Mathematical Geology, Vol. 18, No. 1, 1986

An Interpolation Method Taking into Account Inequality Constraints: I. Methodology[1]

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Several kinds of data can provide information about a variable measured on a one- or two-dimensional space; at some points, the value is known to be equal to a certain number. At other points, the only information may be that the variable is greater or smaller than a given value. The theory of splines provides interpolating functions that can take into account both equality and inequality data. These interpolating functions are presented. The parallel between splines and kriging is reviewed, using the formalism of dual kriging. Coefficients of dual kriging can be obtained directly by minimizing a quadratic form. By adding some inequality constraints to this minimization, an interpolating function may be calculated which takes into account inequality data and is more general than a spline. The method is illustrated by some simple one-dimensional examples.

KEY WORDS: kriging, splines, mapping, prediction.

INTRODUCTION

Definition of the Problem

The problem of mapping under inequality constraints is common in the oil industry. Consider, for instance (Fig. 1) a well that has been drilled through the producing interval of a reservoir but has reached the oil-water contact before the bottom of the interval. When this happens, drilling is often stopped in order to prevent any communication with the aquifer, and also to save drilling costs. The only information about the gross thickness at the well is that it is greater than the penetrated thickness. This information must be taken into account when mapping isopachs. The general problem that this paper addresses is stated as:

Find an interpolating function $z^*(x)$ of the variable $z(x)$ such that

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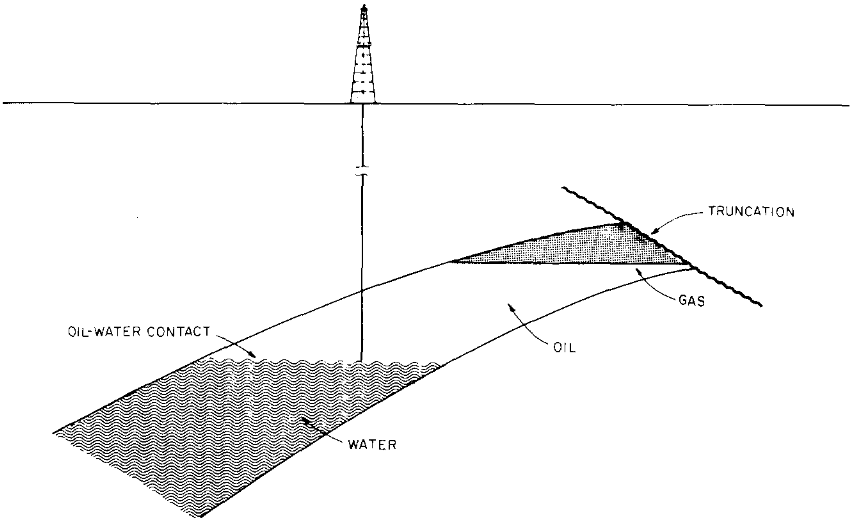

Fig. 1. When mapping gross thickness of a formation, the only information available at the well is that gross thickness is greater than penetrated thickness.

$$
\begin{aligned}
z^*(x_\alpha) &= z_\alpha \quad \text{for } \alpha = 1, .., n \quad &&\text{(exact data)} \\
z^*(x_\alpha) &\geq z_\alpha \quad \text{for } \alpha = n+1, .., n+p_1 \quad && \\
& && \text{(inequality data)} \qquad (1) \\
z^*(x_\alpha) &\leq z_\alpha \quad \text{for } \alpha = n+p_1+1, \ldots, n+p_1+p_2 &&
\end{aligned}
$$

x can be a point in one or two dimensions, $p = p_1 + p_2$ is the total number of inequality constraints, and at some points x_α, the inequality constraint can be two-sided

$$z_{\alpha 1} \leq z^*(x_\alpha) \leq z_{\alpha 2} \qquad (2)$$

In that case, location x_α appears twice in (1).

In order to facilitate reading of this paper, an outline of the developments follows.

The problem of interpolation under inequality constraints has been solved already in the framework of spline theory. This solution is presented. Then, using the fact that spline interpolation is a particular case of kriging, the spline solution is generalized. The generalization is performed in several steps. First, existing works about kriging and inequality data are reviewed. A convenient way to deal with this problem is to treat kriging as defining an interpolating function rather than a "point by point" estimator. That is why formalism of "dual kriging" is presented briefly. Coefficients of dual kriging are shown to minimize a norm similar to the norm minimized by the spline function. Thus, results of the theory of splines can be generalized to the kriging interpolator, which takes into account inequality data, and also is more general than a spline function. The last part of the paper presents simple applications of the method. The reader more interested in applications than theory can read that part first.

SPLINE FUNCTIONS UNDER INEQUALITY CONSTRAINTS

The theory of splines under inequality constraints is presented in order to prepare for the generalization to kriging. In this paper we restrict ourselves to the so-called "thin plate" splines, which are most often used in interpolation problems.

Splines in R and R^2

If x_α, either a point in R or R^2, is in R^2, its two coordinates are $x_{\alpha 1}$ and $x_{\alpha 2}$. The "constrained" thin plate splines are defined as functions minimizing

$$\text{In } R \qquad \int_a^b f''^2(x)\,dx \tag{3}$$

$$\text{In } R^2 \qquad \iint_{R^2}\left[\left(\frac{\partial^2 f}{\partial x^2}\right)^2 + \left(\frac{\partial^2 f}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2\right] dx\,dy \tag{4}$$

under the conditions

$$\begin{aligned} f(x_\alpha) &= z_\alpha && \text{for } \alpha = 1, \ldots, n \\ f(x_\alpha) &\geq z_\alpha && \text{for } \alpha = n+1, \ldots, n+p_1 \\ f(x_\alpha) &\leq z_\alpha && \text{for } \alpha = n+p_1+1, \ldots, n+p \end{aligned} \tag{5}$$

