STATIC CONFIGURATION WITH AN ULTRARELATIVISTIC EQUATION OF STATE AT THE CENTER

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A class of solutions is obtained for the Einstein equations for liquid spheres with finite values of the energy density and pressure at the center. The fourth Tolman solution and the Adler solution belong to this class. A new solution is presented for which a merger is made with external empty space. Configuration parameters are calculated for an ultrarelativistic equation of state at the center.

1. All exact static spherically-symmetric solutions of the Einstein equations, known at the present time in the presence of a substance, are obtained without preliminary assignment of the equation of state. In this case the system of Einstein equations consists of three equations

$$e^{-\lambda}(\nu' r+1)-1=r^2P(r),\ e^{-\lambda}(1-r\lambda')-1=-r^2\varepsilon(r),\qquad (1)$$
$$\nu'=-2P'/(P+\varepsilon),$$

where e^{λ} and e^{ν} are metric coefficients of the desired metric

$$ds^2=e^{\nu}c^2dt^2-e^{\lambda}dr^2-r^2(d\theta^2+\sin^2\theta d\varphi^2),\qquad (2)$$

ε and P are the energy density and the pressure (we include $8\pi\gamma\ c^{-4}$ in ε and P). The equation of state should be the fourth equation. However, as a rule the system of equations can only be solved approximately. Consequently, a certain additional condition which permits obtaining exact solutions is chosen as the fourth equation. Thus, all static solutions known at the present time are obtained (for instance, [1-5], with the exception of the solutions with the equations of state $\varepsilon=-P$; $\varepsilon=-3P$; $\varepsilon=\text{const}$). The disadvantage of such a method is that the equation of state can even be not physical in the solutions obtained. Hence, the question of selecting such an additional condition as will be known to result in an equation of state with definite properties, is important. It is shown in this paper what selection of the additional condition will result in finite values of the energy density and pressure at the center. New exact solutions are obtained for the Einstein equations which satisfy this condition. The possibility is investigated of an ultrarelativistic equation of state at the center of the configuration.

2. The metric coefficients can be eliminated from the system (1), and one equation can be obtained

$$P'rv^{-1}(v-1)(v-1/2)+P(rv'-v-2v^2+1)+r^{-2}(rv'+v-2v^2)=0,\qquad (3)$$

where

$$v=P'r(P+\varepsilon)^{-1}.\qquad (4)$$

If (4) is selected as the initial condition, then (3) is a linear differential equation in P, i.e., by giving v, we have P, and therefore, the energy density v and the metric coefficients. Let us analyze the behavior of the function v near the center under the condition that

$$P(0)=P_0=\text{const},\quad \varepsilon(0)=\varepsilon_0=\text{const}\qquad (5)$$

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(P and ε are continuous and differentiable functions). To do this, we take account of (4) and convert (3) to the form:

$$\varepsilon/P = (v-1)^{-1}(v-1/2)^{-1}[P^{-1}r^{-2}(2v^2-v-rv')+v+2v^2-1-rv']-1. \quad (6)$$

It follows from (6) that as $r \to 0$ and (5) is taken into account, the function v should behave as

$$v_0(r) = -Kr^2, \quad (7)$$

where K is some positive constant. The negative sign of v assures the decrease in pressure with the growth of r. From (6) and (7) we obtain an equation of state at the center of the configuration in the form

$$\varepsilon_0 = 3(2K - P_0). \quad (8)$$

To satisfy the ultrarelativistic equation of state $\varepsilon_0 = 3P_0$ at zero, it is necessary to set $K = P_0$. If the desired functions are differentiable arbitrarily often in the neighborhood of zero, then a solution in the form of a series can be obtained. We present the first three terms of the expansion for the pressure

$$P = P_0(1-2P_0r^2+2P_0\sqrt{3P_0}r^3).$$

3. Let us select v so that P (from (3)) could be expressed in terms of the power-law function

$$v = -\lambda y^2(1+y^2)^{-1}, \quad (9)$$

where $y = rr_0^{-1}$, and r_0 is a constant with a dimensionality of a length; λ is a dimensionless constant ($\lambda = Kr_0^2$).

