MASSLESS EQUATION FOR AN ANTISYMMETRIC TENSOR FIELD

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Starting from the formalism of covariant spin projection operators we present a general derivation and analysis of massless equations with zero helicity for an antisymmetric tensor field. We show that the minimal electromagnetic interaction for gauge-invariant equations is inconsistent.

1. The description of a massless field with zero helicity by an antisymmetric tensor field $F^{\mu\nu}$, given for the first time in [1], seems, at first glance, surprising because $F^{\mu\nu}$ contains only spin 1. However, if one takes into account the fact that helicity is related to the spin projection and that there exists, among others, also a zero projection for the unit spin, then in principle it is possible to describe also zero helicity. Since normally the $F^{\mu\nu}$ field is used to describe a photon with helicity ±1, an object with zero helicity was called notohp in [1].

Massless equations for zero helicity using an antisymmetric tensor field $F^{\mu\nu}$ have been considered in recent works [2-6]. An antisymmetric tensor field is of interest from the point of view of supersymmetric models [2] because addition of a spinor field allows one to describe a massless supermultiplet with helicities 0, 1/2.

The possibility of considering the antisymmetric tensor field $F^{\mu\nu}$ as a gauge field with zero helicity is not trivial. It is related to the fact that the $F^{\mu\nu}$ field is a direct sum of two irreducible fields, whose projection operators do not contain nonlocalities higher than $\Box^{-1}$. In this paper we shall show that, by use of the general formalism of covariant spin projection operators [7], it is possible to give a very simple derivation and analysis of a general massless equation for an antisymmetric tensor field $F^{\mu\nu}$. We present a general gauge transformation and a restriction to a source.

We shall show that the minimal electromagnetic interaction is inconsistent for all gauge fields, not only for an antisymmetric tensor field. The gauge fields are electrically neutral or have nonminimal interaction.

2. We represent an antisymmetric tensor field $F^{\mu\nu}$ as a direct sum $F_1 \oplus F_2$, where F_1 transforms according to the representation 1 = (1, 0) and F_2 according to 2 = (0, 1). For $F^{\mu\nu}$ it is possible to write the following general second-order equation:

$$\Box \begin{vmatrix} aP^1_{11} & bP^1_{12} \\ cP^1_{21} & aP^1_{22} \end{vmatrix} \begin{vmatrix} F_1 \\ F_2 \end{vmatrix} = 0, \tag{1}$$

where the P^s_{ij} are covariant spin projection operators satisfying [7]

$$P^s_{ij}\, P^{s'}_{kl} = \delta_{ss'}\delta_{jk}P^s_{il}, \tag{2}$$

and a, b, c are certain free parameters. With our choice of parameters the equation can be derived from a Lagrangian and for b = c the equation is P-invariant. Without loss of generality it is possible to assume that all the parameters are real.

We consider a gauge transformation of Eq. (1). As a gauge field we choose a vector field $h^\mu(x)$, which will be denoted by h_3. The representation 3 = (1/2, 1/2) is the unique representation that agrees with both representations 1 and 2 for gauge transformations which are linear in derivatives. A general gauge transformation is written in the form:

$$\begin{vmatrix} F_1 \\ F_2 \end{vmatrix} \to \begin{vmatrix} F_1 - \alpha i\sqrt{\Box}\, P^1_{23} h_3 \\ F_2 + \beta i\sqrt{\Box}\, P^1_{13} h_3 \end{vmatrix}. \tag{3}$$

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where the parameters α and β are to be determined.

We require gauge invariance. Using Eq. (2) we find that Eq. (1) is gauge-invariant if and only if a, b, and c satisfy

$$a^2 = bc \tag{4}$$

and α and β are related by the equation

$$\beta = -a\alpha/b. \tag{5}$$

In terms of the Gel'fand–Yaglom formalism the condition (4) means that the determinant of the spin-block corresponding to spin one is equal to zero.

