CANONICAL QUANTIZATION OF SINGULAR THEORIES

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1. INTRODUCTION

It is well known that the recently most popular theories in physics of gauge fields belong to the so-called singular Lagrangian theories. In the general case these theories contain relations (between the coordinates and velocities in the Lagrangian formalism or coordinates and momenta in the Hamiltonian formalism) and degeneracy of the equations of motion, so that solution of the latter include functional arbitrariness. Dirac's method [4], based on a Hamiltonian formalism, canonical quantization, was used successfully [1-3] to quantize theories of the type mentioned (particularly specific theories, such as Yang-Mills theories, gravitation, etc.). Thus, corresponding continual representations for the S-matrix were naturally obtained by this method. (We are not concerned here with the parallel development of the Lagrangian quantization formulation of singular theories.) Nevertheless, it must be pointed out that the general formulation of the method of canonical quantization of singular theories has not yet achieved its final form. This is related to a number of unsolved problems, generated at the classical level already. To these belong, in particular, problems of correspondence between the presence of first type connections in Hamiltonian formulation and Lagrangian gauge invariance, the occurrence of a certain number of Lagrange multipliers not determined in the Dirac method, the presence of first type connections and the degree of degeneracy of the Lagrange equations of motion, problems of physical variables, problems of supplementary conditions in the Lagrangian and Hamiltonian formalisms, etc.

The present work is devoted to studying the structure of singular theories in the Hamiltonian and Lagrangian formalisms, as applied to the method of canonical quantization. For clarity of exposition we first review the Dirac method. We further establish the structure of the Hamiltonian and of the connections in a special canonical coordinate system. This makes it possible to find a correspondence between the degree of degeneracy of the Lagrange equations, between first type connections and the presence of a certain number of Lagrange multipliers undetermined by the Dirac method. The physical interpretation of a degenerate theory is discussed, the physical sector of gange theories is described in the Hamiltonian formulation, and the gauge problem is considered from this point of view, in particular, canonical gauges in the Hamiltonian formulation. Finally, a one-to-one correspondence is established between first kind connections and a gauge invariant Lagrangian; and at the same time the structure of generators of gauge transformations is described. The treatment is carried out on the example of classical systems with a finite number of degrees of freedom; nevertheless, all basic conclusions and results are easily carried over, by well-known procedures, to field theory, which is, ultimately, the main topic of interest.

2. HAMILTONIAN FORMULATION OF SINGULAR THEORIES

Consider a classical mechanical system, described by a Lagrangian dependent only on generalized coordinates and velocities $L = L(q, \dot{q})$, $= (q_1, \ldots, q_n)$. We are interested in generalized theories for which the rank of the Hessian is lower than the number of degrees of freedom:

$$\operatorname{rank}\left\|\frac{\partial^2 L}{\partial \dot{q}\partial \dot{q}}\right\| = R,\ n - R = m > 0. \tag{1}$$

In this case we assume that relationship (1) retains its form in some neighborhood, for example, of vanishing coordinates and velocities. This implies that the rank of the Hessian is determined by the part of the Lagrangian which is quadratic in coordinates and velocities L_0, $(L = L_0 + \Delta L)$. In this we mention a number of restrictions on the shape of Lagrangians considered here. Besides, all our statements will refer, generally speaking, only to some

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neighborhood of vanishing coordinates and velocities (momenta). This will not be mentioned again specifically in the following. It is easily verified that the presence of some connections between coordinates and velocities is already a consequence of (1) in the Lagrangian formulation. Relationship (1) is also the reason that the equations

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{2}$$

do not determine all $\dot{q}$ as unique functions **of** q and p, and it seems impossible to carry out the standard transition to the Hamiltonian formulation, in which the quantization problem is solved by well-known methods [5]. Nevertheless, some analog Hamiltonian formulation can be constructed, as we will show, and by its help the problem of canonical quantization of singular theories is solved.

The original equations required by us are naturally generated by stating the variational problem for the conditional extremum of the action [6]

$$S = \int L(q, v)\, dt,\ L(q, v) = L(q, \dot{q})|_{\dot{q}=v}$$

with the supplementary conditions $\dot{q}_i = v_i$. Obviously, this problem is equivalent to the problem of **an unconditional extremum of the functional†**

$$S = \int [L(q, v) + p_i(\dot{q}_i - v_i)]\, dt,$$

where all q, p, v are independent functions of time, subject to variation. The corresponding equations on the extrema are

$$\dot{q}_i = v_i,\ \dot{p}_i = \frac{\partial L}{\partial q_i},\ p_i = \frac{\partial L}{\partial v_i}. \tag{3}$$

It is easily verified that in the sector of variables q the system of equations (3) is fully equivalent to the Lagrange equations

$$\frac{\delta S}{\delta q_i} = \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0,\ S = \int L(q, \dot{q})\, dt. \tag{4}$$

If the Hessian theories were nonvanishing, from the last group of equations (3) the functions v could be expressed in terms of q and p. Substituting them in the first two groups of Eqs. (3), we would reach Hamiltonian equations. In the given case (1) this cannot be done literally. Nevertheless, we introduce the function $H^* = p_i v_i - L(q, v)$. With its aid Eqs. (3) can be written as follows:

$$\dot{q}_i = \{q_i, H^*\},\ \dot{p}_i = \{p_i, H^*\},\ \frac{\partial H^*}{\partial v_i} = 0. \tag{5}$$

(In (5) {...} is the ordinary Poisson bracket, subject only to the variables (q, p).) **Consider** in more detail the structure of the equations obtained. First, without loss of generality we renumber the coordinates in such a manner that in the matrix (1) the minor of maximum rank is located in the top left corner. This is always possible, since in a symmetric matrix there exists a major minor of maximum rank. In this case the coordinates q and the corresponding quantities p, v are partitioned into two groups, which we often denote as follows:

$$X_\alpha = q_\alpha,\ \Pi_\alpha = p_\alpha,\ V_\alpha = v_\alpha,\ \alpha = 1,\dots R, \tag{6}$$

$$x_a = q_{R+a},\ \pi_a = p_{R+a},\ \lambda_a = v_{R+a},\ a = 1,\dots m,$$

$$\det\left\| \frac{\partial^2 L}{\partial V \partial V} \right\| \neq 0. \tag{7}$$

Due to condition (7), from the equation

$$\Pi = \frac{\partial L}{\partial V},\ \left(\frac{\partial H^*}{\partial V} = 0\right) \tag{8}$$

one can find all

$$V = \bar{V}(q, \Pi, \lambda). \tag{9}$$

Substituting (9) into the remaining equations $\pi = \partial L/\partial\lambda$, $[(\partial H^*/\partial\lambda) = 0]$, we obtain relations of the form

†Repeated subscripts always imply summation.

$$\Phi_a^{(1)} = \pi_a - f_a(q, \Pi) = 0, \ a = 1, \ldots m, \tag{10}$$

not containing the function λ. Indeed, if one of the functions contained λ, this would contradict (1). Thus, relations (10) for the coordinates and momenta follow directly from the last group of Eqs. (5). These relations are called primary.

It can be shown that if (9) is substituted into the expression for H*, we reach the following results:

$$H^*|_{V=\bar{V}} = H^{(1)} = H + \lambda_a \Phi_a^{(1)}, \tag{11}$$

where

$$H = \left(\frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L(q, \dot{q})\right)\Bigg|_{V=\bar{V}}.$$

Despite the fact that Eqs. (2) do not allow one to express all $\dot{q}$ in terms of q, p in the case under consideration, the Hamiltonian H depends under conditions (2) only on coordinates and momenta [4]; more precisely, only on q and Π.

Equations (5) can now be written in the form

$$\dot{q}_i = \{q_i, H^{(1)}\}, \ \dot{p}_i = \{p_i, H^{(1)}\}, \ \Phi_a^{(1)} = 0. \tag{12}$$

This equation is handled by the Dirac method [4]. It is seen that the velocities λ, not determined from Eqs. (2), must be identified with the Lagrange multipliers undetermined by this method.

