NUMERICAL SOLUTION OF THREE-DIMENSIONAL THERMOPLASTICITY PROBLEMS FOR BODIES OF COMPLEX SHAPE

V. M. Pavlychko

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One of the numerical methods for solving spatial thermoplasticity problems [5] is the finite element method. A three-dimensional thermoelastic-plastic analysis of a number of bodies of different configuration is made on its basis [1, 2, 4]. The necessity occurs here to solve large systems of linear algebraic equations (SLAE). The direct methods applied in the papers mentioned above result in large machine time expenditures and require a considerable volume of electronic computer storage.

We use gradient methods to solve thermoplasticity problems since only nonzero SLAE matrix elements are used in them. The quantity of these elements per matrix row is determined only by the mesh topology and is independent of the quantity of nodes, which permits a significant saving in electronic computer resources when investigating bodies of complex geometry.

1. Formulation of the Problem and Introduction of the Finite-Element Approximation

Let us consider a body of arbitrary shape subjected to the action of a nonstationary temperature field T and volume forces $\vec{K}(K_1, K_2, K_3)$. Surface forces $\vec{t}_n(t_{n1}, t_{n2}, t_{n3})$ act on part of the body surface Σ_t, while displacements $\vec{u}^*(u^*_1, u^*_2, u^*_3)$ are given on the remainder of the body surface Σ_u. The temperature value T_0 corresponds to the natural (unstressed) state of the body. The mechanical properties of the body material depend on the temperature.

We will solve the problem under consideration in a quasistatic formulation. The creep strains are assumed neglibibly small as compared with the instantaneous elastic-plastic strains. For this, the range of variation of the load, temperature, and time in all the body elements should lie below the surface of the conditional creep limit [5].

Let us assume that the appearance of plastic strains in the body is possible under the action of the mentioned load and loading is realized along rectilinear strain trajectories or those deviating slightly from them, for which the following equations of state are valid [5]

$$\sigma_{ij} = 2G^*\varepsilon_{ij} + 3\lambda^*\varepsilon_0\delta_{ij} - \sigma^a_{ij}, \tag{1.1}$$

where σ_{ij} and ε_{ij} are stress and strain tensor components, respectively

$$\sigma^a_{ij} = 2G^*\varepsilon^{1p}_{ij} + K\varepsilon_T\delta_{ij}; \tag{1.2}$$

$$\lambda^* = (K - 2G^*)/3; \tag{1.3}$$

$$G^* = \begin{cases} S/2\Gamma & \text{(under active loading)}; \\ G & \text{(under elastic unloading)}; \end{cases} \tag{1.4}$$

$\varepsilon_{ij}{}^{1p}$ are the plastic strain components occurring in a body element up to the time of unloading (we set $\varepsilon_{ij}{}^{1p} = 0$ under active loading) $\varepsilon_T = \alpha_T(T - T_0)$ is the purely thermal strain, α_T is the coefficient of linear thermal expansion, K is the volume expansion modulus, G is the shear modulus, $S = (s_{ij}s_{ij}/2)^{1/2}$ is the tangential stress intensity, $s_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$; $\Gamma = (e_{ij}e_{ij}/2)^{1/2}$ is the shear strain intensity, $e_{ij} = \varepsilon_{ij} - \varepsilon_0\delta_{ij}$; $\varepsilon_0 = \varepsilon_{kk}/3$; δ_{ij} is the Kronecker delta. Summation between 1 and 3 is performed over repeated subscripts in the monomial expressions if other limits of their variation are not given.

In the general case the tangential stress intensity S is a functional of the shear strain intensity Γ and the temperature T [5], but this dependence is described simply by

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functions under the assumptions made. It is assumed that the latter is independent of the kind of stress state and is determined from test data on uniaxial tension of cylindrical specimens for different fixed temperature values.

