GEODESIC CURVATURES IN KERR SPACE

I. M. Dozmorov and G. V. Lutsenko

UDC 530.12:531.51

The structure of the sources of a gravitational field in Schwarzchild and Kerr spaces is investigated using the method of geodesic curvature. The curvature is calculated in Schwarzchild space for an isotropic and time-like congruence and in Kerr space for two isotropic congruences. An analysis of the curvature is made.

Geodesic curvature is utilized in [1] to analyze the structure of the source of a gravitational field in Schwarzchild space. In this paper, geodesic curvature is considered for determining the physical singularities in the Schwarzchild and Kerr solutions using the Newman–Penrose (NP) formalism [2] (see also [3, 4]).

The quasiorthogonal tetrad formed by the four linearly independent isotropic vectors l_μ, n_μ, m_μ, and $\bar{m}_\mu$, satisfying the orthogonality condition

$$l_\mu n^\mu = -m_\mu \bar{m}_\mu = 1 \tag{1}$$

(the line denotes complex conjugation) is utilized in the NP formalism. The rest of the contractions of these vectors vanish. It is convenient to write the tetrad in the form $Z_{m\mu} = (l_\mu, n_\mu, m_\mu, \bar{m}_\mu)$, where the tetradic indices are denoted by Latin letters (except i, j) and the covariant indices are denoted by Greek letters. All of the indices vary from 1 to 4.

The Kerr solution [5] is written in the NP formalism in the following manner [6]:

$$\begin{gathered}
g^{\mu\nu} = l^\mu n^\nu + l^\nu n^\mu - m^\mu \bar{m}^\nu - \bar{m}^\mu m^\nu, \\
l^\mu = \delta_2^\mu, \quad n^\mu = \delta_1^\mu + U\delta_2^\mu, \quad m^\mu = \omega\delta_2^\mu + \xi^i \delta_i^\mu, \\
l_\mu = \delta_\mu^1 + a\sin^2\Theta\,\delta_\mu^4, \quad n_\mu = -U\delta_\mu^1 + \delta_\mu^2 - a\sin^2\Theta\,(U+1)\,\delta_\mu^4, \\
m_\mu = -1/\sqrt{2}(r + ia\cos\Theta)\,(\delta_\mu^3 + i\sin\Theta\,\delta_\mu^4), \quad (i,\ \kappa = 1, 3, 4), \\
U = -\frac{1}{2} - \frac{1}{2}m(\rho + \bar{\rho}), \quad \omega = \omega^0\bar{\rho}, \quad \xi^i = \xi^{0i}\bar{\rho}, \\
\xi^{03} = -\frac{1}{\sqrt{2}}, \quad \xi^{04} = -\frac{i}{\sqrt{2}\sin\Theta}, \quad \omega^0 = -\xi^{01} = \frac{-i}{\sqrt{2}}a\sin\Theta.
\end{gathered} \tag{2}$$

The coordinates $x^\mu = (u, r, \Theta, \varphi)$, where u is the retarded time, are utilized; r is the affine parameter along the isotropic geodesic and characterizes the distance from the source; Θ and φ are polar angular coordinates. The superscript "0" denotes that the function is independent of r. The nonzero rotation coefficients of Ricci and the components of the Weyl tensor are

$$\begin{gathered}
\rho = -\frac{1}{r + ia\cos\Theta}, \quad \alpha = -\bar{\beta} = \frac{\cos\Theta}{2\sqrt{2}\sin\Theta}\rho, \quad \gamma = \frac{1}{2}m\rho^2, \\
\mu = \frac{1}{2}(\bar{\rho} + m\rho\bar{\rho} + m\rho^2), \quad \nu = \frac{i}{\sqrt{2}}a\sin\Theta\, m\rho^3, \\
\Psi_0 = \Psi_1 = 0, \quad \Psi_2 = m\rho^3, \quad \Psi_3 = \frac{3}{\sqrt{2}}iam\rho^4\sin\Theta, \quad \Psi_4 = -3\rho^5 a^2 m\sin^2\Theta.
\end{gathered} \tag{3}$$

All-Union Scientific-Research Institute of Optical Physical Measurements. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Fizika, No.12, pp.142-145, December, 1974.

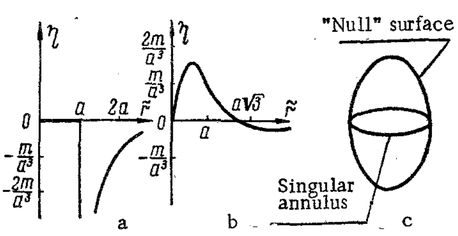

Fig. 1

The Schwarzchild solution is obtained from the Kerr solution by letting a equal zero.