Following Dirac [4], consider the time conservation conditions of the primary connections, which by account of the equations of motion (12) are reduced to the form

$$\dot{\Phi}_a^{(1)} = \{\Phi_a^{(1)}, H^{(1)}\} = \{\Phi_a^{(1)}, H\} + \{\Phi_a^{(1)}, \Phi_b^{(1)}\}\lambda_b = 0. \tag{13}$$

All functions λ can be determined from (13) if

$$\det\|\{\Phi^{(1)}, \Phi^{(1)}\}\| \neq 0. \tag{14}$$

When condition (14) is not satisfied, only part of Eqs. (13) determines several of the functions λ, while the other part equates to zero several functions of q, p, and, consequently, are connections. New connections can appear among them, not functionally dependent on the primary ones. From the time conservation conditions similar to these connections one can again determine several λ and establish some new connections. Continuing this process, we finally reach a situation when new connections will not be obtained. We name the whole set of connections, functionally independent of each other and of the primary connections, secondary connections, and denote them by $\Phi^{(2)}$. The set of all connections, both primary and secondary, **is simply denoted** by $\Phi = (\Phi^{(1)}, \Phi^{(2)})$. We note that for quadratic Lagrangians the connections are nothing else than linear combinations of coordinates and momenta. For general Lagrangian types considered by us the connections differ from the connections of the corresponding quadratic theories only by nonlinear corrections. In particular, the hypersurface of connections passes through the point q = p = 0. We will always assume that the nonlinear corrections do not increase the ranks of matrices (of the type of Jacobians, Poisson brackets) in the exact theory as compared with the corresponding quantities in the quadratic theory. These assumptions refer to the restrictions on the shape of the Lagrangians discussed here.

Two cases are possible:*

$$\det\|\{\Phi, \Phi\}\| \neq 0, \tag{15}$$

$$\det\|\{\Phi, \Phi\}\| = 0, \ [\Phi] - \mathrm{rank}\|\{\Phi, \Phi\}\| = M > 0. \tag{16}$$

We consider the case (15), to which also refers the situation described by condition (14).

In this case λ can be determined from the time conservation conditions of all connections, which on account of the equations of motion (12) can be written as:

$$\dot{\Phi}_l = \{\Phi_l, H^{(1)}\} = \{\Phi_l, H\} + \{\Phi_l, \Phi_a^{(1)}\}\lambda_a = 0. \tag{17}$$

Recalling that $\Phi^{(1)}$ is part of Φ, we obtain

$$\lambda_a = -(\{\Phi, \Phi\})^{-1}_{al}\{\Phi_l, H\}, \ a = 1, \ldots m,$$

*Here [Φ] is the amount of Φ; similar notation is also used in what follows.

$$(\{\Phi, \Phi\})^{-1}_{ll'}\{\Phi_{l'}, H\} = 0, \ l \neq 1,...m.$$

This result makes it **possible** to write Eqs. (12) as a system of differential equations, determining only the trajectories in phase space:

$$\dot{q}_i = \{q_i, H\}_D, \ \dot{p}_i = \{p_i, H\}_D, \ \Phi_l = 0. \tag{18}$$

Here we denote by $\{...\}_D$ the Dirac bracket [4]:

$$\{A, B\}_D = \{A, B\} - \{A, \Phi_l\}(\{\Phi, \Phi\})^{-1}_{ll'}\{\Phi_{l'}, B\}.$$

The system of independent equations of all connections $\Phi = 0$ describes some hypersurface in phase space, which in what follows will be called the hypersurface of all connections. If the system of independent equations $\Psi = 0$, $[\Psi] = [\Phi]$ describes the same hypersurface, **with** Φ and Ψ connected nonsingularly on the surface of transformation connections, then the set of functions Ψ will be called equivalent to the system of connections Φ. It can be shown that if in (18) all connections Φ are replaced by the system of equivalent connections Ψ, then an equivalent system of equations is obtained. (Also in Eq. (12) all connections $\Phi^{(1)}$ can be accurately replaced by an equivalent system of connections.) The following theorem [7] is important for what follows.

<u>THEOREM I.</u> Let there be given a system of independent connections $\Phi = 0$,

$$\operatorname{rank} \frac{D(\Phi)}{D(q, p)} = [\Phi], \ \text{with} \quad \det \|\{\Phi, \Phi\}\| \neq 0;$$

then there exists a time-independent canonical transformation from the variables (q, p) to the variables (Ω, ω), $[\Omega] = [\Phi]$, $\Omega = (\Omega_{q_\alpha}, \Omega_{p_\alpha})$, $\omega = (\omega_{q_a}, \omega_{p_a})$, such that an equivalent system of connections in the new variables is* $\Omega = 0$.

In the variables (Ω, ω) the Dirac bracket with Ω connections acquires a particularly simple form:

$$\{A, B\}_D^{\omega, \Omega} = \{A, B\}^{\omega}, \tag{19}$$

where $\{...\}^{\omega}$ is the Poisson bracket in the variables ω. Due to this fact Eqs. (18) in the variables (Ω, ω) can be written as:

$$\dot{\omega} = \{\omega, H(\omega)\}^{\omega}, \ \Omega = 0, \ H(\omega) = H|_{\Omega=0}. \tag{20}$$

We turn now to case (16). It can be shown that there exist here linear combinations of connections in the manifold M, commuting in the sense of the Poisson bracket with all connections, which, according to Dirac's terminology, are first type connections. In particular, the whole system of connections Φ can be chosen in such a manner that some of the Φ will directly be first type connections. In what follows we assume that our system of connections is precisely such, that part of the connections of Φ, which are first type connections, will be sometimes denoted by χ. The connections not possessing the properties mentioned are commonly called second type connections. Thus, in case (15) we dealt only with second type connections.

If it were possible in case (16) that

$$\operatorname{rank}\|\{\Phi, \Phi^{(1)}\}\| = [\Phi^{(1)}], \tag{21}$$

this would imply that all λ can be determined from Eqs. (17). The satisfaction of (21) also implies that among the primary connections there are no first type connections. We show, however, that in case (16) condition (21) cannot be satisfied, so that among the first type connections there **surely exist** primary connections in the manifold $\mu = [\Phi^{(1)}] - \operatorname{rank}\|\{\Phi, \Phi^{(1)}\}\|$, in which **connections part of** the functions λ in the same manifold is not determined from Eqs. (17). Moreover, we show that these functions λ are even not determined by the whole system of equations of motion (12). For what follows we need a theorem, being a gen-

*The subscripts q, p at the variables Ω, ω determine whether these variables are "coordinates" or "momenta" conjugate to them, and have no relation to the original variables q, p.

$$\{\Omega_{q_\alpha}, \Omega_{p_\beta}\} = \delta_{\alpha\beta}, \ \{\omega_{q_a}, \omega_{p_b}\} = \delta_{ab},$$

$$\{\Omega_{q_\alpha}, \Omega_{q_\beta}\} = \{\Omega_{p_\alpha}, \Omega_{p_\beta}\} = \{\omega_{q_a}, \omega_{q_b}\} = \{\omega_{p_a}, \omega_{p_b}\} = 0.$$

eralization of Theorem I, which is formulated as follows [8]:

THEOREM II. Let there be given a system of independent connections $\Phi = 0$,

$$\operatorname{rank} \frac{D(\Phi)}{D(q,\, p)} = [\Phi],$$

the first of which are called primary and are denoted by $\Phi^{(1)}$. Then there exists a time-independent canonical transformation from the variables (q, p) to the variables*

$$(\varphi_q^{(1)},\ \varphi_p^{(1)};\ \xi,\ \varkappa;\ Q^{(1)},\ P^{(1)};\ \varphi_q^{(2)},\ \varphi_p^{(2)};\ Q^{(2)},\ P^{(2)};\ \omega_q,\ \omega_p) \tag{22}$$

such that the equivalent system of all connections in these variables is

$$\Omega = 0,\ \Omega = (\varphi^{(1)},\ \xi,\ \varkappa,\ P^{(1)},\ \varphi^{(2)},\ P^{(2)}), \tag{23}$$

while

$$\Omega^{(1)} = 0,\ \Omega^{(1)} = (\varphi^{(1)},\ \varkappa,\ P^{(1)}) \tag{24}$$

is an equivalent system of primary connections, and

$$\Omega^{(2)} = 0,\ \Omega^{(2)} = (\xi,\ \varphi^{(2)},\ P^{(2)}) \tag{25}$$

is an equivalent system of secondary connections.

As already mentioned, Theorem II is a generalization of Theorem I, and this generalization is concerned with two aspects. First, the satisfaction of condition (15) is not required, which makes it possible to apply this theorem to theories with first type connections; and, secondly, the possibility occurs of such a choice of new variables, so as to retain the partition of connections into primary and secondary. Obviously, in the new momentum variables $P = (P^{(1)}, P^{(2)})$ are first type connections, where $P^{(1)}$ are first type primary connections, and $P^{(2)}$ are secondary. It follows from the equivalence of connections (23) to all connections Φ, and of connections (24) to the primary connections $\Phi^{(1)}$ that the ranks of the corresponding matrices, consisting of Poisson brackets, coincide, and, consequently, both the manifold of all first type connections and the manifold of first type primary connections in these and the other variables coincide. This implies that

$$[P] = [\chi] = M,\ [P^{(1)}] = \mu.$$

The manifolds of primary and of all connections, and, in particular, connections of the first and second type in the full theory and in its quadratic variant also coincide (this, in fact, is an assumption; we will consider only such theories). Thus, in a quadratic theory the transformation of Theorem II is linear.