We realize the solution of the thermoplasticity problem in a Cartesian coordinate system on the basis of the Lagrange variational equation [5], which, under the assumption of invariability of the volume and surface forces, the additional stresses σ^a_{ij}, and the variable elasticity parameters G^* and λ^* with relationships (1.1)-(1.4) taken into account, take the form

$$\delta\text{Э} = 0, \tag{1.5}$$

where

$$\text{Э} = \int_V \left(2G^*\Gamma^2 + \frac{3}{2}K\varepsilon_0^2\right) dV - \int_V (\sigma^a_{ij}\varepsilon_{ij} + K_i u_i)\, dV - \int_{\Sigma_t} t_{ni} u_i d\Sigma. \tag{1.6}$$

The Cauchy relationships

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1.7}$$

and kinematic boundary conditions

$$u_i = u_i^* \text{ on } \Sigma_u. \tag{1.8}$$

must be added to the variational formulation of the problem (1.5) and (1.6).

We use tetrahedral finite elements with a linear approximation of the displacements to discretize the functional (1.6). We assume that the body volume V is represented here by the sum of M tetrahedra, the body surface Σ is the sum of L triangles for a total quantity N of nodes. Let x_{ip} be coordinates and u_{ip} the displacement of the node numbered p ($p = \overline{1, N}$). Then the displacements of an arbitrary point of the m-th element whose vertices have the numbers n_1, n_2, n_3, n_4 are evaluated from the formula

$$\{u_i\}_m = u_{ip}(D^m_{pj}x_j + D^m_{p4})/D^m \quad (p = n_1, n_2, n_3, n_4), \tag{1.9}$$

where

$$D^m = \det \begin{Vmatrix} x_{1n_1} & x_{2n_1} & x_{3n_1} & 1 \\ x_{1n_2} & x_{2n_2} & x_{3n_2} & 1 \\ x_{1n_3} & x_{2n_3} & x_{3n_3} & 1 \\ x_{1n_4} & x_{2n_4} & x_{3n_4} & 1 \end{Vmatrix}; \tag{1.10}$$

D_{pj}^m are algebraic cofactors to x_{jp} in D^m, D_{p4}^m is the algebraic cofactor to one in the row of D^m that contains coordinates of the node numbered p ($p = n_1, n_2, n_3, n_4$), x_j are coordinates of a point within the element. Here and henceforth, it is assumed that the quantities within the braces with subscript m are determined within the m-th element.

Taking account of the approximation (1.9), the Cauchy relationships (1.7) yield expressions for the strain in an element

$$\{\varepsilon_{ij}\}_m = \frac{1}{2D^m}(u_{ip}D^m_{pj} + u_{jp}D^m_{pi}) \quad (p = n_1, n_2, n_3, n_4). \tag{1.11}$$

Let us replace the integral over the body volume V in (1.6) by a sum of integrals over volumes of all the elements and the integral over the surface Σ_t by a sum of integrals over all the loaded boundaries of the finite elements. Taking the average of the volume force and temperature values in each element and the values of the surface loads in each triangle belonging to the surface Σ_t, and integrating, we obtain the discrete analog of the functional (1.6)

$$\text{Э} = \sum_{m=1}^{M} (\text{Э}^m_0 - \text{Э}^m_V) - \sum_{l=1}^{L} \text{Э}^l_\Sigma, \tag{1.12}$$

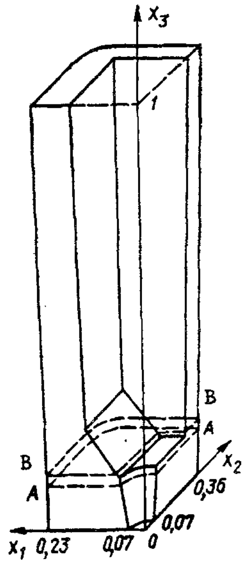

Fig. 1

where

$$Э_0^m = \left(\frac{1}{3}\langle G^*\rangle_m\{\Gamma^2\}_m + \frac{1}{4}\langle K\rangle_m\{\varepsilon_0^2\}_m\right)|D^m|; \qquad (1.13)$$

$$Э_V^m = \left(\frac{1}{6}\langle\sigma_{ij}^a\rangle_m\{\varepsilon_{ij}\}_m + \frac{1}{24}(u_{in_1} + u_{in_2} + u_{in_3} + u_{in_4})\langle K_i\rangle_m\right)|D^m|; \qquad (1.14)$$

$$Э_\Sigma^l = \frac{1}{3}(u_{ik_1} + u_{ik_2} + u_{ik_3})t_{ni}^l S_l; \qquad (1.15)$$

the angular brackets $<\ldots>_m$ denote the mean values of the corresponding quantities in the element numbered m, t_{ni}^{ℓ} are the mean values of the surface load components in the triangle numbered ℓ formed by the nodes k_1, k_2, k_3, and S_ℓ is the area of this triangle.