In order to formalize this problem mathematically, two normed spaces X and Y and a linear operator T mapping X into Y are defined, so that the spline function is *the function* f *of* X *which minimizes* $\|Tf\|^2$ *under constraints (5).* For instance, in one dimension, X, Y, and T are:

X functions defined on the interval $[a, b]$ which have their second derivatives in $L^2\,[a, b]$

Y $L^2\,[a, b]$, with the usual norm: $\|f\|^2 = \int_a^b f^2(t)\,dt$

T associates to a function f of X its second derivative

$$T(f) = f''$$

In two dimensions, the definition of X, Y, and T is more complicated and can be found in Duchon (1975). Let us mention only that T is a second-order differential operator, which associates to each function f the quadruplet of functions

$$T(f) = \left(\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x\, \partial y}, \frac{\partial^2 f}{\partial y\, \partial x}\right) \tag{6}$$

When n "exact" points of coordinates (x_α, z_α) (resp. $x_{\alpha 1}, x_{\alpha 2}, z_\alpha$ in R^2) are not on the same line (resp. plane in R^2), Laurent (1972, Chap. 9) has shown that a unique constrained spline $\sigma(x)$ is characterized by the relation

$$\langle T\sigma, Tf \rangle = \sum_{\alpha=1}^{n+p} b^\alpha f(x_\alpha) \qquad \text{for all } f \text{ in } X \tag{7}$$

The left member of the equation is the scalar product between $T\sigma$ and Tf, associated with the norm of space Y. Notice that α varies from 1 to $n + p$. The exact data play the same role as inequality data.

Now, consider functions $K_i(\cdot, x_\alpha)$ defined by

$$\text{In } R \qquad K_1(x, x_\alpha) = K_1(x - x_\alpha) = |x - x_\alpha|^3 \tag{8}$$

$$\text{In } R^2 \qquad K_2(x, x_\alpha) = K_2(x - x_\alpha) = [(x_1 - x_{\alpha 1})^2 + (x_2 - x_{\alpha 2})^2] \cdot \text{Log}[(x_1 - x_{\alpha 1})^2 + (x_2 - x_{\alpha 2})^2]^{1/2} \tag{9}$$

Functions $K_i(\cdot, x_\alpha)$ associate a real number to points of R and R^2. They belong in each case to space X and are mapped by the application T into space Y, so that

$$\langle TK_i(\cdot, x_\alpha), Tf \rangle = f(x_\alpha) \qquad \text{for all } f \text{ in } X \tag{10}$$

For R, the proof of (10) is easy and can be found in Laurent (1972) or Prenter (1975). For R^2, it is more difficult and has been established by Duchon (1975).

Consider now eqs. (7) and (10). If we take for σ a linear combination of functions $K_i(\cdot, x_\alpha)$ for $\alpha = 1$ to $n + p$, eq. (7) clearly will be satisfied. We even can add to this linear combination a function of the kernel of T (that is, a function n such that $Tn = 0$) and eq. (7) still will be satisfied. In R, T is the second derivative, and the kernel of T is composed of polynomials of order 1. The spline function then can be written

$$\sigma_1(x) = c_0 + c_1 x + \sum_{\alpha=1}^{n+p} b^\alpha K_1(x - x_\alpha) \tag{11}$$

In R^2, expression (6) shows that the kernel of T is composed of polynomials of order 1, as in R. Therefore, the spline function in R^2 can be written

$$\sigma_2(x_1, x_2) = c_0 + c_1 x_1 + c_2 x_2 + \sum_{\alpha=1}^{n+p} b^\alpha K_2(x - x_\alpha) \tag{12}$$

The expression of the constrained spline in R is from Laurent (1972). In R^2, Duchon (1975) found the expression of the spline in the case where all data are exact. But eq. (7) and his paper evidently imply that (12) applies to constrained splines.

The balance of the calculations for spline functions will be made in R^2, but can be applied easily to R.

Equation (12) shows that the constrained spline depends linearly on $(n + p + 3)$ coefficients c_0, c_1, c_2, $b^1, \ldots, b^{n+p}$. How can these coefficients be calculated? First, let us apply eq. (7) to functions $f(x) = 1$, $f(x) = x_1$, and $f(x) = x_2$, which are basis functions of the kernel of T.

We obtain

$$\sum_{\alpha=1}^{n+p} b^\alpha = \sum_{\alpha=1}^{n+p} b^\alpha x_{\alpha 1} = \sum_{\alpha=1}^{n+p} b^\alpha x_{\alpha 2} = 0 \tag{13}$$

which is a set of three equations in $(n + p + 3)$ unknowns. Applying eq. (7) to the spline function $\sigma(x)$ itself, we find $(n + p)$ additional equations which determine all unknowns.

$$\|T\sigma\|^2 = \langle T\sigma, T\sigma \rangle = \sum_{\alpha=1}^{n+p} b^\alpha \sigma(x_\alpha) \tag{14}$$

Replacing $\sigma(x)$ by its analytic expression (12) and using eqs. (13), eq. (14) can be shown to imply

$$\|T\sigma\|^2 = \sum_{\alpha=1}^{n+p} \sum_{\beta=1}^{n+p} b^\alpha b^\beta K_2(x_\alpha - x_\beta) \tag{15}$$

(In the following, $K_2(x_\alpha - x_\beta)$ will be noted simply $K_{\alpha\beta}$). Equation (15) is interesting. It shows that the expression to be minimized, $\|T\sigma\|^2$, is a quadratic form of unknown coefficients b^α. Minimization is constrained by (5), that is, $\sigma(x)$ should satisfy the exact and inequality data. But $\sigma(x_\alpha)$, for $\alpha = 1$ to $(n + p)$, is simply a linear function of unknown coefficients:

$$\sigma(x_\alpha) = c_0 + c_1 x_{\alpha 1} + c_2 x_{\alpha 2} + \sum_{\beta=1}^{n+p} b^\beta K_{\alpha\beta} \tag{16}$$

Coefficients of the spline apparently can be obtained by solving a classical problem of quadratic programming: minimization of the quadratic form (15) under linear conditions (5) (The three eqs. (13) can be shown to result from minimization of the quadratic form).