We turn now to Eqs. (12), which in the new variables, due to the transformations being canonical, retain their structure, where instead of the equations $\Phi^{(1)} = 0$ we must write Eqs. (24). The Hamiltonian function $H^{(1)}$ is represented as follows in the new variables:

$$H^{(1)} = H_0 + \lambda_{\varphi(1)} \cdot \varphi^{(1)} + \lambda_\varkappa \cdot \varkappa + \lambda_{P(1)} P^{(1)} + A \cdot P^{(2)} + B \cdot \xi + C \cdot \varphi^{(2)} + \Delta H. \tag{26}$$

Here $H_0 = H^{(1)}|_{\Omega=0} = H|_{\Omega=0}$; ΔH is the part of $H^{(1)}$ which is quadratic and higher in secondary connections of $\Omega^{(2)}$ and is independent of the primary connections $\Omega^{(1)}$, and A, B, and C are independent of the connections of the Ω functions.† Equation (26) is an expansion of $H^{(1)}$ in a series in the variable connections. We investigate the structure of $H^{(1)}$ in the form (26). First, it can be established that H_0 depends on the variables ω only. By the construction of the function H_0 it could also depend on the variables $Q = (Q^{(1)}, Q^{(2)})$; however from the time conservation of the connections P

$$\{H^{(1)},\ P\} = \frac{\partial H_0}{\partial Q} + \{\Phi\} = 0,$$

where the notation $\{\Phi\}$ was introduced for arbitrary terms proportional to connections, and from the independence of H_0 on connections follows the Q-independence of H_0. Thus, H_0 =

*In (22) the points with the commas differ from each other by a canonical pair (coordinate, momentum); see also the comment to Theorem I concerning the subscripts q, p of the variables Ω, ω, and other possible subscripts, such as $(\xi, \varkappa) = (\xi_i, \varkappa_i)$ are omitted. The same abbreviations are used in all subsequent equations.

†We assume that A varies not less than linearly with its variable. This can always be supplemented by shifting the variable by a constant.

$H|_{\Omega=0} = H(\omega)$.

Similar considerations on the basis of the relations

$$\{H^{(1)}, \varkappa\}=B+\{\Phi\}=0,\ \{H^{(1)}, \varphi^{(2)}\}=\pm C+\{\Phi\}=0$$

make it possible to conclude that the functions B and C vanish identically.

We write now the conservation conditions of all connections (23) on the equation of motion, which will, obviously, be similar to Eqs. (17):

$$\{\varphi^{(1)}, H^{(1)}\} = \pm \lambda_{\varphi^{(1)}} = 0,\ \{\xi, H^{(1)}\} = \lambda_{\varkappa} = 0, \tag{27}$$

$$\{H^{(1)}, \varkappa\} = \frac{\partial \Delta H}{\partial \xi} = 0,\ \{\varphi^{(2)}, H^{(1)}\} = \pm \frac{\partial \Delta H}{\partial \varphi^{(2)}} = 0, \tag{28}$$

$$\{H^{(1)}, P\} = \frac{\partial A}{\partial Q} P^{(2)} + \frac{\partial \Delta H}{\partial Q} = 0. \tag{29}$$

We know that the secondary connections of the theory are a consequence of Eqs. (17) and of the primary connection equations. Since the partition into primary and secondary connections is conserved in the new variables, Eqs. (25) must be a consequence of (24) and (27)-(29), or, in different words, (25) is the solution of system (27)-(29) under condition (24).

We note that within the assumptions earlier made on the structure of connections in the full theory and in its quadratic variant the solution (25) must also hold in the quadratic theory. For the latter A are simply linear functions, and ΔH is quadratic in the secondary connections and, consequently, independent of the other variables, since in this case $H^{(1)}$ is quadratic in the new variables due to the linearity of the canonical transformation, established by Theorem II for the quadratic case. Consider the approximate quadratic theory. In this case (29) contains only secondary connections of the **first** type $P^{(2)}$, and secondary connections of the second type ξ and $\varphi^{(2)}$ are contained only in Eqs. (28), whose number equals exactly the number of these connections. Consequently, the secondary connections of the first type can in this case be a consequence of Eqs. (29) only, while the secondary connections of the second type are a consequence of Eqs. (28) only. (If $P^{(2)} = 0$ is not a consequence of (29) only, then Eqs. (29) are dependent, their number is smaller than [P], **but** then the total number of Eqs. (28), (29) is smaller than the total number of secondary connections, which is impossible.) This implies that the generation process of secondary connections of the first type starts with commutation of the functions $H^{(1)}$ with primary connections of the first type $P^{(1)}$, and is then extended by commutation of $H^{(1)}$ with the generated secondary connections of the first type, etc. The secondary connections of the second type ξ and $\varphi^{(2)}$ are first generated by commutation of $H^{(1)}$ **with the primary connections of the second type, and then extended by commutation of $H^{(1)}$ only with the generated secondary** connections there must necessarily exist primary connections of the first type (and **among** second type connections – second-type primary connections). Although this observation was made for the quadratic approximation, it also remains valid in the complete theory, since the structure of connections in the complete theory and in its quadratic approximation are identical (see remark preceding Eq. 26). Thus, among first-type connections there must necessarily exist first-type primary connections, and among second-type connections – second-type primary connections. Since this statement is independent of the choice of canonical variables, this proves that conditions (16) and (21) are incompatible. This also implies that the functions λ are not determined from the conditions of time conservation of connections μ, where μ equals the number of first-type primary connections. In the variables of (22) this is easily seen from Eqs. (27)-(29), from which $\lambda_{\varphi^{(1)}}$ and $\lambda_{\varkappa}$ are determined, while $\lambda_{P^{(1)}}$, corresponding to first-type primary connections, are not determined (they generally drop out of these equations). Moreover, it can now be shown that $\lambda_{P^{(1)}}$ are also not determined from the complete system of equations of motion. For this we write the equations of motion in the new variables. They are of the form

$$\dot{\omega} = \{\omega, H(\omega)\}^{\omega},\ \Omega = 0, \qquad \dot{Q}^{(1)} = \lambda_{P^{(1)}},\ \dot{Q}^{(2)} = A. \tag{30}$$

We recall that the functions A may depend on Q and ω. The equations for ω for Hamiltonians with a determining **Hamilton** function H(ω) are also independent of the remaining variables.

It is now seen that the remaining equations, not counting the connection equations, can be considered as equations for Q. These equations have solutions for any functions $\lambda_{p(\iota)}$. Thus the functions $\lambda_{p(\iota)}$ of the full system of equations of motion are not determined, and appear in the solutions of the equations as arbitrary functions, the Hamiltonian theory of degeneracy. It is easily seen that $\lambda_{p(\iota)}$ [10] occupy an important role in the original function (that is, $\mathrm{rank} D\lambda/D\lambda_{p(\iota)} = [\lambda_{p(\iota)}] = \mu$). Since the equations of motion contain $\dot{x} = \lambda$ (see Eqs. (3) and (6)), this implies that the solutions of the Hamilton equations for x essentially contain μ arbitrary functions of time. It follows from the equivalence of the Hamiltonian and Lagrangian equations that the solutions of the Lagrangian equations are also degenerate, while their degree of degeneracy is at least μ or, which is the same, the number of first-type primary connections in the Hamiltonian formulation. It is shown below that imposing μ conditions reduces the degeneracy of the Hamiltonian, and, consequently, also of the Lagrangian equations. We also obtain the result that the solutions of the Lagrangian equations contain precisely μ arbitrary functions of time. It is natural to expect that in this case there exist μ variables for which there are no equations, and that there exist μ relations between the equations of motion. It is shown in Section 6 that these assumptions are correct.

3. PHYSICAL INTERPRETATION OF DEGENERATE THEORIES

We now discuss the possibility of describing physical systems on the basis of degenerate theories. In this case we consider physical systems for which the following is assumed.

One can introduce the concept of a system state at each moment of time, such that assigning the state at one moment of time determines the state at remaining moments of time. All physical quantities referring to the system described at a given moment of time are single-valued functions of state. The state is completely determined by assigning all possible physical quantities corresponding to the system at the given moment of time.