Therefore, according to (1.11) and (1.13)-(1.15) the functional (1.12) is a function of discrete values of the displacement components at the nodal points of a finite-element mesh. Then differentiation of (1.12) with respect to the nodal displacements with the relationships (1.1)-(1.4), (1.11), and (1.13)-(1.15) taken into account yields an expression for the gradient component of the functional (1.12).

$$g_{ip} = \frac{1}{6}\sum_m \langle\sigma_{ij}\rangle_m D_{pj}^m \operatorname{sign} D^m - \frac{1}{24}\sum_m \langle K_i\rangle_m |D^m| - \frac{1}{3}\sum_l t_{ni}^l S_l \quad (p = \overline{1, N}). \qquad (1.16)$$

Here the summation in the first two sums is over all finite elements containing nodes numbered p, and in the last overall loaded faces of the finite elements containing this node.

2. Construction and Approval of the Algorithm to Solve the Problem

We trace the body loading history by separating the whole loading process into a number of stages in such a manner that the times delimiting them would, if possible, agree with the times of the change in direction of the strain process for the separate body elements from active loading to unloading, and conversely [5]. We use the method of variable elasticity parameters to construct the algorithm of the solution in each loading stage. In a first approximation we set here $G^* = G$, $\lambda^* = (K - 2G)/3$. As values of the plastic strains $\varepsilon_{ij}{}^{1p}$ at the unloading time we use their value at the end of the preceding stage (we set $\varepsilon_{ij}{}^{1p} = 0$ in the first stage). We find the displacement of the nodes of the finite-element mesh u_{ip} ($p = \overline{1, N}$) by minimizing the functional (1.12) by using gradient methods. We check the convergence of these latter by the diminution of the absolute value of the gradient vector (1.16) relative to its value for zero displacements