When no inequality constraints exist, the result is the well-known linear system of splines for exact data only (Duchon, 1975)

$$\sum_{\alpha=1}^{n+p} b^\alpha = \sum_{\alpha=1}^{n+p} b^\alpha x_{\alpha 1} = \sum_{\alpha=1}^{n+p} b^\alpha x_{\alpha 2} = 0 \tag{17}$$

where $\sigma(x_\alpha) = z_\alpha$ for $\alpha = 1, \ldots, n$.

Therefore, in conclusion, the spline function satisfying exact and inequality data can be obtained by the two following steps (in two dimensions):

- calculate the coefficients c_0, c_1, c_2, and b^α which minimize (15) under conditions (5)
- compute the spline function after (12), with functions $K_2(\cdot, x_\alpha)$ given by (9).

The method used to solve the quadratic minimization problem is presented and discussed in Kostov and Dubrule (1986).

GENERALIZATION TO KRIGING

Comments on Existing Methods

Barnes and Johnson (1984) presented a method for solving the kriging system under inequality constraints on kriging weights, rather than on interpolated values. Limic and Mikelic (1984) showed how to add positivity constraints on kriging weights in order to obtain positive estimates when estimating positive variables. Mallet (1980) applied a quadratic programming algorithm to a problem of multiple linear regression under linear constraints. Mallet was not specifically referring to kriging in his paper. Nevertheless, kriging is a multiple regression of unknown value $Z(x)$ against data $Z(x_\alpha)$, and Mallet's method can be simply applied to kriging. All these methods add inequality constraints to the classical kriging system. Kriging weights λ^α minimize

$$\text{Variance}\left[Z(x) - \sum_{\alpha=1}^{n} \lambda^\alpha Z(x_\alpha)\right]$$

not only under usual nonbias conditions of universal kriging (Matheron, 1971)

$$\sum_{\alpha=1}^{n} \lambda^\alpha f^1(x_\alpha) = f^1(x) \qquad \text{for all } 1$$

but also under constraints:

$$\lambda^\alpha \geq 0 \qquad (\text{for all } \alpha \text{ between 1 and } n)$$

(with Barnes and Johnson, 1984, or Limic and Mikelic, 1984), or

$$z'(x) \leq \sum_{\alpha=1}^{n} \lambda^\alpha z(x_\alpha) \leq z''(x)$$

(by generalizing Mallet's method, 1980, to kriging).

By expressing variance in terms of the λ^{α} and using quadratic programming techniques, this kind of problem can be solved. Can we apply a technique similar to Mallet's to our problem? Unfortunately, the problem is that the constraint on value $z(x)$ can only be used when estimating the value at point x itself. When estimating $z(x + \epsilon)$ at a point close to x, information about bounds on $z(x)$ are not available. One can use only exact data $z(x_{\alpha})$. The consequence of this problem is shown (Fig. 2) if we consider four exact values (abscissa 2, 3, 5, 6) whereas at $x = 4$ the value is known only to be greater or equal to 9. When estimating points around x (even at a small distance ϵ from x), the method can only take into account exact data from four points (location 2, 3, 5, 6), which explains why interpolated values are so small. Then, just at $x = 4$, the estimate "jumps" to take a value greater or equal to 9 (that is, exactly 9 here). This is not acceptable, because if the value is so large at $x = 4$, it is probably also large at $x = 3.99$, $x = 4.01$, etc. We want the information at x to be used also when estimating values surrounding x. Two more comments can be made:

- The behavior of the interpolator is similar to what happens when the variogram model has a large nugget effect: the estimated value varies discontinuously between data points and estimated points. However, the behavior we see (Fig. 2) has nothing to do with this nugget effect discontinuity: *it would happen even with a zero nugget effect.* The problem comes from "point by point" formulation of kriging.
- Works by Barnes and Johnson (1984), Limic and Mikelic (1984), or Mallet (1980) have many useful applications. What is demonstrated here is that they are not suitable for mapping applications where one needs an interpolator at all the points of a given domain. A satisfactory interpolating function (Fig. 3) has been produced by the spline method described earlier. This method calculates a *global function* honoring both exact data and inequality conditions. As a result, these conditions are built into the function and have an impact not only at the inequality points themselves, but also in their neighborhood.

In conclusion, classical formalism of kriging is difficult to use here because it works point by point. The solution seems to belong to methods, such as splines, which calculate global interpolating functions rather than point estimates. That is why the formalism of dual kriging, which expresses kriging as a global interpolating function, is going to be useful.