On the other hand, let there exist some theory which is determined by the set of variables η and the equations of motion $M[\eta] = 0$. Such a theory will be denoted by $(\eta; M)$. If the theory is degenerate, the same initial data generally corresponds to a set of different trajectories $\eta(t)$. (The initial data to the equations of motion of a degenerate theory are conditionally called the set of all variables and their derivatives, selected at the given moment of time together with the equations of motion. This set will also be called the instantaneous trajectory state. Finally, if the equations of motion make it possible to express the whole set of coordinates and derivatives at the given moment of time only in terms of part of them, as initial data it is sufficient to select precisely that part which is usually selected in the nondegenerate cases.)

Thus, at first glance a noncorrespondence is generated between timewise causally related states of a physical system and the functional arbitrariness in the solutions of the degenerate theory, occurring in the absence of one-to-one correspondence between the instantaneous trajectory states of the degenerate theory. To overcome this noncorrespondence, and to describe compatibly physical systems of the type mentioned within degenerate theories one can adopt the following natural interpretation, which practically consists of two points: a) the state of the physical system, and consequently all physical quantities uniquely related with the instantaneous trajectory state of the corresponding theory; b) all physical quantities coincide at simultaneous points of intersecting trajectories of the theory. (Two trajectories η and η' are called intersecting if their instantaneous states coincide at some moment of time. In what follows this fact will be denoted as $\eta \cap \eta'$.)

Point (b) guarantees the independence of physical quantities of the arbitrariness associated with solutions of the degenerate theory, and reconciles point (a) with causal time evolutions of the physical state. It follows from point (a) that any physical quantity A can be described by functions of the form $A(\eta, \dot{\eta}, \ddot{\eta}, \ldots)$. In this case point (a) imposes a restriction on the possible shape of these functions. More precisely,

$$A(\eta, \dot{\eta}, \ldots) = A(\eta', \dot{\eta}', \ldots), \ \forall t, \ \forall \eta \cap \eta', \ M[\eta] = M[\eta'] = 0. \tag{31}$$

Functions of the instantaneous trajectory state satisfying Eq. (31) will be called physical, and will be denoted in the following by A^{Φ}.

We call different trajectories η equivalent if all physical quantities coincide on these trajectories. Thus, the whole set of trajectories η is decomposed into classes of equivalent trajectories. It can then be concluded that to each physical state s there corresponds a class of equivalent trajectories there corresponds one and the same physical state. Here by physical state we imply the set of instantaneous physical states transforming to each other in time. In what follows we often understand a physical state in the extended sense. We denote by $K_s(\eta; M)$ the class of equivalent trajectories of the theory $(\eta; M)$ corresponding to the state s. In these theories what was said above can be written as:

$$A^{\Phi}(\eta, \dot{\eta},\ldots) = A^{\Phi}(\eta', \dot{\eta}',\ldots), \ \forall t, \ \forall A^{\Phi}, \ \forall \eta, \ \eta' \in K_s(\eta; M), \ \forall s. \tag{32}$$

It is clear that all intersecting trajectories occur in one class of equivalent trajectories.

We call physical functions A_1^{Φ} and A_2^{Φ} equivalent if their values coincide on any equivalent trajectories of any classes $K_s(\eta; M)$. Obviously, if a physical quantity A is described by some function A^{Φ}, it can also be described by any other equivalent function. Thus, all physical functions are decomposed into classes of equivalent functions. Let $F_A(\eta; M)$ be a class of equivalent functions of the theory $(\eta; M)$; then

$$A_1^{\Phi}(\eta, \dot{\eta},\ldots) = A_2^{\Phi}(\eta', \dot{\eta}',\ldots), \ \forall t, \ \forall \eta, \ \eta' \in K_s(\eta; M), \ \forall s, \ \forall A_1^{\Phi}, \ A_2^{\Phi} \in F_A(\eta; M). \tag{33}$$

We address now the following problem. Can the physical system **described** by the theory $(\eta; M)$ be described within some other theory, for example, $(\xi; N)$? The answer to this question must be assumed positive if for an arbitrary class $K_s(\eta; M)$ and any class $F_A(\eta; M)$ of the theory $(\eta; M)$ there exist in the new theory $(\xi; N)$ some classes $K_s(\xi; N/\eta; M)$ of trajectories ξ and classes $F_A(\xi; N/\eta; M)$ of functions $B_A(\xi, \dot{\xi}, \ldots)$, such that

$$\begin{aligned} &B_A(\xi, \dot{\xi},\ldots) = A^{\Phi}(\eta, \dot{\eta},\ldots), \ \forall t, \ \forall \eta \in K_s(\eta; M), \\ &\forall A^{\Phi} \in F_A(\eta; M), \ \forall \xi \in K_s(\xi; N/\eta; M), \\ &\forall B_A \in F_A(\xi; N/\eta; M). \end{aligned} \tag{34}$$

The classes of trajectories $K_s(\xi, N/\eta; M)$ and functions $F_A(\xi; N/\eta; M)$ will be called a physical sector of the theory $(\xi; N)$ with respect to the theory $(\eta; M)$. It must be noted that the theory $(\xi; N)$ considered regardless of the theory $(\eta; M)$ would produce, generally speaking, its class of equivalent trajectories $K_{\zeta}(\xi; N)$ and another compound of physical functions and their equivalent classes $F_B(\xi; N)$. The theories $(\xi; N)$ possessing a physical sector equivalent to the theory $(\eta; M)$ will be simply called in what follows physically equivalent theories.

Practically, it is necessary to transform from a description of a physical system in terms of a degenerate theory to a description in terms of a nondegenerate theory. Obviously, the corresponding nondegenerate theory must be physically equivalent to the original theory. This transition will be called gauge application to the theory.

In our problem there is no detailed and exhaustive description of all possible gauges. We dwell only on some of them. We consider a class of gauges, naturally called minimal. In applying such gauges we transform from the theory $(\eta; M)$ to a physically equivalent theory $(\eta; G)$ with the same set of variables, while for any physical quantity A it is possible to mention a physical function A^{Φ}, describing it both in the theory $(\eta; M)$ and in the theory $(\eta; G)$. That is, the classes $F_A(\eta; M)$ and $F_A(\eta; G/\eta; M)$ necessarily intersect:

$$F_A(\eta; M) \cap F_A(\eta; G/\eta; M) \neq 0, \ \forall A.$$

In the transition to such a gauge we have, removing the degeneracy, minimally changed the originally degenerate theory, remaining within the original variables and the original physical functions. Minimal gauges can occur without violating the equations of motion. This implies that the trajectories of the physical sector satisfy the original equations of motion. In the language of classes $K_s(\eta; G/\eta; M)$ and $K_s(\eta; M)$ this imples that

$$K_s(\eta; G/\eta; M) \in K_s(\eta; M).$$

Important among the gauges are those which can be called rigid. In this case each class $K_s(\xi; N/\eta; M)$ contains only one trajectory. The rigid minimal gauge which does not violate the equations of motion is quite obvious. Obviously, such gauging reduces to the fact that from each class $K_s(\eta; M)$ of the original theory one representation is somehow removed, so that each physical state is reached from one trajectory of the original theory.

Thus, the transition to some gauge makes it possible to describe the original physical system in terms of a nondegenerate theory. By the construction this description is fully equivalent to the original theory; therefore the physical responses are independent of the choice of the gauge. The possibility of describing the same physical system within different gauges will be called gauge invariance.

One often considers some set of minimal gauges $G_1, G_2, \ldots, G_k, \ldots$, such that the intersection of all classes $F_A(\eta; M)$, $F_A(\eta; G_i/\eta; M)$, $i = 1, \ldots k, \ldots$, which we denote by $\bar{F}_A$, is not empty for any physical quantity A. This imples that for each physical quantity A there exist functions A^Φ, which describe it both in the original theory and in any of the gauges G_i, $i = 1, \ldots k, \ldots$. That is,

$$A^\Phi(\eta, \dot{\eta}, \ldots) = A^\Phi(\eta_i, \dot{\eta}_i, \ldots), \ \forall t, \ \forall A^\Phi \in \bar{F}_A, \ \forall A, \\ \forall \eta \in K_s(\eta; M), \ \forall \eta_i \in K_s(\eta; G_i/\eta; M), \ \forall i. \tag{35}$$

In this class of gauges the gauge invariance appears in the fact that the function describing the physical quantity is independent of the gauge. In this case the property (35) is commonly called gauge invariance of physical functions.

In conclusion, we note that a wider implication of the transition to some gauge is also possible, as a transition from a given theory (not necessarily degenerate) to a physically equivalent nondegenerate theory.