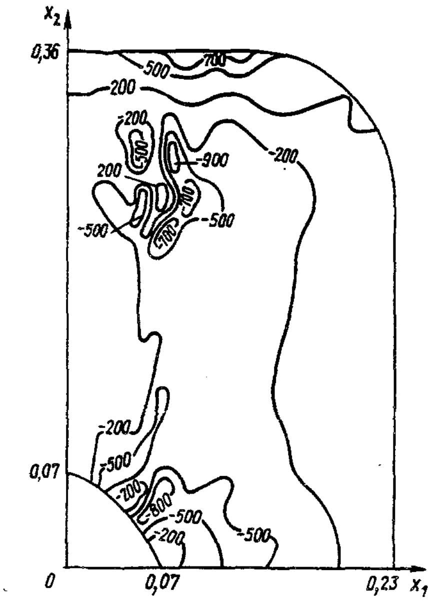

Fig. 2

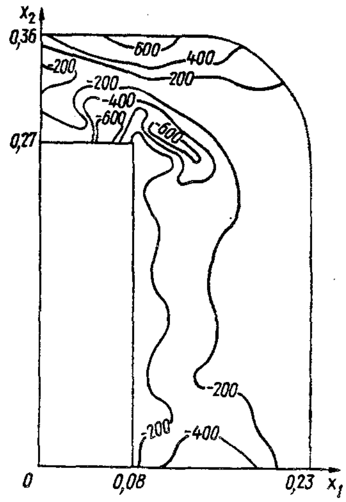

Fig. 3

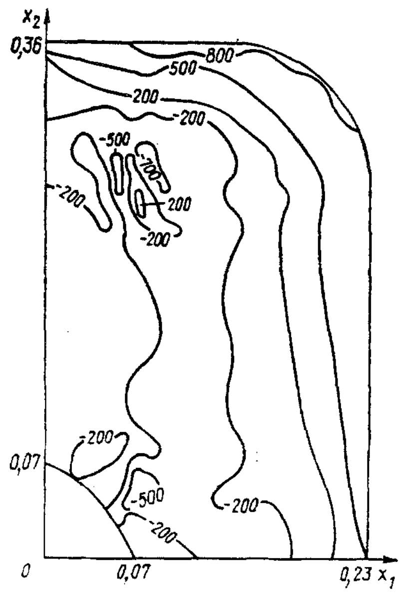

Fig. 4

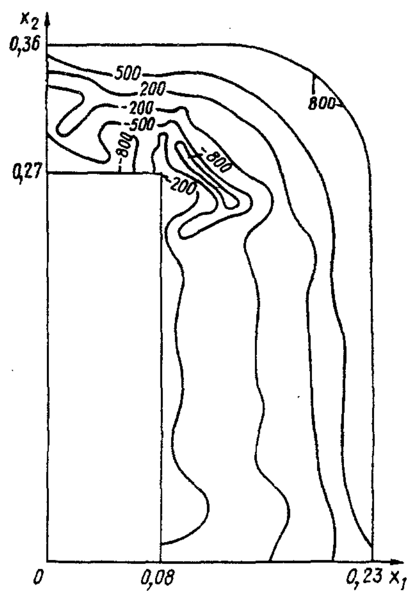

Fig. 5

$$\varkappa_1 = \frac{(g_{ip}g_{ip})^{1/2}}{(g_{ip}g_{ip})^{1/2}\big|_{\vec{u}\equiv 0}} \quad (p = \overline{1, N}). \tag{2.1}$$

We take account of the kinematic boundary conditions (1.8) by using given values of the displacements for the calculation of the gradient (1.16). Since the given displacements do not vary, then the gradient components (1.16), obtained by differentiating the functional (1.12) with respect to these displacements must be made zero.

We determine the strains ε_{ij} (1.11) from the displacements found, and from them we calculate the strain deviator components e_{ij} and the shear strain intensity Γ in all the finite elements. Then by using the functional dependence $S = S(\Gamma, T)$, we find the tangential stress intensity S and we refine the variable elasticity parameters G^* and λ^* (1.3) and (1.4). We determine the directivity of the process by the increment in the plastic strain intensity in a first approximation. If this increment is non-negative, then we set $\varepsilon_{ij}{}^{1p} = 0$ for all the subsequent approximations of this stage. Afterwards we turn to the second approximation, etc.

We cut off the successive approximations when the condition

$$(\Gamma^{(n)} - \Gamma^{(n-1)})/\Gamma^{(n)} \leqslant \varkappa_2, \quad (2.2)$$

is satisfied in all the elements, where $\Gamma^{(n-1)}$ and $\Gamma^{(n)}$ are values of the shear strain intensity at two successive iterations, and $\varkappa_2$ is the given accuracy.

Approval of the method elucidated above and its comparison with the method utilized in [2] were realized in a number of problems for prismatic bodies, published in [5]. The computations performed for a different quantity of nodes for both uniform and nonuniform partition of the body volume permit making the following deductions.

1. All the tested gradient methods (steepest descent, conjugate gradients, upper relaxation methods) are well recommended for solving problems on a uniform mesh and for a weak change in G^* within the body limits. The most effective of these, the conjugate gradient method, permits the solution to be obtained considerably more rapidly than the other method [2].

2. The nonuniformity of the partitioning and the strong change in G^* result in strong prolateness of the equipotential surfaces of the functional (1.12) in the space of the displacements u_{ip}. This substantially reduces the efficiency of the conjugate gradient method which it is expedient to use only for final refinement of the solution in this case. It is more preferable to use the upper relaxation method in the first iterations in G^*.

3. It is sufficient to limit oneself to a diminution of the parameter $\varkappa_1$ (2.1) by 4-5 orders of magnitude (by 2 orders in certain problems) in each approximation in G^*. A more exact estimate can be obtained in each specific case by investigating the spectral properties of the Hess matrix of the functional (1.12).

