+m)} &= \varphi\,(X\ldots X^{(2n-1)},\ x\ldots x^{(n+m-1)} Y\ldots Y^{(n+m-1)},\ u\ldots u^{(2m-1)}); \\ u^{(2m)} &= \psi\,(X\ldots X^{(2n-1)},\ x\ldots x^{(n+m-1)},\ Y\ldots Y^{(n+m-1)},\ u\ldots u^{(2m-1)}). \end{aligned} \tag{29}$$

In the singular theory the Lagrange equations do not reduce to the form (29) in the x, y variables. An analogous assertion is valid even for an arbitrary quantity of different n_i. The nondegeneracy of the Hessian (9) is the condition assuring the possibility of reducing the Lagrange equations (2) to a normal form of the type (29) even in this case.

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