4. CANONICAL GAUGES

In light of the discussion above consider the interpretation of a singular theory with first type connections. We start the analysis in a special coordinate system, in which the variables of the theory are the canonical coordinates (22) and the functions λ. The contraction of all these variables will be denoted by $\eta = (\Omega, Q, \omega, \lambda)$. According to the assumptions made in Section 3, we assume that each physical quantity A can be described by a function of the form $A(\eta, \dot{\eta}, \ldots)$. Since it follows from the equations of motion (30) that $\Omega = \lambda_\chi = \lambda_{\varphi^{(1)}} = 0$, all the derivatives with respect to ω can be expressed only in terms of ω, all derivatives with respect to $Q^{(1)}$ can be expressed only in terms of Q, ω, and the derivatives with respect to $Q^{(1)}$, and the functions $\lambda_{p^{(1)}}$ and all their derivatives can be expressed only in terms of the derivatives with respect to $Q^{(1)}$, then among the functions equivalent to the functions $A(\eta, \dot{\eta}, \ldots)$ there are always functions of the form

$$A(\omega, Q, \dot{Q}^{(1)}, \ddot{Q}^{(1)}, \ldots). \tag{36}$$

We use the following to establish what restrictions are imposed on the structure of the functions (36) by condition (32). Consider the trajectory of variables ω, Q, whose state at $t = t_0$ and $t = t_0 + \Delta$ is given by the set of values $(\omega_0, Q_0, \dot{Q}_0^{(1)}, \ddot{Q}_0^{(1)}, \ldots)$ and $(\omega_0 + \Delta\omega, Q_0 + \Delta Q, \dot{Q}^{(1)} + \Delta\dot{Q}^{(1)}, \ddot{Q}^{(1)} + \Delta\ddot{Q}^{(1)}, \ldots)$, respectively. The equations of motion (30) **are** such* that ΔQ, $\Delta\ddot{Q}^{(1)}, \ldots$ can be quantities assigned ahead. Consequently, there exists a set of intersecting trajectories at $t = t_0$, differing from each other at $t = t_0 + \Delta$ by the values Q, $\dot{Q}^{(1)}$, $\ddot{Q}^{(1)}$, ... Condition (32) requires that any physical functions coincide on these trajectories. This implies that for $\Omega = 0$ physical functions of the form (36) must be independent of Q, $\dot{Q}^{(1)}$, $\ddot{Q}^{(1)}$, ... Hence also follows that in each class of physically equivalent functions there exist necessarily functions of the form $A(\omega)$. Differently speaking, any physical quantity A may describe functions of the variables ω only, $A = A(\omega)$. Therefore the variables ω are naturally called physical. It is easily established that any physical quantity can also be described by functions of the variables q, p only. This is indeed so, since an assignment of q, p uniquely determines ω. We write down the structure of the physical functions $A^\Phi(q, p)$. Transforming in these functions to the variables Ω, Q, ω, one can write

$$A^\Phi(q, p) = A(\omega) + \Lambda(\Omega, Q, \omega)\cdot\Omega, \tag{37}$$

where $A(\omega)$ is a function of the variables ω, describing the same physical quantity, so that both $A^\Phi(q, p)$ and Λ are functions without singularities at $\Omega = 0$. Consequently,

$$\frac{\partial A^\Phi}{\partial Q} = \{A^\Phi, P\} = \frac{\partial \Lambda}{\partial Q}\cdot\Omega. \tag{38}$$

We now recall that P is a system of connections, equivalent to all first type connections χ, while Ω is a system of connections, equivalent to all connections Φ. By definition equivalent connections are expressed in terms of each other by means of nonsingular matrices, so

*Restrictions on the size of this paper do not allow us to prove this statement.

that (38) can be rewritten in the form

$$\{A^{\Phi}(q, p), \chi(q, p)\}=\{\Phi\}. \tag{39}$$

Thus, it has been established that any physical quantity can be described by functions of q, p only. By Eq. (37) all equivalent physical functions of q, p differ from each other only by terms proportional to connections. The physical functions of q, p must satisfy Eq. (39).

Further, one can describe classes of equivalent trajectories. According to the discussion above, one class contains all trajectories for which the variables ω coincide. These sets of trajectories differ from each other only by the variables Q (Ω = 0 for all trajectories). The transition to a near trajectory, differing from the given one only by the variables Q, can be represented by an infinitely small canonical transformation with generating functions $\delta W = \chi_\alpha \cdot \delta\xi_\alpha$, where $\delta\xi_\alpha$ are arbitrarily small functions of time. Thus, all first type connections generate nonphysical changes of state!

Now, having discussed in detail the structure of degenerate theories in the Hamiltonian formulation, we write several gauges in general form. For example, one of the possible gauges is the theory in the variables ω with equations of motion

$$\dot{\omega}=\{\omega, H(\omega)\}^{\omega} \tag{40}$$

and with physical functions $A(\omega)$ of the original classes of physically equivalent functions.

Consider gauges in the variables Ω, Q, ω. To achieve the purpose of gauging we must remove the degeneracy, i.e., remove the functional arbitrariness in the solutions of the equations of motion (30). For this, generally speaking, it is sufficient to introduce certain functions $\lambda_{p(1)}$ in these equations. We equate them, for example, to zero. This is equivalent to the fact that the equations of motion are supplemented by the conditions $Q^{(1)}$ = const at first type primary connections. It hence follows, in particular, that the degeneracy of solutions of Lagrangian equations cannot exceed μ, and consequently, by the proof in Sec. 2, is exactly equal to μ. Which functions can be used to describe physical quantities in such a gauge? Obviously, these can be all original functions $A^{\Phi}(\Omega, Q, \omega)$ of the form (37). Indeed, we have not violated the original equations of motion, particularly the equations Ω = 0. The condition Ω = 0 guarantees the coincidence of values of these functions on trajectories with values of functions $A(\omega)$ of the variables ω, for which the equations are also not violated. Thus, we have an example of a minimal gauge without violation of the equations of motion. However, the gauge considered is not rigid. Indeed, Eqs. (30) for $Q^{(2)}$ are of first order. For given functions $Q^{(1)}$ and ω they have a set of solutions according to the possibility of selecting different initial data. Consequently, in the gauge under consideration there exists a set of trajectories, equivalent from the point of view of the original theory. The variables ω coincide in them, but the variables $Q^{(2)}$ are different.

It is now clear that if we wish to apply a minimal rigid gauge, we must somehow fix all variables Q. The equations of such a gauge can, for example, be

$$\dot{\omega}=\{\omega, H(\omega)\}^{\omega}, \ \Omega=0, \ Q-\Psi(\omega)=0. \tag{41}$$

In this case all physical quantities can be primarily described by the original functions (37). As is seen, the gauge under consideration consists of the fact that the differential equations in the variables Q are replaced by additional conditions of the form

$$Q-\Psi(\omega)=0 \tag{42}$$

in all first-type connections. We note now that relations (42) are connections, so that in Eqs. (41) the hypersurface of connections is described by the equations

$$\Omega=0, \ Q-\Psi(\omega)=0 \tag{43}$$

and, consequently, the whole system of connections is of second type. Moreover, Eqs. (41) are equations of some Hamiltonian theory with second-type connections (43) and with a Hamiltonian

$$H^{(1)}=H(\omega)+\lambda_{\Omega}\cdot\Omega+\lambda_Q\cdot(Q-\Psi(\omega)).$$

It is easily verified that all λ are determined from the conditions of time conservation of connections (43); these quantities simply vanish here, so that the equations of motion

$$\dot{\omega}=\{\omega, H^{(1)}\}, \ \Omega=0, \ Q-\Psi(\omega)=0 \tag{44}$$

indeed have the form (41).

This observation makes it possible to formulate a similar gauge in the variables q, p. In these variables this gauge is

$$\dot{q} = \{q, H^{(1)}\}_D, \ \dot{p} = \{p, H^{(1)}\}_D, \ \boldsymbol{\Phi} = 0, \tag{45}$$

where $\boldsymbol{\Phi} = (\Phi, \Phi^G)$, while Φ are all connections of the original theory, and Φ^G is some system of connections (supplementary conditions), selected in such a manner that the whole system of connections $\boldsymbol{\Phi}$ be of second type, i.e.,

$$\operatorname{rank} \frac{D(\boldsymbol{\Phi})}{D(q, p)} = [\boldsymbol{\Phi}], \ \det\|\{\boldsymbol{\Phi}, \boldsymbol{\Phi}\}\| \neq 0. \tag{46}$$

Gauges of this type are called canonical. We note that it follows from (46) that

$$[\Phi^G] = [\chi], \ \det\|\{\Phi^G, \chi\}\| \neq 0, \tag{47}$$

so that the supplementary conditions are in some sense associated with all first-type connections, which seems quite clear in the variables (Ω, Q, ω).