Formalism of Dual Kriging

Originally, the kriged value $z^*(x)$ at a point x was defined as a weighted average of n values $z_{\alpha} = z(x_{\alpha})$ measured at n surrounding data points. Matheron (1971) showed that one also could interpret kriging as an interpolating function, a linear combination of some elementary basis functions. If we consider an

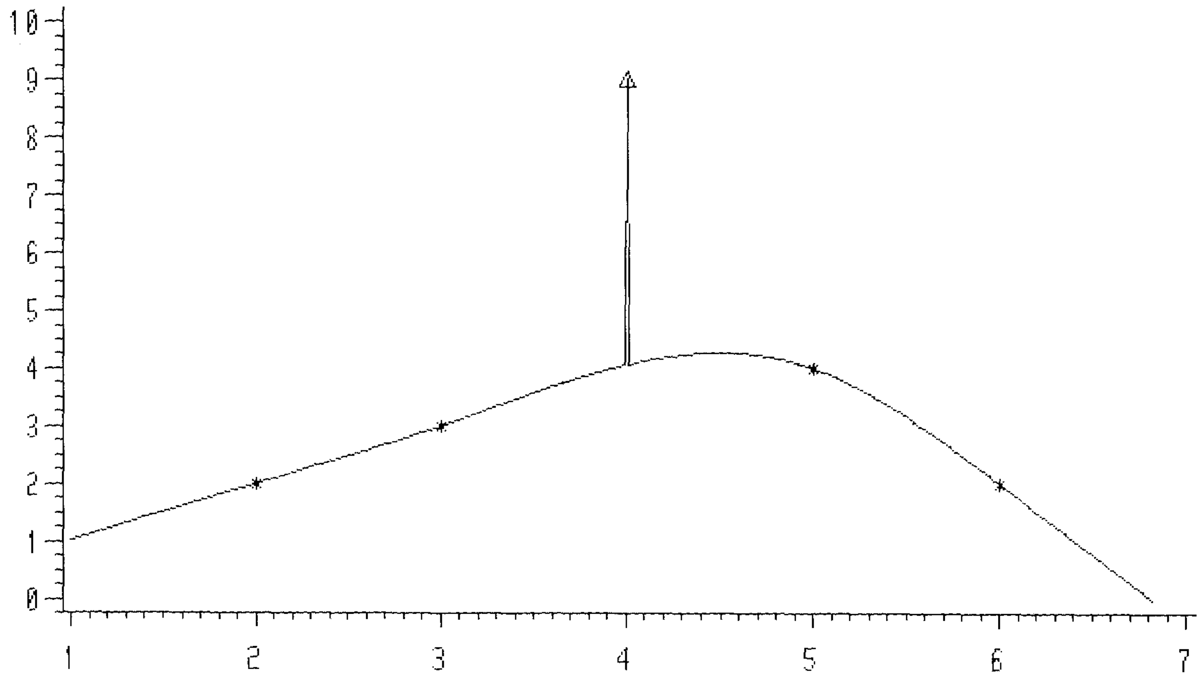

Fig. 2. Asterisks are exact data, whereas the triangle is a lower bound. The interpolator obtained by any existing method uses the inequality constraint only when estimating the constrained point itself. (Figs. 2–8 have been produced by commercial package SAS/GRAPH.)

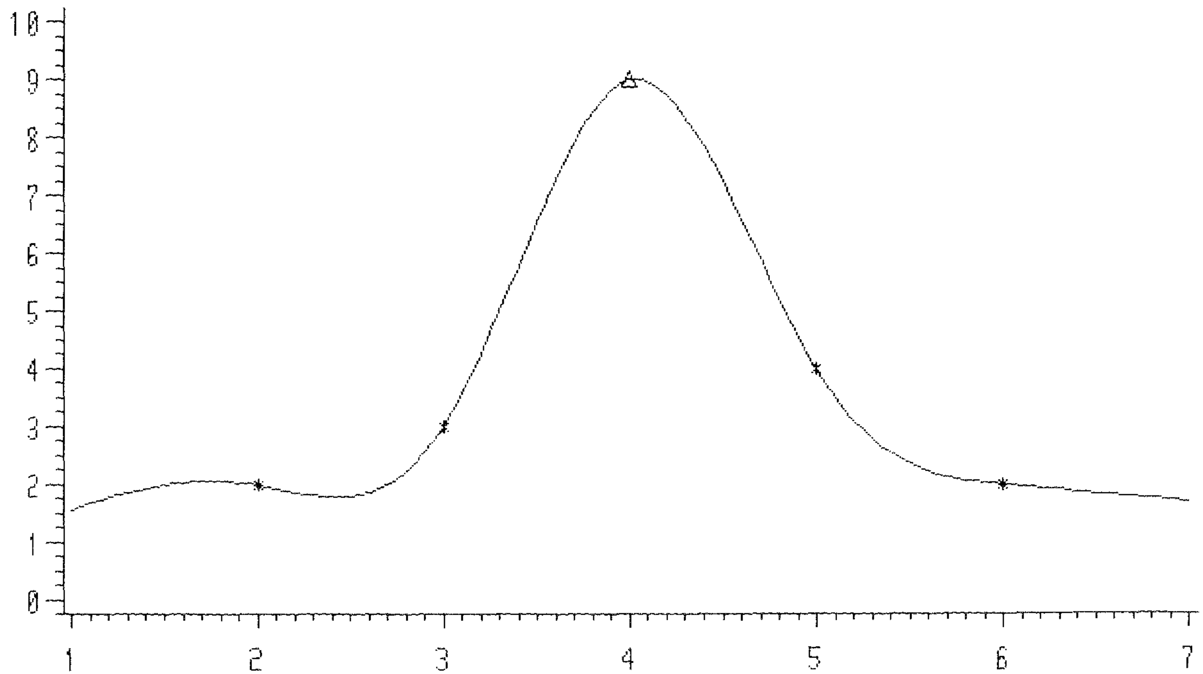

Fig. 3. Interpolation obtained with a constrained spline: information about the value at a constrained point also is taken into account when estimating surrounding points. Symbols as in Fig. 2.

intrinsic random function of order k (k − IRF) $Z(x)$ (Delfiner, 1975) with a generalized covariance $K(h)$, $z^*(x)$ can be written

$$z^*(x) = \sum_{l=0}^{m} c_l f^l(x) + \sum_{\alpha=1}^{n} b^\alpha K(x - x_\alpha) \tag{18}$$

where coefficients b^α and c_l are solutions of the dual kriging system

$$z^*(x_\beta) = \sum_{l=0}^{m} c_l f^l(x_\beta) + \sum_{\alpha=1}^{n} b^\alpha K(x_\alpha - x_\beta) = z_\beta \qquad (\beta = 1, \ldots, n)$$

$$\text{where } \sum_{\alpha=1}^{n} b^\alpha f^l(x_\alpha) = 0 \qquad (l = 0, \ldots, m) \tag{19}$$

(Functions f^l are monomials of degree $\leq k$ in the coordinates of x. For instance, in R^2, when $k = 1$, three monomials exist: $f^0(x_1, x_2) = 1, f^1(x_1, x_2) = x_1$, $f^2(x_1, x_2) = x_2$. The following simplified notations will be used: $f^l_x = f^l(x)$ and $f^l_\alpha = f^l(x_\alpha)$.