3. Example of a Computation

A quadrangular vessel with a conical hole in the center of the bottom, fabricated from the heat-resistant alloy ÉI-395 (the thermomechanical characteristics are presented in the monograph [5]) is subjected to slow heating without resulting in the origination of unloading. At the end of the heating the temperature at the outer surface equals the initial value of 20°C while it is 820°C at the inner and varies linearly over the wall and bottom thickness. Consequently, the computation of the thermoplastic stress state of the body can be performed in one stage. It is assumed that the vessel is mounted in a rigid horizontal plane and that points of its base do not emerge outside the limits of this plane during deformation. The presence of symmetry conditions permits limiting oneself to analysis of the stress state of the quarter of the vessel displayed in Fig. 1. The normal shifts and tangential stresses in the planes $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$ are here assumed to equal zero while the remaining part of the body surface is free of external load.

Discretization of the body volume into finite elements was performed automatically by the method elucidated in [3]. Partitioning of the body into hexahedral cells in the form of curvilinear parallelepipeds (4 cells over the wall thickness and 18 over the height) was realized in the first stage. Then each of the cells was partitioned into 5 tetrahedra. In all 6920 finite elements were obtained for 1910 nodes. The analysis was performed by the method elucidated above with accuracy $\varkappa_2 = 0.01$.

Figures 2 and 3 show the stress distribution σ_{11} (MPa) at points of the planes marked in Fig. 1 by the dashed lines A and B, respectively. The stresses σ_{33} (MPa) are presented in Fig. 4 (plane A) and Fig. 5 (plane B).

Analysis of the results shows that the normal stresses reach a maximum at points lying near the outer body surface, in the neighborhood of the hole in the bottom, and in the zone of wall connection to the bottom. The maximal values of the stresses σ_{11} and σ_{33} here exceed the maximal values of the stress σ_{22} by almost three times while the maximal values of the tangential stress are an order below the normal stress. The most complex nature of the stress distribution is in the neighborhood of the hole and near the inner surface of the bottom. Plasticity domains occur at these same sites and are developed in the whole bottom thickness near the hole. In order to verify the reliability of the results, satisfaction of the static boundary conditions was investigated. The stress on the free part of the body surface, obtained by linear extrapolation of their values at points near the boundary, does not here exceed 3-4% of the maximal stress. This is a guarantee of satisfactory compliance with the boundary conditions.

The computation results described above permit making the deduction that the method developed can be used successfully for a thermoplastic three-dimensional analysis of bodies of complex geometry.

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ELASTIC EQUILIBRIUM IN A LAYER INHOMOGENEOUS WITH DEPTH

A. N. Borodachev

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It is well known that boundary value problems of the theory of elasticity are difficult to solve analytically in the inhomogeneous case. Therefore various simplified models of the inhomogeneities of an elastic material have been widely used. The most often-used assumption is that the Poisson coefficient of the material is a constant, while the shear modulus (or modulus of elasticity) varies with one of the coordinates as a power law or an exponential [3-5, 9].

In the present paper we consider a somewhat more general model of an inhomogeneous elastic material, in which the shear modulus is assumed constant, while the Poisson coefficient (or the modulus of elasticity) depends on one of the coordinates in an arbitrary manner. In this model one can introduce stress functions which satisfy second-order partial differential equations with constant coefficients.

With the help of the stress functions and the two-dimensional Fourier transform we construct the general solution (in transform space) of the problem for the equilibrium of a layer inhomogeneous with depth for different types of boundary conditions on its surfaces. The case when the surfaces of the layer are free from tangential stresses is discussed in detail. We obtain an explicit expression for the characteristic kernel for a layer lying on a perfectly rigid substrate, in the absence of friction.

1. General Solution of the Problem. We consider the equilibrium of an infinite elastic layer $S = \{x^0 : |x| < \infty, 0 \le x_3 \le h\}$, where $x^0 = (x_1, x_2, x_3) \equiv (x, x_3)$, $x = (x_1, x_2)$, $0 < h < \infty$ is the thickness of the layer, and x_1, x_2, x_3 are rectangular coordinates. The shear modulus $\mu > 0$ of the layer is constant, while the Poisson coefficient $\nu(x_3)$ is an arbitrary, but sufficiently smooth, function satisfying the usual condition $-1 < \nu(x_3) < 1/2$ [10]. In this case the modulus of elasticity of the material $E(x_3) = 2\mu[1 + \nu(x_3)]$ is a positive-definite function of the coordinate x_3.

Institute of Mechanics, Academy of Sciences of the Ukrainian SSR, Kiev.

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