From (3) we obtain the following expression for P

$$P = r^{-2}[-1-1/2\,y^2(1+y^2)^{1-\lambda}(1+(1+\lambda)y^2)^{-2/(1+\lambda)}(1+(1+2\lambda)y^2) \times (C+4\int(1+y^2)^{\lambda-1}(1+(1+\lambda)y^2)^{(1-\lambda)/(1+\lambda)}y^{-3}dy. \quad (10)$$

which is a solution of the Einstein equation, expressed in quadratures, for liquid spheres under the conditions P_0 = const, ε_0 = const. The case of integrability of the integral in (10) is: 1) λ is an integer, and 2) $(1-\lambda)/(1+\lambda)$ is an integer. In the second case, however, λ is always negative, which does not correspond to a physical equation of state. In the first case ($\lambda > 0$) $\lambda = 1$ is the fourth Tolman solution [1], while $\lambda = 2$ is the Adler solution [2]. The new solutions are for $\lambda \geqslant 3$. Let us present the solution for $\lambda = 3$.

$$\begin{aligned} P(r) &= (1/2r_0^2)(1+y^2)^{-2}[9(1-y^2)-C(1+7y^2)(1+4y^2)^{-1/2}],\\ \varepsilon(r) &= (3/2r_0^2)(1+y^2)^{-2}[3+y^2+C(1+3y^2)(1+4y^2)^{-3/2}],\\ e^{\nu} &= B(1+y^2)^3;\ e^{-\lambda} = 1-1/2y^2(1+y^2)^{-1}[3+C(1+4y^2)^{-1/2}], \end{aligned} \quad (11)$$

where B is an arbitrary constant. For an ultrarelativistic equations of state at the center of the configuration C = 3. The equation of state will be physical if

$$3 \leqslant c < 9 \quad (12)$$

($P_0 = 0$ for c = 9).

4. In order to merge the solution obtained with the external Schwarzchild solution

$$ds^2 = (1-r_g/r)c^2dt^2-(1-r_g/r)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta d\varphi^2) \quad (13)$$

it is necessary that the following conditions be satisfied

$$P(r_{\text{boun}}) = 0;\ e^{\nu}_{in}(r_{\text{boun}}) = e^{\nu}_{out}(r_{\text{boun}}); \quad (14)$$

$$e^{\lambda}_{in}(r_{boun}) = e^{\lambda}_{out}(r_{boun});\ (e^{\nu}_{in}(r_{boun}))' = (e^{\nu}_{out}(r_{boun}))', \tag{14}$$

three of which are independent. The condition $P(r_{boun}) = 0$ yields the equation

$$(1+4z)^{1/2} = (c/9)(1+7z)(1-z)^{-1}, \tag{15}$$

where $z = r^2_{boun} r_0^{-2}$. The remaining merger conditions yield

$$r_g/r_{boun} = 6z/(1+7z) \tag{16}$$

and

$$B = (1 - r_g r^{-1}_{boun})(1+z)^{-3}. \tag{17}$$

It follows from (16) and (15) that the ratio r_g/r_{boun} is determined uniquely by the equation of state at the center of the configuration. From (15) we have for the ultrarelativistic equation of state at the center (C = 3)

$$z = 0{,}3017,\quad r_g/r_{boun} = 0.582. \tag{18}$$

This is the maximum value of the ratio between the gravitational radius and the radius of the configuration. The greater the C (C < 9), i.e., the more the equation of state at the center differs from the ultrarelativistic, the smaller the ratio of r_g to r_{boun}. From (17) we determine the constant B. Therefore, only the dimensional constant r_0, which can be expressed in terms of the value of the energy density at the center, remains undetermined. For instance, for the ultrarelativistic equation of state $r_0 = 3\varepsilon_0^{-1/2}$. Therefore, giving the energy density at the center determines all the parameters of the configuration.

If the radius of the configuration is selected equal to 10 km, then the gravitational radius equals 6 km, i.e., the mass of the configuration equals two solar masses. The central density equals 10^{15} g/cm^3, therefore, the configuration parameters are close to the parameters of neutron stars.

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