Developments will use the formalism of the theory of k − IRF, because this makes comparison with splines easier. However, previous remarks about dual kriging apply to ordinary and universal kriging (Journel and Huijbregts, 1978) by replacing the generalized covariance $K(h)$ by $-\gamma(h)$, where $\gamma(h)$ is the variogram. For instance, in the case of ordinary kriging, the kriging interpolator can be written

$$z^*(x) = c_0 - \sum_{\alpha=1}^{n} b^\alpha \gamma(x - x_\alpha) \tag{20}$$

(in the case of simple kriging with zero mean, we would have $c_0 = 0$). Many methods of interpolation calculate an interpolating function $g(x)$ which depends on a certain number of parameters. These parameters are determined by conditions that $g(x)$ honors the data, $g(x_\alpha) = z_\alpha$, for $\alpha = 1, \ldots, n$ and often by other particular conditions. Equations (18) and (19) define kriging in a similar manner and make the comparison of kriging with other interpolators easy (see Dubrule, 1981). In particular, the parallel between splines and dual kriging appears clearly. Matheron (1981) and Salkauskas (1982) compared splines and kriging in a rather abstract framework. Dubrule (1981) applied Matheron's results to classical problems of interpolation. Watson (1984) used a simple method to establish the parallel between the two methods. Comparison of eqs. (17), (18), and (19), shows that the spline interpolator is equal to:

- kriging of a 1 − IRF (linear trend) with generalized covariance $K(h) = |h|^3$ in R.
- kriging of a 1 − IRF with generalized covariance $K(h) = |h|^2 \log |h|$ in R^2.

This parallel is discussed in Dubrule (1984): the spline function appears to be a particular kriging interpolator, for which covariance, instead of being obtained from data, is fixed a priori. This "spline covariance" is $|h|^3$ in R and $|h|^2 \log |h|$ in R^2.

Quadratic Form Minimized by Coefficients of Dual Kriging

Earlier, the spline function was determined in two steps. First, we showed that the norm to be minimized was given by expression (15). Then, (15) was minimized under constraints (5). The right-hand side of (15) can be calculated for the kriging interpolator as well, merely by replacing spline covariance $K_2(h)$ with generalized covariance $K(h)$. Of course, this expression cannot be interpreted as a norm, as it was for the spline. However, do coefficients of dual kriging minimize

$$\sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} b^\alpha b^\beta K(x_\alpha - x_\beta) \tag{21}$$

under conditions $z^*(x_\alpha) = z_\alpha$? Introducing n Lagrange parameters d^α, this is equivalent to the minimization of the following expression, with respect to coefficients c_l, b^α, d^α

$$\sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} b^\alpha\, b^\beta\, K_{\alpha\beta} - 2 \sum_{\alpha=1}^{n} d^\alpha \left(\sum_{l=0}^{m} c_l f_\alpha^l + \sum_{\beta=1}^{n} b^\beta\, K_{\alpha\beta} - z_\alpha \right) \tag{22}$$

It is easy to check, by calculating partial derivatives of (22) with respect to all coefficients, that one first obtains

$$b^\alpha = d^\alpha \qquad \text{for all } \alpha$$

and that the other relations are nothing more than (19); that is, equations which characterize coefficients of dual kriging! This is an interesting result: it is not only kriging weights λ^α which minimize a quadratic form (the estimation variance). Coefficients of dual kriging themselves minimize a quadratic form, which is equal to $\|T\sigma\|^2$ in the particular case of a spline covariance. The direct consequence of this property is that the approach used with splines for inequality data can be generalized to kriging: we will minimize the quadratic form (21) not only under equality, but also under inequality constraints.

Last Theoretical Remark

We have shown that coefficients of dual kriging minimize a quadratic expression, but we have not been able to interpret this expression. This expression has been justified only in the case of splines. We know (Matheron, 1981) that to any generalized covariance $K(h)$ can be associated an operator T and two Hilbert spaces X and Y such that the kriging interpolator $z^*(x)$ is equal to the spline function relative to X, Y, and T (we are referring to generalized splines, where T can be any kind of operator, not only the second derivative, as it is for thin plate splines). In this case, the minimized quadratic form is equal to $\|Tz^*\|^2$.

However, this is only a formal equivalence, without much practical meaning, because "only in rare cases can the metric associated with a random function be defined in terms of differential operations" (Matheron, 1981).

A recent work by Journel (1986) shows that results presented above can also be obtained by working in the framework of random functions only, without any reference to dual kriging or to the theory of splines. Journel's "soft kriging" approach can take into account inequality data, or more generally information about "prior probability distribution" at data points.

SIMPLE EXAMPLES

A one-dimensional data set (Fig. 4) with $n = 9$ exact data and $p = 18$ inequality constraints has $p_1 = 12$ lesser bounds and $p_2 = 6$ greater bounds. Two points ($x = 4$ and $x = 6$) have two-sided inequality constraints. When only a standard interpolator package (for exact data only) is available, people often use the following iterative approach to take into account inequality data:

- A first interpolator (or map, in two dimensions) is produced using exact data only.
- Values obtained by this interpolator at inequality points are compared with greater and/or lesser bounds.
- Any maximum/minimum which is not honored is added to the exact data set and considered as an exact datum. The interpolator then is recalculated for the revised data set.

