THE PHYSICAL INTERPRETATION OF THE PARAMETERS IN THE GENERALIZED KERR—NUT SOLUTION. I

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The interpretation of the parameters in the generalized Kerr—NUT solution is considered by investigating the dynamical characteristics of the field corresponding to a given solution. It is shown that the interpretation obtained of these parameters differs in some aspects from the one generally accepted. It is assumed that the reason for this is the unfortunate choice of the frame of reference utilized in obtaining the dynamical characteristics of the field.

In 1966, Demianski and Newman obtained a new solution of Einstein's gravitation equations [1]. This solution possesses a number of very interesting and, in a certain sense, even extraordinary properties which makes its investigation extremely urgent. The point is, it generalizes the two solutions which have at the present time numerous physical and astrophysical applications. They exist in terms of the Kerr [2, 3] and Newman—Unti—Tamburino (NUT) [4] metrics. The solution considered here was obtained by means of a complex coordinate transformation, where the Schwarzchild metric was utilized as the original metric. Incidentally, let us note that until recently this unusual technique was applied purely intuitively and its utilization was only justified by the correctness of the results obtained. The procedure for obtaining the solutions of Einstein's equations by means of a complex coordinate transformation has received a rigorous, mathematical basis only in one of Newman's recent papers [5].

The Demianski—Newman (DN) solution is obtained by using the Newman—Penrose formalism [6], where a quasiorthogonal tetrad of isotropic vectors

$$Z^{\mu}_{m} = (l^{\mu}, n^{\mu}, m^{\mu}, \overline{m}^{\mu}), \quad l_{\mu} n^{\mu} = - m_{\mu} \overline{m}^{\mu} = 1, \tag{1}$$

$$g^{\mu\nu} = l^{\mu} n^{\nu} + l^{\nu} n^{\nu} - m^{\mu} \overline{m}^{\nu} - \overline{m}^{\mu} m^{\nu}. \tag{2}$$

is utilized in the notation of the field equations. The contravariant components of the Newman—Penrose tetrad, corresponding to the DM metric, have the form

$$l^{\mu} = \delta^{\mu}_{1}, \quad n^{\mu} = U\delta^{\mu}_{1} + X^{a}\delta^{\mu}_{a},$$
$$m^{\mu} = \omega\delta^{\mu}_{1} + \xi^{a}\delta^{\mu}_{a} \quad (a = 0, 2, 3). \tag{3}$$

Here the following symbols are introduced:

$$U = -\frac{1}{2}[1 + m(\rho + \bar{\rho}) - 2b(a\cos\theta + b)\rho\bar{\rho} + (e^2 + g^2)\rho\bar{\rho}],$$
$$\omega = \frac{ia\sin\theta}{\sqrt{2}}\bar{\rho}, \quad \xi^0 = -\frac{iA}{\sqrt{2}\sin\theta}\bar{\rho},$$
$$\xi^2 = -\frac{\bar{\rho}}{\sqrt{2}}, \quad \xi^3 = -\frac{i}{\sqrt{2}\sin\theta}\bar{\rho}, \quad X^0 = 1, \quad X^2 = X^3 = 0,$$
$$\rho = -\frac{1}{r - i(a\cos\theta + b)}, \quad A = a\sin^2\theta + 4b\sin^2\frac{\theta}{2}. \tag{4}$$

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That is, the vectors (3) were obtained in [1] by a complex coordinate transformation of the initial vectors corresponding to the Schwarzchild solution. It is now easy to obtain the metric of the solution in question, utilizing the relationship (2). One should stipulate that this procedure is not performed entirely correctly in [1].

The expression for the Demianski–Newman metric with corrected errors, introduced in their paper, has the form

$$g^{\mu\nu}=\left\|\begin{matrix} -\dfrac{A^2\rho\bar{\rho}}{\sin^2\theta} & 1+aA\rho\bar{\rho} & 0 & -\dfrac{A\rho\bar{\rho}}{\sin^2\theta} \\ \cdot & -2U-a^2\rho\bar{\rho}\sin^2\theta & 0 & a\rho\bar{\rho} \\ \cdot & \cdot & -\rho\bar{\rho} & 0 \\ \cdot & \cdot & \cdot & -\dfrac{\rho\bar{\rho}}{\sin^2\theta} \end{matrix}\right\|. \tag{5}$$

The determinant of the metric tensor (5) equals

$$\det g^{\mu\nu}=-\frac{\rho^2\bar{\rho}^2}{\sin^2\theta}. \tag{6}$$

Let us also cite the covariant components $g_{\mu\nu}$, substituting them into the following expression for the square of the 4-interval:

$$\begin{gathered} ds^2=-2U\,du^2+2du\,dr+2\,(a\sin^2\theta-2AU)\,du\,d\varphi \\ -2Ad\,rd\varphi-\frac{1}{\rho\bar{\rho}}d\theta^2-\left[\frac{\sin^2\theta}{\rho\bar{\rho}}+A\,(2AU+2a\sin^2\theta)\right]d\varphi^2, \\ x^\mu=(u,\,r,\,\theta,\,\varphi). \end{gathered} \tag{7}$$