Transforming in (45) to the variables (Ω, Q, ω), it can be shown directly that in these variables the equations of motion have the form (41), i.e., (45) is a minimal rigid gauge. By the selection of suitable supplementary conditions, they can sometimes be supplemented in such a manner that the canonical gauge does not violate the equations of motion.

It follows from the discussion above that the physical functions $A^{\Phi}(q, p)$ satisfying conditions (39) describe physical quantities in any of the canonical gauges. In this sense the functions are gauge invariant with respect to a choice of canonical gauges. This result can also be proved directly by remaining within the variables q, p. It is therefore natural to call conditions (39) a condition of gauge invariance of physical quantities.

It must be emphasized that the solutions of the equations in an arbitrary canonical gauge do not satisfy the equations of the original theory. In this case we say that the gauge violates the equations of motion. In this case, however, the equations are violated only for nonphysical variables Q, while the physical sector of the theory is not violated. The possibility of violating the equations for nonphysical variables Q leads to the fact that among the physically equivalent theories with the same state of physical functions $A^{\Phi}(q, p)$ there exists a theory described by the equations

$$\dot{q} = \{q, H^{(1,2)}\}, \ \dot{p} = \{p, H^{(1,2)}\}, \ \Phi = 0,$$
$$H^{(1,2)} = H + \lambda_l \Phi_l,$$

where Φ are all connections of the original theory.

Differently stated, the Hamiltonian $H^{(1)}$ can be supplemented by all secondary connections with corresponding λ factors, while the physical sector of the theory is not changed! This statement is most simply verified in the variables (Ω, Q, ω). The corresponding equations of motion are obtained from Eqs. (30) by replacing all functions A by the functions λ.

5. QUANTIZATION

One possible point of view consists of the fact that to construct a quantum theory of some system it is sufficient to "quantize" some of the corresponding physically equivalent classical theories. Considering the problem interesting us from this point of view, it can be said that to quantize a singular theory, generally speaking, it is sufficient to "quantize" it in some gauge. Even here the most suitable candidate, at first glance, is the gauge (40), which is the standard Hamiltonian theory in variables ω with Hamiltonian $H(\omega)$. This gauge occurs for singular degenerate theories with first type connections, as well as in its special case of theories with second-type connections. In this gauge the quantization is standard. We label all ω by operators $\hat{\omega}$ with canonical commutation relations

$$[\hat{\omega}_{q\alpha}, \hat{\omega}_{p\beta}] = i\delta_{\alpha\beta}, \ [\hat{\omega}_{q\alpha}, \hat{\omega}_{q\beta}] = [\hat{\omega}_{p\alpha}, \hat{\omega}_{p\beta}] = 0, \tag{48}$$

where the energy operator $\hat{H}$ is constructed from the classical Hamiltonian $H(\omega)$.*

*Here and in the following we do not touch on the well-known problems of comparing classical functions with operators.

Sometimes, however, finding the variables ω can be a technically nontrivial **problem.*** For this reason a formulation is necessary in terms of the original variables q, p. For quantization of singular theories of general form it is sufficient to formulate the procedure of quantization for theories with **second**-type connections only, since, as we saw, for theories with first-type connections there always exists a canonical gauge.

The corresponding compatible quantization theory with second-type connections is as follows [4]: all q, p are labeled by operators $\hat{q}$, $\hat{p}$, for which the following operator equalities are satisfied

$$[\hat{\eta}, \hat{\eta}'] = i\{\eta, \eta'\}_D\Big|_{\substack{\eta=\hat{\eta}\\ \eta'=\hat{\eta}'}}, \quad \Phi(\hat{\eta}) = 0, \quad \eta = (q, p). \tag{49}$$

The energy operator $\hat{H}$ is constructed from the classical Hamiltonian H(q, p). The quantization (49) of theories with **second**-type connections is called Dirac quantization. The existence of canonical gauges earlier proved for theories with first-type connections makes their Dirac quantization also possible.

It must be noted that in the general case of Dirac quantization the problem arises of realization of the operator relations (49), as well as the problem of proving the independence of constructing the quantum theory of the choice of one or another canonical gauge.

Consider now how Dirac quantization appears in the special variables (Ω, ω). Since in the variables mentioned the Dirac bracket has the form (19), the canonical commutation relations (48) are satisfied for the operator $\hat{\omega}$, and, besides,

$$[\hat{\omega}, \hat{\Omega}] = [\hat{\Omega}, \hat{\Omega}] = 0, \quad \hat{\Omega} = 0. \tag{50}$$

The energy operator $\hat{H}$ is constructed from the classical Hamiltonian H in the variables Ω, ω, while from condition (50) it simply coincides with the energy operator constructed from the operator H(ω). It is easily argued that despite the formal presence of additional degrees of freedom, related to the variables Ω, Dirac quantization in the variables (Ω, ω) and quantization in the gauge (40) practically lead to identical quantum theories. It can be verified that Dirac quantization satisfies the correspondence principle in the sense that for $h \to 0$ the Heisenberg equations can be reduced to a form coinciding with the form of the classical equations of motion.

Finally, we note that all results given in the present paper can be generalized to cases when the Lagrangian depends not only on ordinary commuting (boson) variables, but also on anticommuting (fermion) variables, formed by some Grassman algebra [9]. In this case all equations retain their form if the derivatives with respect to coordinates are always assumed to be right-handed, and with respect to momentum — left-handed, while the Poisson bracket is redefined as follows [2]:

$$\{A, B\} = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - (-1)^{n_A \cdot n_B}\frac{\partial B}{\partial q_i}\frac{\partial A}{\partial p_i},$$

where n_A and n_B are the Grassman parities of the quantities A and B (for A even $n_A = 0$, for A odd $n_A = 1$, etc.), while in quantizing fermion variables commutators are replaced by anticommutators.

6. CONCLUSION. FIRST-TYPE CONNECTIONS AND GAUGE INVARIANCE OF THE LAGRANGIAN

It follows from the results of Sec. 2 that if there exist first-type connections in the Hamiltonian formulation of a singular theory, **then** the solutions of both Hamiltonian and Lagrangian equations of motion contain a functional arbitrariness in the number of first-type primary connections, which are necessarily present in this case. What expresses this fact in the Lagrangian formulation? We show that here there exists a one-to-one relationship with the invariance of the action (4) with respect to gauge transformations

$$q \to q' = G(q, \xi), \quad (S(q) = S(q')), \tag{51}$$

where in the general case G is a functional of q and μ of arbitrary functions of time $\xi_\alpha(t)$,

*Most of the well-known theories have a special structure, allowing one, not having found the variables Ω, ω, to establish quite simply the shape of the Hamiltonian H(ω), which is, obviously, sufficient for the purpose of quantization.

$\alpha = 1, \ldots, \mu$, where all functions ξ appear essentially in G [10], so that for an infinitely small transformation (51).

$$q_i' = q_i + R_i^a \xi_a \tag{52}$$

implies independence of the generators R_i^{α} (that is, from $n_{\alpha} R_i^{\alpha}$ it must follow that $n_{\alpha} = 0$). We note that (52) is a contracted description of the expression $q_i'(t) = q_i(t) + \int R_i^{\alpha}(t, t') \xi_{\alpha}(t')dt'$, where $R_i^{\alpha}(t, t')$ is in the general case a functional of q. This form is also used in what follows.

Thus, let the action (4) be invariant with respect to transformation (51); then it is also invariant with respect to the infinitely small transformations (52). Due to independence of the generators R_i^{α} this leads to the presence of μ identities of the form

$$R_i^a \frac{\delta S}{\delta q_i} \equiv 0, \tag{53}$$

which, obviously, is an identity between Lagrange equations, so that the number of independent equations of motion is decreased, at least by μ. Therefore, the solutions of Lagrangian (and consequently, also Hamiltonian) equations of motion contain a functional arbitrariness not smaller than μ. It can be shown that in this case there exists a nonsingular replacement of variables $q \to (x^{\perp}, x^{\parallel})$, $[x^{\parallel}] = \mu$, so that the action S is generally independent of the variables $x^{\parallel}$. One form of this replacement is, for example:

$$q = G(q, \xi)\Big|_{\substack{q=\varphi(x^{\perp}) \\ \xi = x^{\parallel}}},$$

where $\varphi(x^{\perp})$ is the solution of the system of equations $W(\varphi) = 0$, $[W] = \mu$, expressed in terms of the $[q] - \mu$ independent variables $x^{\perp}$. Thus, $x^{\parallel}$ is generally separated from the equations of motion, and the presence of arbitrariness in the solutions becomes particularly transparent.