Problems are associated with this method. A spline may be used to interpolate between exact data only (Fig. 5). Inequalities which were not honored on Fig. 5 may be added to the exact data set, and the spline recalculated (Fig. 6). Two apparent problems with this method are:

- At abscissum 6, the interpolated value should be between 2 and 4. Although it was initially (Fig. 5), it is not subsequently (Fig. 6)!
- The interpolator should not be equal to lesser bounds at abscissa 8 to 10. Only the constraint at abscissum 7 should be used, which should automatically guarantee an interpolated value greater than the three minima at abscissa 8 to 10.

The spline function obtained by the method presented in this paper (Fig. 7) estimates more realistic values at abscissa 8 to 10. The function is equal to the lesser/greater bound only for some inequality constraints. Intuitively, the algorithm seems to "pick" the constraints it needs and interpolates them exactly, with the result that other constraints are satisfied automatically. We also made the following test: we picked only inequality constraints which were exactly honored on Fig. 7, merged them with exact data, and reran a standard

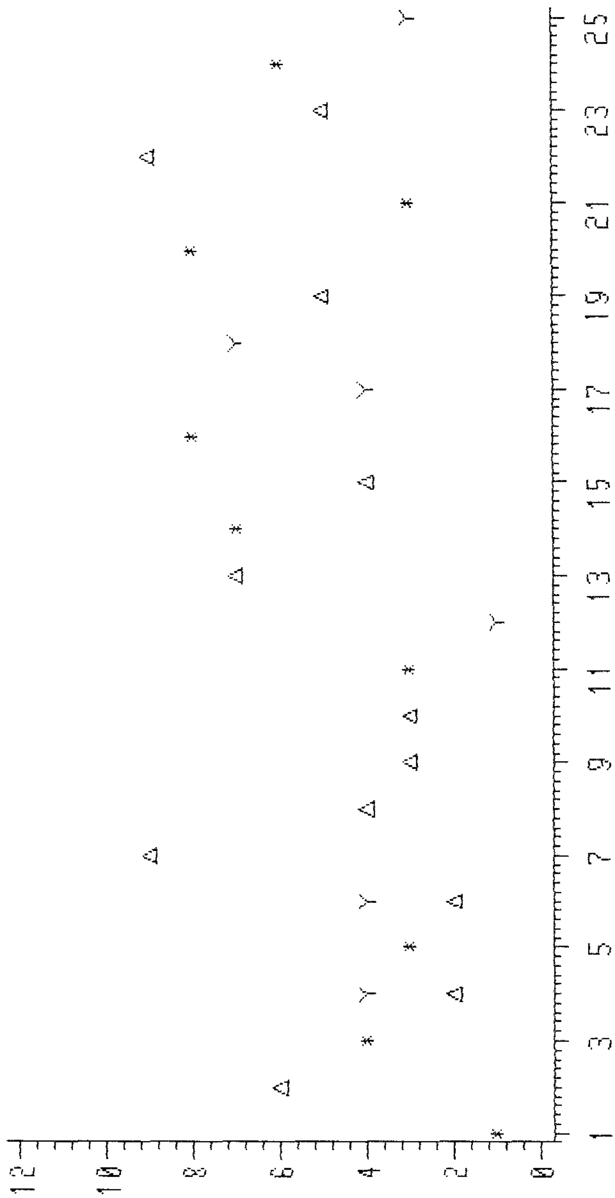

Fig. 4. Asterisks are exact data (at x = 1, 3, 5, 11, 14, 16, 20, 21, 24) whereas triangles are lesser bounds (at x = 2, 4, 6, 7, 8, 9, 10, 13, 15, 19, 22, 23) and "Y" are greater bounds (at x = 4, 6, 12, 17, 18, 25).