The nonzero Ricci rotation coefficients corresponding to (5) have the form

$$\begin{gathered} \rho=-\frac{1}{r-i\,(a\cos\theta+b)}, \quad \alpha=-\bar{\beta}=\frac{\operatorname{ctg}\theta\rho}{2\sqrt{2}}, \\ \mu=\frac{1}{2}\,[\bar{\rho}+(m-ib)\,\rho^2+(m-ib)\,\rho\bar{\rho}]+\frac{1}{2}\,(e^2+g^2)\,\rho^2\bar{\rho}, \\ \gamma=\frac{1}{2}\,[(m-ib)\,\rho^2-(e^2+g^2)\,\rho^2\bar{\rho}], \\ \nu=-\frac{ia\sin\theta}{\sqrt{2}}\,[(m-ib)\,\rho^3+(e^2+g^2)\,\rho^3\bar{\rho}]. \end{gathered} \tag{8}$$

And, finally, the contracted components of the Weyl and Maxwell tensors equal

$$\Psi_2=(m-ib)\,\rho^3+(e^2+g^2)\,\rho^3\bar{\rho}, \quad \Psi_3=-\frac{3}{\sqrt{2}}ia\sin\theta\,(m-ib)\,\rho^4-\frac{3}{\sqrt{2}}ia\sin\theta\,(e^2+g^2)\,\rho^4\bar{\rho}, \tag{9}$$

$$\begin{gathered} \Psi_4=-3a^2\sin^2\theta\,(m-ib)\,\rho^5-3a^2\sin^2\theta\,(e^2+g^2)\,\rho^5\bar{\rho}, \\ \Phi_1=\frac{(e+ig)}{\sqrt{2}}\,\rho^2, \quad \Phi_2=-ia\sin\theta\,(e+ig)\,\rho^3. \end{gathered} \tag{10}$$

As is seen from Eqs. (4) and (7), the DN metric depends on the parameters m, a, b, e, and g. When a = b = g = 0, it becomes the Reisner–Nordstrom metric; when a = g = 0, a "charged" NUT space is obtained, and when b = g = 0, we have the Kerr solution for an electric charge. Thus, this solution unifies diversified classes of greatest importance from the point of view of the physical applications of the spaces (let us mention that Demianski was also able to generalize this solution in a recent paper [7]).

The metric (7) is related to the D type according to the classification of Petrov. It permits the existence of geodesic rays with nonzero divergence and curl ($\rho \neq \bar{\rho} \neq 0$). The fact that its components do not depend on the time and the angle φ indicates that the corresponding physical space permits two kinds of motion described by the following solutions of Killing's equations:

$$\underset{(1)}{\xi^\mu}=\delta^\mu_0, \quad \underset{(2)}{\xi^\mu}=\delta^\mu_3. \tag{11}$$

In [1], the five parameters (m, a, e, b, g) are interpreted as follows: m is identified with the source, a with its angular momentum per unit mass, e with the electric charge of the source, b with the mass of a type of magnetic monopole, and g with its magnetic charge. The fact that when b = g = 0 the metric (7) describes a Kerr space with a charge serves as the basis for the first three statements and this, it would appear, eliminates any ambiguity in the physical interpretation of the parameters m, a, and e. The purpose of our work, in particular, will be to show that even in this case definite confusion exists in the interpretation of these parameters. When $a = 0$, the DN solution describes a charged NUT space. In addition, in this case, on the basis of the structure of Maxwell's tensor (10), one can assume that e corresponds to the electric charge of the source, while g corresponds to a magnetic charge which, as is known, is lacking in the classical electrodynamics of Maxwell [8]. The similarity in the structure of the Maxwell (10) and Weyl (9) tensors allows one to say that b is a mass of the "magnetic" type [1]. A similar interpretation of these parameters in the NUT solution is correct in general and we shall give a rigorous proof of this in a future paper. However, to begin with, let us elucidate the physical meaning of the quantities m and a occurring in Eq. (7).

If one calculates the total energy of the field created by the source of this metric, the following expression is obtained:

(12)

($\varkappa$ is Einstein's constant).

This not exactly conventional result is nevertheless physically correct, In the fundamental paper [9], for example, it is remarked that "in all global laws we deal with the total energy, 'total' meaning summation over the entire space and all forms of energy." Thus, the constant m in the metric (7) is correctly interpreted to be the total energy of DN space, and not the total energy of the material mass as is assumed in [1].