We write Eq. (1) in the form $\pi F = 0$. If condition (4) is satisfied, then there exists an operator Q^z having the property $Q^z\pi = 0$. In the case of an external source J the operator Q^z gives for the equation $\pi F = J$ a restriction to the source $Q^z J = 0$. The operator Q^z has the form

$$Q^z = \sqrt{\Box}\,|\gamma P^1_{31}\ \delta P^1_{32}|, \tag{6}$$

where

$$\delta = -\gamma a/c. \tag{7}$$

In this way we see that in the formalism of covariant spin projection operators it is easy to write a general equation, gauge transformation, and a restriction to a source. It follows from (4), (5), and (7) that a, b, and c are nonzero. Since it follows from (4) that $a = \mp\sqrt{bc}$, it is always possible to choose a gauge $\pm\sqrt{\frac{c}{b}}P^1_{31}F_1 + P^1_{32}F_2 = 0$, for which Eq. (1) reduces to a zero mass equation $2a\,\Box\,F^{\mu\nu} = 0$.

In the case of massive equations $\pi F + m^2 F = 0$ the condition (4) means that the given equation describes one particle with spin one and mass $m/\sqrt{2a}$. Massive equations for $F^{\mu\nu}$ are considered in [8-10]. In [6] it is pointed out that the spin has a discontinuity when one goes to the massless limit. This idea is slightly inaccurate because in taking such a limit a zero spin projection is eliminated. This is the reason why helicities ±1 appear in notohp interactions [1]. $F^{\mu\nu}$ contains only spin 1 and this explains why an attempt to describe a massive spin zero field by the $F^{\mu\nu}$ field was unsuccessful [2].

3. Using the explicit forms of the operators P^s_{ij} (see Appendix) we write the results of the previous section in covariant form. We write a general form of Eq. (1)

$$\frac{2a-b-c}{2}\Box F^{\mu\nu} + (b+c)(\partial^\mu\partial_\varkappa F^{\varkappa\nu} - \partial^\nu\partial_\varkappa F^{\varkappa\mu}) + \frac{i(c-b)}{4}(\partial^\mu\partial_\rho\varepsilon^{\rho\nu}{}_{\varkappa\lambda}F^{\varkappa\lambda} - \partial^\nu\partial_\rho\varepsilon^{\rho\mu}{}_{\varkappa\lambda}F^{\varkappa\lambda} + 2\varepsilon^{\mu\nu\rho}{}_\varkappa\partial_\rho\partial_\sigma F^{\varkappa\sigma}) = 0. \tag{8}$$

Equation (8) is invariant with respect to the gauge transformation

$$F^{\mu\nu} \to F^{\mu\nu} + \frac{\alpha+\beta}{2}(\partial^\mu h^\nu - \partial^\nu h^\mu) + \frac{i(\alpha-\beta)}{2}\varepsilon^{\mu\nu\rho}{}_\sigma\partial_\rho h^\sigma. \tag{9}$$

Restriction to the antisymmetric source $J^{\mu\nu}$ is the following:

$$(\gamma+\delta)\,\partial_\mu J^{\mu\nu} + \frac{i(\gamma-\delta)}{2}\varepsilon^{\mu\nu}{}_{\rho\sigma}\partial_\mu J^{\rho\sigma} = 0. \tag{10}$$

Concrete choices of the parameters a, b, and c give various forms of the equation that all describe zero mass with helicity 0. In what follows we give only P-invariant equations. Then b = c and, from Eq. (4), we have two possibilities: b = −a or b = a.