The equation of the geodesic curvature, according to [7], has the form

$$\frac{\delta^2 V^i}{\delta u^2} + R^i_{j\kappa m} U^j V^\kappa U^m = 0, \quad U^i = \frac{\partial x^i}{\partial u}, \quad V^i = \frac{\partial x^i}{\partial v}, \tag{4}$$

where $\delta/\delta u$ is the absolute derivative; u is the canonical parameter along the geodesic; v is a parameter which is constant on each of the geodesics. The vectors U_μ (the vector tangent to the geodesic) and V_μ are related in the following manner:

$$U_{\mu;\nu} V^\nu = V_{\mu;\nu} U^\nu. \tag{5}$$

We shall define the geodesic curvature as

$$\eta^\mu = \frac{\delta^2 V^\mu}{\delta u^2} = - R^\mu_{\nu\sigma\rho} U^\nu V^\sigma U^\rho. \tag{6}$$

Let us consider an isotropic geodesic congruence in Schwarzchild space such that $U_\mu = l_\mu$. As is known, in order that the isotropic vector l_μ would be tangent to the geodesic, it is necessary to satisfy the condition k = 0 [2] and

$$l_{\mu;\nu} l^\nu = (\epsilon + \bar{\epsilon})\, l_\mu. \tag{7}$$

One can transform the coefficient $l_\mu \to \varphi l_\mu$ to zero by a change in scale $(\varepsilon + \bar{\varepsilon})$. The vector l_μ is a geodesic for the Schwarzchild solution written in the form (2).

Let us define the vector V_μ in the form

$$V_\mu = a l_\mu + n_\mu + c m_\mu + \bar{c} \bar{m}_\mu, \tag{8}$$

where a and c are functions of r and Θ. Using Eq. (5), we obtain $a = -mr^{-1}$ and $c = -c^0 r$. Then the geodesic curvature is written in the following form:

$$\eta_\mu = - R_{\mu\nu\sigma\rho} l^\nu V^\sigma l^\rho = - R_{\mu 121} = (\Psi_2 + \bar{\Psi}_2)\, l_\mu. \tag{9}$$

Substituting $\Psi_2 = -mr^{-3}$ into Eq. (9), we obtain

$$\eta_\mu = -(2m/r^3) l_\mu. \tag{10}$$

Let us now determine the geodesic curvature in Schwarzchild space for a time-like geodesic congruence. Let us define the 4-vector velocity in the form

$$U_\mu = \frac{1}{\sqrt{2}} (l_\mu + n_\mu). \tag{11}$$

Let us satisfy the geodesic equation $u_{\mu;\nu} u^\nu = 0$ with the aid of the Lorentz rotations $\tilde{l}^\mu = \varphi l_\mu$, $\tilde{n}_\mu = \varphi^{-1} n_\mu$.

Let us define the vector V_μ in the form

$$V_\mu = a l_\mu + (1 + a) n_\mu + c m_\mu + \bar{c} \bar{m}_\mu. \tag{12}$$

Solving Eq. (5) and substituting the expression obtained for V_μ and Eq. (11) into Eq. (6), we obtain

$$\eta_\mu = \frac{1}{2} (\Psi_2 + \bar{\Psi}_2)(l_\mu - n_\mu) = - \left(\frac{m}{r^3}\right)(l_\mu - n_\mu). \tag{13}$$

Comparing expressions (10) and (13), we conclude that a unique physical singularity occurs in Schwarzchild space as $r \to 0$, in agreement with the results in [1].

It is especially interesting to analyze the behavior of the geodesic curvature in Kerr space. Let us consider the isotropic geodesic congruence with the tangent vector $U_\mu = l_\mu$ [the vector l_μ is the geodesic for the tetrad (2)].

Let us define a vector V_μ similar to (8):

$$V_\mu = a l_\mu + n_\mu + c m_\mu + \bar{c} \bar{m}_\mu.$$

Solving Eq. (5), we obtain $a = \frac{1}{2} m(\rho + \bar{\rho})$ and $c = c^0/\bar{\rho}$. Then the geodesic curvature assumes a form similar to (9):

$$\eta_\mu = (\Psi_2 + \overline{\Psi}_2)\, l_\mu.$$

Substituting Ψ_2 from (3) into this expression, we obtain the formula for the geodesic curvature in Kerr space

$$\eta_\mu = -\left[\frac{2mr\,(r^2 - 3a^2\cos^2\Theta)}{(r^2 + a^2\cos^2\Theta)^3}\right] l_\mu. \tag{14}$$