More complicated is the proof that if the solutions of the Lagrange equations contain a certain number of arbitrary functions, then there exists a gauge invariance of action of the form (51) with a number of important parameters exactly equal to the number of these arbitrary functions. In this case it is sufficient to establish the invariance of the action with respect to infinitely small transformations (52); then, as shown in [11], these transformations can always be integrated by parts. The proof of existence of infinitely small transformations obviously reduces to a proof of presence of identities between equations of motion of the form (53) with independent generators R_i^{α}. For simplicity we consider this problem for an aribtrary system of first-order ordinary differential equations, where the scheme of the proof is also easily generalized to second-order equations, to which belong the Lagrange equations, or the results obtained can be applied directly to equations of motion of arbitrary order (including equations of theories with high-order derivatives), if the latter are written in the form of first-order equations. Thus, let

$$F^i(\dot{\eta}, \eta) = 0, \quad [\eta] = [F] \tag{54}$$

be a system of first-order differential equations, with

$$\operatorname{rank} \frac{D(\boldsymbol{F})}{D(\dot{\eta})} = \rho < [\boldsymbol{F}]. \tag{55}$$

In the case (55) Eqs. (54) cannot be represented in a form decomposed in the derivatives $\dot{\eta}$. Condition (55) implies that $\mathbf{F}$, considered as functions of $\dot{\eta}$, are dependent, so that they can be decomposed into two groups:

$$\boldsymbol{F} = (F_1, f_1), \ \operatorname{rank} \frac{D(f_1)}{D(\dot{\eta})} = [f_1] = \rho,$$

with $F_1 = \alpha_1 \cdot f_1 + u$, where u are some functions of η only. Let them be dependent, i.e.,

$$\operatorname{rank} \frac{D(u)}{D(\eta)} = r < [u].$$

We denote the independent functions of u by Φ_1:

$$\operatorname{rank} \frac{D(\Phi_1)}{D(\eta)} = [\Phi_1] = r.$$

One can then write (assuming that all functions encountered are analytic and do not contain undetermined terms):

$$u_1=\beta_1\cdot\Phi_1,\ \mathrm{rank}\ \beta_1=[\Phi_1],\ F_1=\alpha_1\cdot f_1+\beta_1\cdot\Phi_1. \tag{56}$$

We now **decompose all** functions F_1 into two groups: $F_1 = (\bar{F}_1, \underline{F}_1)$, so that $[\underline{F}_1] = [\Phi_1]$, $\underline{F}_1 = \underline{\alpha}_1\cdot f_1 + \underline{\beta}_1\cdot\Phi_1$, $\det \underline{\beta}_1 \neq 0$. It is easily verified that the original functions $F = (\bar{F}_1, \underline{F}_1, f_1)$ are related to the functions $\boldsymbol{F}_1 = (\bar{F}_1, f_1, \Phi_1)$ by nonsingular matrices $\varkappa^{0,1}$:

$$\boldsymbol{F}=\varkappa^{0,1}\boldsymbol{F}_1,\ \boldsymbol{F}_1=\varkappa^{1,0}\boldsymbol{F},\ \varkappa^{1,0}=(\varkappa^{0,1})^{-1}.$$

Therefore the equations $\boldsymbol{F} = 0$ and $\boldsymbol{F}_1 = 0$ are equivalent. Consider relations (56), in which the functions $\bar{F}_1$ appear. They represent identities between the functions $\boldsymbol{F}_1$, which can be written as

$$\bar{F}_1\equiv\gamma_1\cdot\underline{F}_1,\ \bar{F}_1=\bar{F}_1,\ \underline{F}_1=(f_1,\ \Phi_1)$$

or

$$\boldsymbol{R}_1\cdot\boldsymbol{F}_1\equiv 0,\ \boldsymbol{R}_1=\left\|\begin{matrix}1 & & 0 & \vdots & \\ & 1 & & \vdots & -\gamma_1\\ 0 & & 1 & \vdots & \end{matrix}\right\|.$$

Thus, of the equations $\boldsymbol{F}_1 = 0$, only the equations $\underline{F}_1 = 0$ are important. They are completely equivalent to the original equations (54).

Let now the functions f_1 and $\dot{\Phi}_1$ depend on the variable $\dot{\eta}$, i.e.,

$$\mathrm{rank}\,\frac{D(f_1,\ \dot{\Phi}_1)}{D(\dot{\eta})}=\rho_1<[f_1]+[\Phi_1].$$

There exists then a partition of the function f_1 into two groups: $f_1 = (F_2, f_2)$, so that, similarly to the preceding situation,

$$\mathrm{rank}\,\frac{D(f_2,\ \dot{\Phi}_1)}{D(\dot{\eta})}=[f_2]+[\Phi_1]=\rho_1, \tag{57}$$

$$F_2=\alpha_2\cdot f_2+\nu_1\cdot\dot{\Phi}_1+\mu_1\cdot\Phi_1+\beta_2\cdot\Phi_2,\ \mathrm{rank}\,\beta_2=[\Phi_2],$$

where $\dot{\Phi}_2 = \Phi_2(\eta)$ is a function independent of the function Φ_1:

$$\mathrm{rank}\,\frac{D(\Phi_1,\ \Phi_2)}{D(\eta)}=[\Phi_1]+[\Phi_2].$$

We expand F_2 into $\bar{F}_2$ and $\underline{F}_2$: $|\underline{F}_2| = [\Phi_2]$,

$$\underline{F}_2=\underline{\alpha}_2\cdot f_2+\underline{\nu}_1\dot{\Phi}_1+\underline{\mu}_1\cdot\Phi_1+\underline{\beta}_2\cdot\Phi_2,\ \det\underline{\beta}_2\neq 0,$$

and definite sets of functions $\boldsymbol{F}_2 = (\bar{F}_2, f_2, \Phi_1, \Phi_2)$, related to the functions $\boldsymbol{F}_1$ by the nonsingular matrices:

$$\underline{F}_1=\varkappa^{1,2}\boldsymbol{F}_2,\ \varkappa^{1,2}=\left\|\begin{matrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&A&B&C\end{matrix}\right\|,\ \begin{matrix}A=\underline{\alpha}_2,\ B=\underline{\nu}_1\cdot\dfrac{d}{dt}+\mu_1,\\ C=\underline{\beta}_2,\end{matrix}$$

$$\boldsymbol{F}_2=\varkappa^{2,1}\underline{F}_1,\ \varkappa^{2,1}=\left\|\begin{matrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0,&-C^{-1}A,&-C^{-1}B,&C^{-1}\end{matrix}\right\|.$$

It is significant here that both the direct $\varkappa^{1,2}$ and the inverse $\varkappa^{2,1}$ matrices are local operators of time differentiation. The following identities for $\boldsymbol{F}_2$ follow from relations (57)

$$\bar{F}_2\equiv\gamma_2\cdot\underline{F}_2,\ \bar{F}_2=\bar{F}_2,\ \underline{F}_2=(f_2,\ \Phi_1,\ \Phi_2),$$

or

$$R_2F_2 \equiv 0,\; R_2 = \left\| \begin{matrix} 1 & & 0 & \vdots & \\ & 1 & & \vdots & -\gamma_2 \\ 0 & & 1 & \vdots & \end{matrix} \right\|.$$

Thus, of the equations $\mathbf{F_2} = 0$ only the equations $\mathbf{F_2} = 0$ are important. They are completely equivalent to the original equations (54).

Let now the functions f_2, Φ_1, Φ_2 depend on the variable $\dot{\eta}$. Then, similarly to the preceding situation, new connections $\Phi_3 = \Phi_3(\eta)$ are generated, which are independent of Φ_1 and Φ_2, etc. Finally, at the k-th step the following **equations are generated:** $\underline{F}_\kappa = 0$, $\underline{F}_\kappa = (f_\kappa, \Phi_1, \Phi_2, \ldots, \Phi_\kappa)$, completely equivalent to the original equations (54), and such that

$$\operatorname{rank}\frac{D(f,\,\dot{\Phi})}{D(\dot{\eta})} = [f] + [\Phi],\; f = f_\kappa,\; \Phi = (\Phi_1,\, \Phi_2, \ldots, \Phi_\kappa), \tag{58}$$

while by construction

$$\operatorname{rank}\frac{D(\Phi)}{D(\eta)} = [\Phi]. \tag{59}$$

It is useful to illustrate the procedure described schematically:

$$\begin{aligned} &F = \varkappa^{0,1}F_1, \\ &F_1 = \begin{cases} \overline{F}_1 \equiv \gamma_1 \cdot \underline{F}_1, & R_1 \cdot F_1 \equiv 0, \\ \underline{F}_1 = \varkappa^{1,2}\overline{F}_2, \end{cases} \\ &F_2 = \begin{cases} \overline{F}_2 \equiv \gamma_2 \cdot \underline{F}_2, & R_2 F_2 \equiv 0, \\ \underline{F}_2 = \varkappa^{2,3}F_3, \end{cases} \\ &\cdots\cdots\cdots\cdots \\ &F_\kappa = \begin{cases} \overline{F}_\kappa \equiv \gamma_\kappa \cdot \underline{F}_\kappa, & R_\kappa \cdot F_\kappa \equiv 0, \\ \underline{F}_\kappa = (f,\, \Phi). \end{cases} \end{aligned} \tag{60}$$