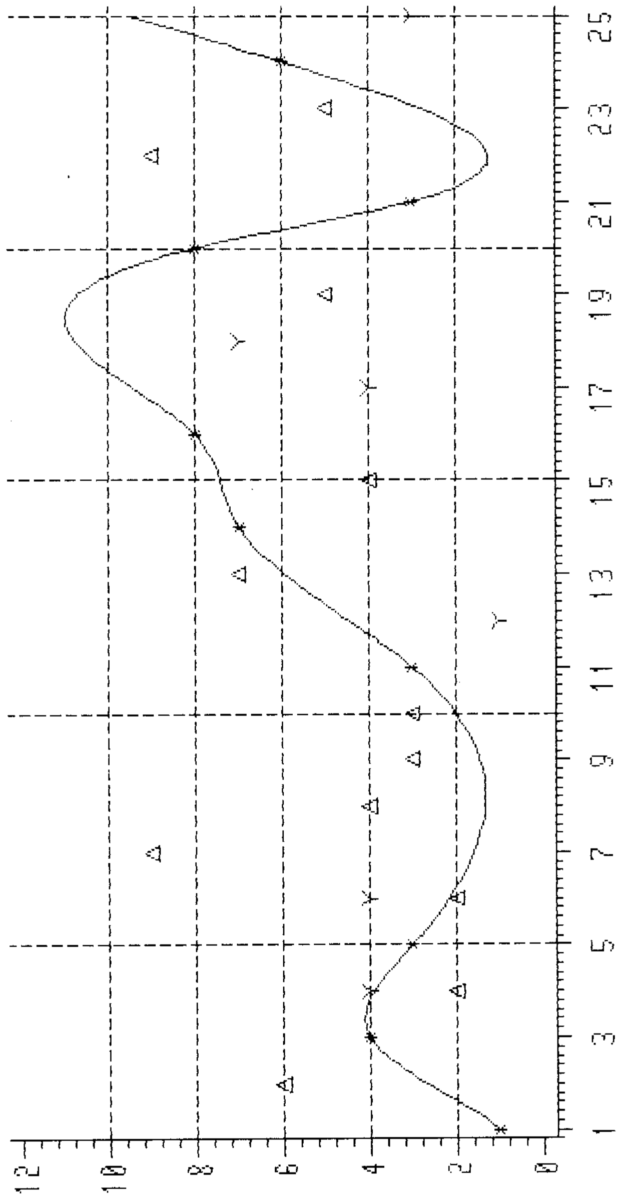

Fig. 5. Interpolating function obtained using only exact data. (The function is the thin plate spline, equal to a kriging in 1.IRF with generalized covariance $K(h) = h^3$). Symbols as in Fig. 4.

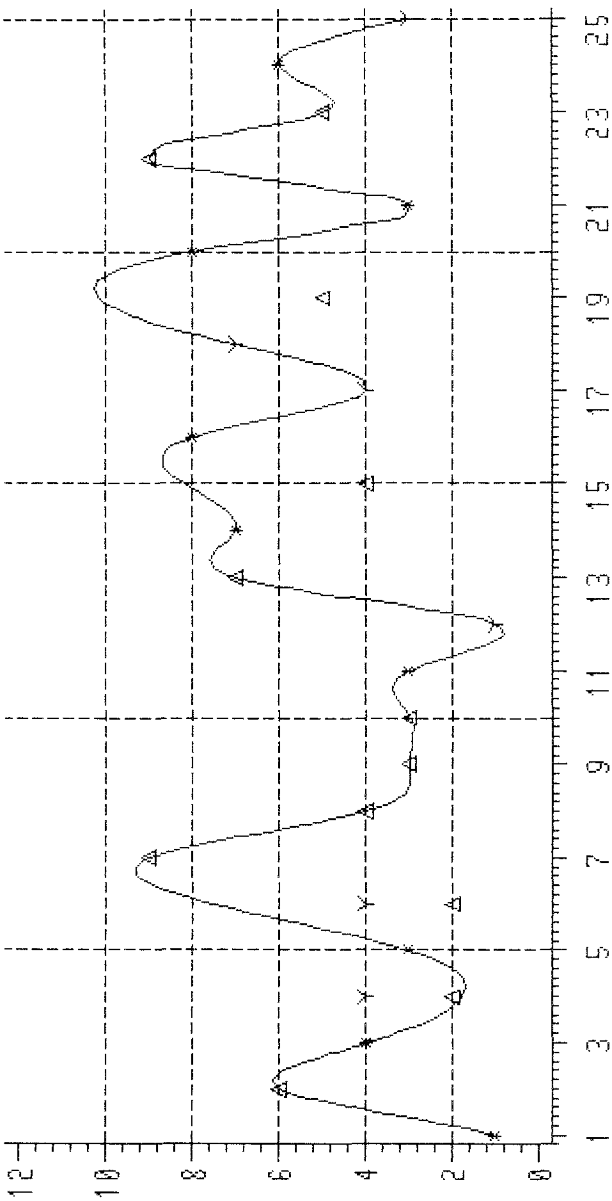

Fig. 6. Function obtained using exact data and inequality constraints which were not honored on Fig. 5. These constraints are considered as exact data. The interpolating function is the spline. Symbols as in Fig. 4.

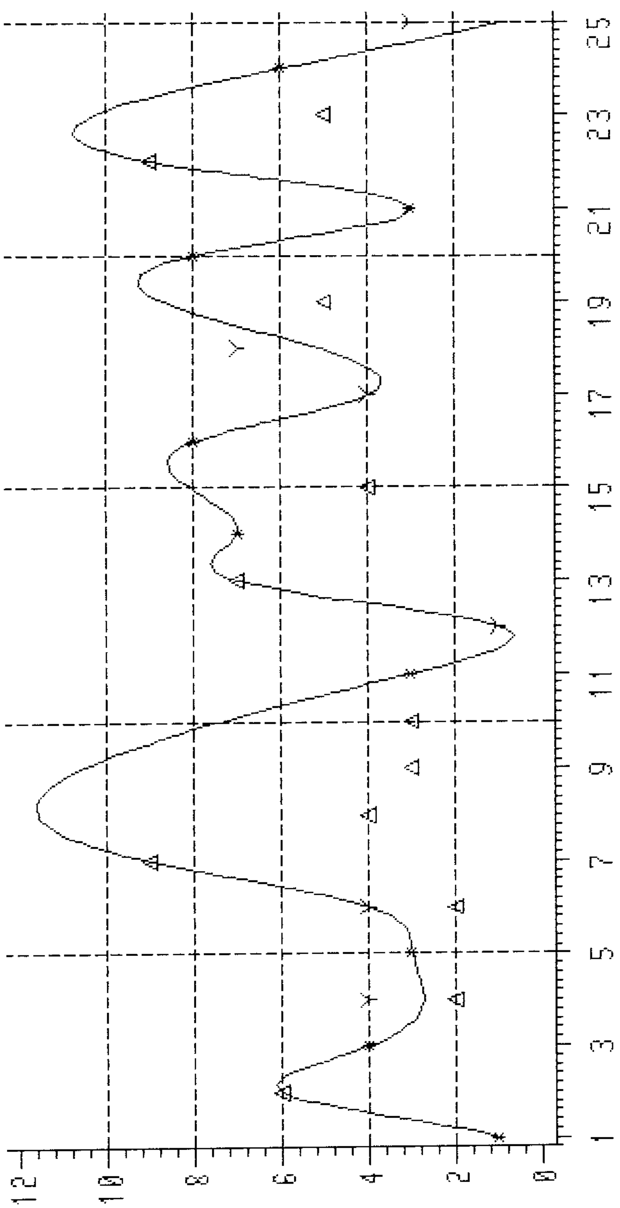

Fig. 7. Function obtained with spline method presented herein. The algorithm uses only "strongest" constraints. Once these constraints are exactly honored, others are satisfied automatically. Symbols as in Fig. 4.

spline program on this set of assumed exact data. We found the same function as the one shown on Fig. 7. In conclusion, one single run of the algorithm performs the two following steps automatically:

Step 1 select the "strongest" inequality constraints. "Strongest" means that, as soon as these inequalities are exactly honored, all the other inequalities are satisfied automatically.