The coordinates r, Θ, and φ are not the usual polar ones since even in the limit of plane space (m = 0) the Kerr metric is not the Minkowski metric in polar coordinates. The following coordinate transformation reduces all of the coordinates to the polar ones $\tilde{r}$, $\tilde{\Theta}$, $\tilde{\varphi}$, $\tilde{u} = t - \tilde{r}$ [8]:

$$\tilde{r}^2 = r^2 + a^2\sin^2\Theta, \quad \mathrm{tg}\,\tilde{\varphi} = \frac{\mathrm{tg}\,\varphi - a/r}{1 + (a/r)\,\mathrm{tg}\,\varphi},$$
$$\cos\tilde{\Theta} = r\cos\Theta/(r^2 + a^2\sin^2\Theta)^{1/2}, \quad \tilde{u} = u - (r^2 + a^2\sin^2\Theta)^{1/2} + r. \tag{15}$$

The situation is such that we have an initial curved space associated with a plane space, in which polar coordinates are defined. We can now analyze the geodesic curvature as a function of the polar coordinates established in the associated plane space.

Analyzing Eq. (14), we conclude: when $\tilde{r} = a$ and $\tilde{\Theta} = \pi/2$, there exists a singular annulus on which $\eta_\mu \to \infty$. Hence, one can consider this annulus as a physical singularity which is unique in Kerr space. Furthermore, it is necessary to mention the existence of a certain "null" surface on which the geodesic curvature changes sign.

Graphs of the equatorial and axial curvatures as a function of $\tilde{r}$ are given in Figs. 1a and 1b, respectively. A cross section of 4-dimensional Kerr space—time, u = const, is given in Fig. 1c.

Let us consider a second isotropic geodesic congruence with the tangent vector $U_\mu = n_\mu$. For the vector n_μ to be a geodesic, it is necessary to satisfy the condition $\nu = 0$. In order to satisfy this condition, a tetradic transformation is possible which leaves the orthogonality condition for the tetradic vectors invariant. This transformation is a zeroth-order rotation of the tetrad about l^μ [3]

$$\tilde{l}^\mu = l^\mu, \quad \tilde{m}^\mu = m^\mu + a l^\mu, \quad \tilde{n}^\mu = n^\mu + a\bar{m}^\mu + \bar{a} m^\mu + a\bar{a} l^\mu, \tag{16}$$

and a is a complex scalar. If the complex scalar a equals $a = (ia\sin\Theta/\sqrt{2})\,\bar{\rho}$, then the functions ν, Ψ_3, and Ψ_4 are eliminated by this transformation. Then the geodesic equation for the isotropic vector n_μ will be

$$n_{\mu;\nu} n^\nu = (\gamma + \bar{\gamma})\, n_\mu. \tag{17}$$

Let us transform the coefficient $(\gamma + \bar{\gamma})$ to zero by means of the scale transformation $l^\mu \to \varphi l^\mu$.

We define a vector V_μ in the form

$$V_\mu = l_\mu + b n_\mu + c m_\mu + \bar{c}\,\bar{m}_\mu, \tag{18}$$

where b and c are functions of r and Θ. One can show that this vector satisfies Eq. (5), the functions b and c not occurring in the expression for the curvature. Thus, for an isotropic geodesic congruence with the tangent vector n_μ we obtain

$$\eta_\mu = (\Psi_2 + \overline{\Psi}_2)\, n_\mu = -\left[\frac{2mr\,(r^2 - 3a^2\cos^2\Theta)}{(r^2 + a^2\cos^2\Theta)^3}\right] n_\mu. \tag{19}$$

Comparing expressions (14) and (19), we conclude that the curvatures are identical for both congruences.

Thus, let us emphasize that the analysis conducted allowed one to uncover the more detailed properties of the structure of the source in Kerr space (the surface for the zeroth-order curvature) than was possible with conventional methods [8]. Additional investigation is required to explain the physical meaning of these properties.

LITERATURE CITED

1. J. Finley, J. Math. Phys., 12, 32 (1971).
2. E. Newman and R. Penrose, J. Math. Phys., 3, 566 (1962).

3. I. M. Dozmorov, Izv. Vyssh. Uchebn. Zaved., Fiz., No.10, 83 (1969).
4. I. M. Dozmorov, Izv. Vyssh. Uchebn. Zaved., Fiz., No.3, 7 (1970).
5. R. Kerr, Phys. Rev. Lett., 11, 237 (1963).
6. I. M. Dozmorov, Izv. Vyssh. Uchebn. Zaved., Fiz., No.11, 16 (1971).
7. J. Synge, Relativity: The General Theory, North-Holland, Amsterdam (1960).
8. E. Newman and A. Janis, J. Math. Phys., 6, 915 (1965).