4. We choose a = −b = 1/2, α = β = 1, γ = δ = 1/2. We obtain an equation [1]

$$\Box F^{\mu\nu} - \partial^\mu\partial_\varkappa F^{\varkappa\nu} + \partial^\nu\partial_\varkappa F^{\varkappa\mu} = 0 \tag{11}$$

with a gauge transformation

$$F^{\mu\nu} \to F^{\mu\nu} + \partial^\mu h^\nu - \partial^\nu h^\mu \tag{12}$$

and a restriction to a source

$$\partial_\mu J^{\mu\nu} = 0. \tag{13}$$

In this case we can choose the gauge

$$\partial_\mu F^{\mu\nu} = 0, \tag{14}$$

hence we obtain the Proca equation with spin 1 for h^μ. In the gauge (14) Eq. (11) reduces to the form $\Box F^{\mu\nu} = 0$. In this gauge in the momentum representation we find that in the system $\kappa^\mu = (\kappa, 0, 0, \kappa)$ one component (F^{12}) remains nonzero, which corresponds to the helicity 0. Here, it is interesting to point out that when the gauge freedom (12) is used for the second time the gauge field h^μ satisfies the massless Proca equation and h^μ becomes in turn a gauge field. This property leads to interesting peculiarities in quantization: since the ghost fields are gauge fields it is necessary to introduce secondary ghosts [3-5].

Equation (11) can be derived from the Lagrangian

$$L = \partial^\rho F^*_{\mu\nu}\partial_\rho F^{\mu\nu} - 2\partial^\mu F^*_{\mu\nu}\partial_\rho F^{\rho\nu}. \tag{15}$$

5. We choose $a = b = 1/2$, $\alpha = -\beta = 1$, $\gamma = -\delta = 1$. We obtain the equation

$$\partial^\mu\partial_\varkappa F^{\varkappa\nu} - \partial^\nu\partial_\varkappa F^{\varkappa\mu} = 0 \tag{16}$$

with the gauge transformation

$$F^{\mu\nu} \to F^{\mu\nu} + i\varepsilon^{\mu\nu\rho}{}_\sigma\partial_\rho h^\sigma. \tag{17}$$

and the restriction to the source

$$i\,\varepsilon^{\mu\nu}{}_{\rho\sigma}\partial_\mu J^\sigma = 0. \tag{18}$$

In the direct analysis of Eq. (16) we introduce a dual tensor $\tilde{F}^{\mu\nu} = \frac{i}{2}\varepsilon^{\mu\nu}{}_{\rho\sigma}F^{\rho\sigma}$ with the gauge transformation

$$\tilde{F}^{\mu\nu} \to \tilde{F}^{\mu\nu} + \partial^\mu h^\nu - \partial^\nu h^\mu. \tag{19}$$

Equation (16) can now be written in the following equivalent form:

$$\Box F^{\mu\nu} - i\varepsilon^{\mu\nu\rho}{}_\sigma\partial_\rho\partial_\varkappa\tilde{F}^{\varkappa\sigma} = 0. \tag{20}$$

In this case there exists a gauge $\partial_\mu\tilde{F}^{\mu\nu} = 0$ in which $\Box F^{\mu\nu} = 0$. In the system $\kappa^\mu = (\kappa, 0, 0, \kappa)$ there is one nonzero component $i\tilde{F}^{12} = F^{03}$.

Equation (16) can be derived from the Lagrangian

$$L = \partial^\mu F^*_{\mu\nu}\partial_\rho F^{\rho\nu}. \tag{21}$$

6. In this section we show the inconsistency of the minimal electromagnetic interaction of gauge fields.

We write the equation for the field $F^{\mu\nu}$ with the source $J^{\mu\nu}$ in the following general form:

$$(\partial_\rho\partial_\sigma\beta^{\rho\sigma})^{\mu\nu}{}_{\varkappa\lambda}F^{\varkappa\lambda} = J^{\mu\nu}. \tag{22}$$

A necessary condition for consistency of Eq. (22) follows from the gauge invariance $F^{\mu\nu} \to F^{\mu\nu} + Q^{\mu\cdot}_{g}\sigma h^\sigma$ of the left-hand side of the given equation: for the consistency of Eq. (22) it is necessary that the source $J_{\mu\nu}$ be gauge invariant with respect to the same gauge transformation. When the source does not depend on the field $F^{\mu\nu}$ the condition given above is satisfied, but when the source depends on the field $F^{\mu\nu}$, it is necessary to verify separately if the source is gauge invariant. We note that the condition for consistency given above does not depend on the form of the equation and is correct for all gauge-invariant equations (second-order, as well as first-order).