Let us mention that in obtaining Eq. (12) we proceeded on the basis of the Denen–Dozmorov theory of reference systems associated with graviinertial observers [10, 11], the course of the argument being entirely analogous to that presented in [12]. The question concerning calculation of the value of the total momentum of the field created by the source of the metric (7) is very interesting. As a generator of the conserved quantities, we utilize the well-known expression of Komar [13]

$$F^{\mu} = \frac{2}{\varkappa}(\xi^{\mu;\nu} - \xi^{\nu;\mu})_{;\nu}, \tag{13}$$

which satisfies a strong conservation law in the form

$$F^{\mu}_{;\mu} = 0. \tag{14}$$

In order to obtain an integral relationship, let us integrate Eq. (14) over a section of a world tube bounded by two spacelike cross sections

$$0 = \int F^{\mu}_{;\mu}\, d\Omega = \oint_{\Sigma_1} F^{\mu}\, dS_{\mu} + \oint_{\Sigma_2} F^{\mu}\, dS_{\mu} + \oint_{\Gamma} F^{\mu}\, d\Gamma_{\mu}. \tag{15}$$

The last integral over the walls of the tube vanishes at a sufficiently great distance from a system of finite sources. Furthermore, assuming that $dS_{\mu} = dV \cdot S_{\mu}$, $S''_{\mu} = -S'_{\mu}$, i.e., the normals transform into one another during the motion of the three-dimensional volume from one cross section to the other, we finally obtain

$$\int_{\Sigma_1} F^{\mu}\, dS_{\mu} = \int_{\Sigma_2} F^{\mu}\, dS_{\mu}. \tag{16}$$

Thus, the integral $\int F^{\mu} dS_{\mu}$ does not depend on the choice of the three-dimensional volume of integration and as a result expresses an integral conservation law. Komar's expression (13) then becomes physically meaningful when we associate the quantities characterizing the symmetry of the space, the Killing vector, with the vector field ξ^{μ}. In the case under consideration, we have, taking into consideration Eq. (13), the following expression for the angular momentum

$$M = \frac{2}{\varkappa}\int_{V}(\underset{(2)}{\xi}{}^{\mu;\nu} - \underset{(2)}{\xi}{}^{\nu;\mu})_{;\nu}\, dS_{\mu} = \frac{4}{\varkappa}\int_{S}\underset{(2)}{\xi}{}^{[\mu;\nu]}\, dS_{\mu\nu}, \tag{17}$$

where ξ^{μ} is given by Eq. (11), and an element of the two-dimensional surface bounding the three-dimensional volume equals [11]

$$dS_{\mu\nu} = n_{[\mu} l_{\nu]}\sqrt{-g}\, dx^2\, dx^3.$$

Here, as in Eq. (15), we applied the Gauss–Ostrogradskii theorem [14].

We shall omit here the simple, but rather lengthy, calculation accompanying the one using Eq. (17). Let us write the final expression for the moment in a sphere of radius r

$$M(r) = \frac{4}{\varkappa} \int_0^{2\pi} d\varphi \int_0^{\pi} \frac{A \sin\theta}{[r^2 + (a\cos\theta + b)^2]} \{r(e^2 + g^2) - m[r^2 - (a\cos\theta + b)^2 - 2rb(a\cos\theta + b)\} d\theta. \quad (18)$$

We obtain the value of the total moment by letting r go to infinity in Eq. (18)

$$M = \lim_{r\to\infty} M(r) = -\frac{8\pi m}{\varkappa} \int_0^{\pi} \left(a \sin^3\theta + 4b \sin\theta \sin^2\frac{\theta}{2}\right) d\theta = -\frac{4}{3}\frac{8\pi ma}{\varkappa} - 4\frac{8\pi mb}{\varkappa}. \quad (19)$$

Let us discuss this result in greater detail. When b = 0, an expression is obtained for the angular momentum of the field described by the Kerr metric. In this case, the result (19) seems to be clearly extraordinary. In fact, the generally accepted interpretation of the Kerr metric is just such that it corresponds to the field of a rotating source with mass m and a total moment $M = -(8\pi ma/\varkappa)$ [15, 16]. In general, a significant number of papers are devoted to calculating the angular momentum of a Kerr space. The expression of Komar (13) was utilized in [17] for this purpose. The usual, correct to a sign, expression $M = (8\pi ma/\varkappa)$ is obtained. However, this result is questionable. The point is that the calculation shown in Eq. (19) actually coincides with the calculations in [17]. Therefore, one must consider as incorrect either the result in [17] or the one obtained by us. (The expression $M = -(4/3)\,8\pi ma/\varkappa$ for the Kerr metric was obtained for the first time in [18]). A thorough examination forces us to consider the result (19) as preferable. Apparently, an error in the calculations was allowed in [17]. One can say the same concerning the results in [19]. One should especially discuss [20], where a check of the calculations given there led us to detect an apparent error in them. Thus, we are faced with the following dilemma. Either the result (19) obtained by us is correct, and one should acknowledge an unsatisfactory, generally accepted, interpretation of the Kerr metric, or utilization of the integral expression (17) in the given case is incorrect. As is pointed out in a recent paper by Dozmorov [21], the second assumption appears to be correct. Nevertheless, the correct expression for the moment is $M = -(8\pi ma/\varkappa)$, obtained on the basis of the theory of graviinertial reference systems and the Bondi coordinates rigidly bound to them [11, 21].

In the same way, a possible and correct interpretation of the parameter b, whose meaning in (19) is not entirely clear, as well as of the rest of the parameters occurring in the Demianski–Newman metric (7), is found. Our next paper will be devoted to this problem.

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