Thus, the original equations (54) are reduced to the equivalent equations

$$f(\dot{\eta},\, \eta) = 0,\; \Phi(\eta) = 0, \tag{61}$$

which, by (58), (59), are indeed independent. For this reason they can be reduced to the form

$$\dot{X} = \varphi(X,\, x,\, \dot{x}),\; Y = \Psi(X,\, x),\; \eta = (X,\, Y,\, x), \tag{62}$$

being the analog of the normal form, i.e., the form in which the equations are solved for the highest derivatives. (For second-order equations the similar result is

$$\ddot{X} = \varphi(X,\, \dot{X},\, Y,\, x,\, \dot{x},\, \ddot{x}),\; \dot{Y} = \Psi(X,\, \dot{X},\, Y,\, x,\, \dot{x}),$$
$$Z = \chi(X,\, Y,\, x),\; \eta = (X,\, Y,\, Z,\, x).)$$

It is seen from Eqs. (62) that if the variables are x, they are not determined by these equations, and in fact there is a functional arbitrariness in their solutions. On the other hand, the following relation holds

$$[x] = [\eta] - [X] = [F] - [F_\kappa],$$

so that the number of variables x equals exactly the number of identities

$$R_l F_l \equiv 0$$

occurring in the procedure (61).

Due to the existence of local matrices $\varkappa^{l,\,l-1} = (\varkappa^{l-1,\,l})^{-1}$, these identities can be reduced to **identities** between the original functions **F.** They are

$$R^a_i F^i \equiv 0,\; [a] = [x], \tag{63}$$

and it is seen from the construction that R^a_i, which we conceptually call generators, are local operators of time differentiation of finite order, depending on the variables η, $\dot{\eta}$, ... only locally. In a detailed description, similar to that carried out following Eqs. (52), the generators must be represented in "matrix" form $R^a_i = R^a_i(t, t')$, where in the given case, as was proved, they are

$$R_i^a(t, t') = \sum_{\kappa=0}^{p} \rho_{i\kappa}^a(\eta(t), \dot{\eta}(t), \ldots) \frac{d^\kappa}{dt^\kappa} \delta(t - t').$$

Using the structure of the matrices R_l and the existence of the matrices $\varkappa^{l,l-1}$, $\varkappa^{l,l+1}$, it can be proved that all generators R_i^a are independent in the sense mentioned earlier.

The results obtained can be applied to the Lagrange equations (4) if the latter are written in the equivalent form of first-order equations

$$\left.\frac{\delta S}{\delta q_i}\right|_{\dot{q}=v} = M_i(q, v, \dot{v}) = 0, \; \dot{q}_i - v_i = 0. \tag{64}$$

In particular, if the Lagrange equations are degenerate, so are Eqs. (64). If the Hamiltonian formulation contains μ primary first-type connections, then, as was shown, the solutions of both the Hamilton and Lagrange equations contain exactly μ identities of the **form**

$$R_{1i}^a M_i(q, v, \dot{v}) + R_{2i}^a(\dot{q}_i - v_i) \equiv 0, \; [a] = \mu, \tag{65}$$

while the generators R_{ki}^a (k = 1, 2) are independent. In identities (65) we put $v_i = \dot{q}_i$, so that we obtain μ identities between the original Lagrange equations

$$R_i^a \frac{\delta S}{\delta q_i} \equiv 0, \text{where } R_i^a = R_{1i}^a|_{v=\dot{q}}. \tag{66}$$

It only remains to show that the generators R_i are independent. For this we consider identity (65) on the hypersurface of variables q, v, $\dot{v}$, considered as independent, determined by the **equations** $M_i(q, v, \dot{v}) = 0$. (Under the assumptions made by us on the Lagrangian structure, vanishing values of all variables are located on this hypersurface.) They are

$$R_{2i}^a(\dot{q}_i - v_i)|_{M=0} \equiv 0. \tag{67}$$

It follows from (67) that on the hypersurface mentioned R_{2i}^a can only have a structure of the form $(\dot{q}_j - v_j)T_{ji}^a$, where T_{ji}^a are matrices antisymmetric in i, j. Following these comments we provide the proof of independence of the generators R_i^a. More precisely, we assume that there exist nonvanishing $\bar{n}_a$, **such that**

$$\bar{n}_a \cdot R_i^a = 0, \tag{68}$$

for example, at the vanishing point of the variables q, $\dot{q}$, ... We then form the combinations $\bar{n}_a R_{\kappa i}^a$, $\kappa = 1,2$, **and consider them at the** vanishing point of all variables (at this point, in particular, $v_i = \dot{q}_i$). Since the vanishing point belongs to the hypersurface M = 0, by **account** of (68) and the established structure of the generators R_2^a we obtain $\bar{n}_a R_{\kappa i}^a = 0$, which contradicts independence of the generators R_{ki}^a at any point. Thus, the generators R_i^a are independent at the vanishing point, and, consequently, by continuity, in some neighborhood of it.

The presence of identities (66) with μ independent generators implies invariance of the action with respect to infinitely small gauge transformations (52) with μ arbitrary functions of time, which also required proof.

It now becomes clear why the possibility of describing a physical system by different nondegenerate theories was called gauge invariance, and the choice of a specific nondegenerate theory was termed gauge application. As was shown, the action corresponding to Hamiltonian theories with first-type connections **is** invariant with respect to gauge transformations (gauge invariant), and, finally, for precisely this reason, relates uniquely the degeneracy of Lagrangian and Hamiltonian equations of motion.

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NEW EXACT ASYMPTOTIC CORRELATIONS BETWEEN THE PHASE AND MODULUS OF THE HADRON SCATTERING AMPLITUDE

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1. PREVIOUS RESULTS: SHORT REVIEW

In 1966, Marten [1], using only the results of axiomatic quantum field theory, obtained a sufficiently strict proof that the upper bound of the forward scattering amplitude F(·) has the form

$$F(E) \sim O(E\ln^2 E),\ E\to\infty \qquad (1.1)$$

and even estimated the multiplicative constants involved in this asymptotic upper bound, for various reactions. In accordance with the accepted terminology, the physical assertions based only on the results of axiomatic quantum field theory, will be called "strict".*

Approximately at the same time Huri and Konoshita showed that the Froissard-Marten bound (1.1) for a cross-symmetric amplitude F(·), (i.e., for the scattering amplitude of a truly neutral particle) can be lowered to some extent if one assumes that in some sense this amplitude is free from strong oscillations, and if the function

$$\rho(E) = \mathrm{Re}F(E)/\mathrm{Im}F(E) = \mathrm{ctg}\,\varphi(E) \qquad (1.2)$$

satisfies respective limitations. A little later, Vernov [3, 4] strengthened these results by means of a certain iterative procedure. In 1969, Lomsadze, Kontrosh and Tokar' [5] showed that all these results can be obtained under much milder limitations on the oscillations of the amplitude. It was also shown in [5] that these results are valid both for the cross-symmetric part $F_+(\cdot) = F_I(\cdot) + F_{II}(\cdot)$, and cross-antisymmetric part $F_-(\cdot) = F_I(\cdot) - F_{II}(\cdot)$ of the amplitude $F_I(\cdot)$ of the scattering of an arbitrary particle I on a target, and the amplitude $F_{II}(\cdot)$ of the scattering of its antiparticle II on the same target. In 1977, Lomsadze and Kelemen [6, 7] extended these considerations to the cross-symmetric function $F_I(\cdot)F_{II}(\cdot)$.

Interesting results in this direction have been obtained by Logunov, Mestvirishvili, and Khrustalev [8], by Fisher, Kolar, and Vrkoc [9], by Fisher and Shishanin [10], and in reviews by Lomsadze [11, 12]. Recently Lomsadze and Lomsadze (Jr.) [13, 14] obtained a number of new conditional (in the above-mentioned sense) upper bounds on the scattering amplitudes with specific multiplicative constants. These conditional upper bounds are obtained with the assumption that the corresponding amplitudes are free from q-power oscillations.

*It has to be stressed, though, that the bound (1.1) cannot yet be considered as "strict", strictly speaking, because it is substantially based on the inequality $A_s(s, t = R) < s^N$ (see [1], Sec. 7), not proven yet within axiomatic quantum field theory.

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