We consider the minimal electromagnetic interaction given by the substitution $\partial_\rho \to D_\rho = \partial_\rho - ieA_\rho$, where A_ρ is the electromagnetic potential. After substituting $\partial_\rho \to D_\rho$ in the free equation $(D_\rho D_\sigma\beta^{\rho\sigma})F = 0$ we obtain an equation of the form (22), where the source $J^{\mu\nu}$ is linear in the field $F^{\mu\nu}$, i.e., $J^{\mu\nu} = j(\partial, A)^{\mu\nu}{}_{\varkappa\lambda}F^{\varkappa\lambda}$. The given source is not gauge invariant with respect to the transformations $F \to F + Q_g h$ because there is a noninvariant term

$e^2(A_\rho A_\sigma \beta^{\rho\sigma})Q_g h$ in the source. The same result is also obtained in the case of the first-order gauge fields $(i\partial_\rho \beta^\rho)\psi = 0$, $\psi \to \psi + Q_g \varepsilon$, because the substitution $\partial_\rho \to D_\rho$ leads to a noninvariant source $J = -e(A_\rho \beta^\rho)\psi$. Since the result does not depend on the concrete form of the equation, it follows that all gauge-invariant fields are inconsistent when there is a minimal electromagnetic interaction.

It is well known that the minimal electromagnetic interaction is a consequence of the gauge invariance of the massive equations. For the massive equation $(\partial_\rho \partial_\sigma \beta^{\rho\sigma})F + m^2 F = 0$ the substitution $\partial_\rho \to D_\rho$ gives the source J given above. Upon the gauge transformation $A^\mu \to A^\mu + \partial^\mu \varphi$ the change in the source J is compensated by the transformation $F \to e^{-ie\varphi}F$. In the mass-less limit $m \to 0$ there is an additional gauge invariance $F \to F + Q_g h$ which breaks the invariance of the source.

APPENDIX

We give the projection operators P^s_{ij} satisfying (2):

$$(P^1_{13})^{\mu\nu}{}_\varkappa = \frac{1}{2\sqrt{\Box}}(\eta^\nu{}_\varkappa \partial^\mu - \eta^\mu{}_\varkappa \partial^\nu + i\partial_\rho \varepsilon^{\rho\mu\nu}{}_\varkappa),$$

$$(P^1_{23})^{\mu\nu}{}_\varkappa = \frac{1}{2\sqrt{\Box}}(\eta^\nu{}_\varkappa \partial^\mu - \eta^\mu{}_\varkappa \partial^\nu - i\partial_\rho \varepsilon^{\rho\mu\nu}{}_\varkappa),$$

$$(P^1_{31})^\mu{}_{\varkappa\lambda} = \frac{1}{2\sqrt{\Box}}(\eta^\mu{}_\lambda \partial_\varkappa - \eta^\mu{}_\varkappa \partial_\lambda + i\partial_\rho \varepsilon^{\rho\mu}{}_{\varkappa\lambda}),$$

$$(P^1_{32})^\mu{}_{\varkappa\lambda} = \frac{1}{2\sqrt{\Box}}(\eta^\mu{}_\lambda \partial_\varkappa - \eta^\mu{}_\varkappa \partial_\lambda - i\partial_\rho \varepsilon^{\rho\mu}{}_{\varkappa\lambda}),$$

where $\Box = \partial_\mu \partial^\mu$ and $\eta^{\mu\nu} = \mathrm{diag}\,(+\,-\,-\,-)$. The remaining operators are calculated from the relations

$$P^1_{11} = P^1_{13}P^1_{31}, \quad P^1_{22} = P^1_{23}P^1_{32}, \quad P^1_{12} = P^1_{13}P^1_{32} \text{ and } P^1_{21} = P^1_{23}P^1_{31}.$$

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