Step 2 run a standard kriging on exact data made of
- original exact data
- strongest constraints

Finally, what the program does is nothing more than "choose" a subset of constraints from the initial data. Note that the method shown on Fig. 6 was doing the same thing, *except that it was not selecting the correct constraints.* Some were not necessary (abscissa 8 to 10 on Fig. 6) and the selected subset was not sufficient. Using these constraints as exact data did not imply that all other constraints were satisfied automatically (abscissum 6). Note that, in case only one inequality constraint is in the original data set, the two methods (presented on Fig. 6 and Fig. 7) are identical: either the inequality is satisfied automatically by the interpolator based on exact data only, and it is not used, or it is not satisfied automatically and it is used as an exact datum by the final interpolator. In optimization theory, this difference between "strongest" in equality constraints and others is derived from the "complementary slackness condition" (for more details, see Kostov and Dubrule, 1986).

The function obtained with a spherical variogram $\gamma(h)$ (range $= 2$) and no trend ($k = 0$) (Fig. 8) corresponds to ordinary kriging in the stationary case (Journel and Huijbregts, 1978). The interpolator can be written

$$z^*(x) = c_0 - \sum_{\alpha=1}^{n} b^{\alpha}\gamma(x - x_{\alpha}) \tag{23}$$

Such a model corresponds to a random function more irregular than that assumed by the spline model. Note that $z^*(x)$ always stays inside the range of data, which was not the case with the spline function (between abscissa 7 and 9 on Fig. 7); assuming that the variable is strongly irregular, the interpolator does not take the risk of extrapolating beyond the minimum and the maximum of the data (see Dubrule, 1984).

CONCLUSION

A method has been developed for interpolation under inequality constraints. This method is an adaptation of kriging. As such, it can provide a wide variety of interpolators, by changing the degree of the trend or covariance. It also can take into account the spatial variability of the variable under study if

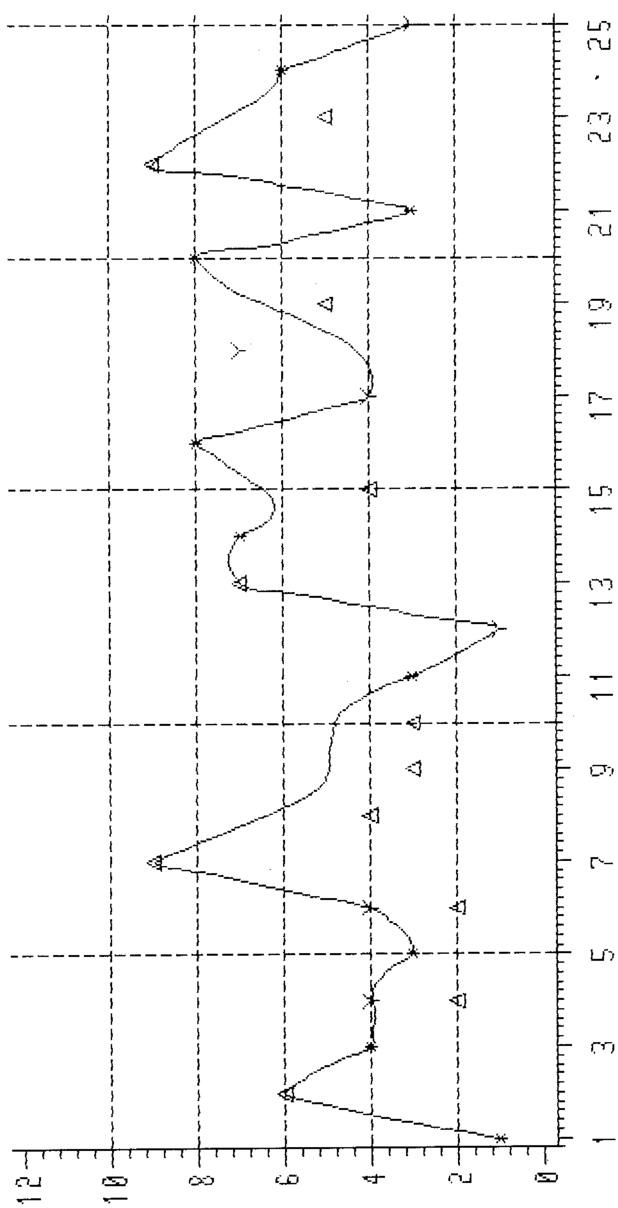

Fig. 8. Function obtained with the method presented herein, using a spherical variogram (range = 2) and $k = 0$. If only exact data existed the interpolator would be ordinary kriging (Journel and Huijbregts, 1978). Symbols as in Fig. 4.

a preliminary structural analysis is performed. This paper has presented the theory for the method, and shown, through a simple one-dimensional example, how the algorithm selects inequality constraints which are important in the interpolation process.

Discussion of computational aspects and application of the method to typical mapping problems are presented in Kostov and Dubrule, 1986.

ACKNOWLEDGMENTS

The authors wish to thank Sohio Petroleum Company for permission to publish this paper.

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