GRAVITOINERTIAL WAVES IN VACUUM.

LINEAR APPROXIMATION

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UDC 530.12:531.51

Zel'manov's method of chronometric invariants is used to develop an approach to the description of gravitoinertial waves of a chosen frame of reference. It is shown that there exist two basically different types of gravitoinertial waves in vacuum. Both types are examined in general relativity in the linear approximation.

INTRODUCTION

Currently, there are various different methods for investigating gravitational wave fields in Einstein's theory. One of them is the method of approximations, different variants of which have been considered by Havas and Goldberg (expansion of the potentials of the gravitational field in a series in the gravitational constant), Bonnor and Rothenberg (the method of two-parameter approximations, which simulates an expansion of the potentials in series in the multipole moments of the system of sources), Bondi and Sachs (separation of the wave zone in the total gravitational field of an isolated emitting source by means of an "information function" defined by an expansion of the metric in inverse powers of the distance from the emitter), and others. The shortcomings of this method are the absence of a proof of the convergence (at least in the wave zone) of the series and also that the fact that it is not covariant, which makes it necessary to use a special (arbitrarily chosen) class of allowed coordinates.

A second method of describing gravitational waves, which is free of these shortcomings, is based on the introduction of a specially postulated (in addition to the gravitational equations) generally covariant criterion for distinguishing a class of gravitational wave fields. However, this approach not only entails an unavoidable ambiguity in the solution of the problem but also leads to difficulties associated with the lack of a direct physical interpretation of the employed generally covariant condition (algebraic or differential) used to determine the wave field. The interpretational difficulties arise because the generally covariant conditions imposed on the curvature of spacetime in order that it should describe gravitational waves do not admit an interpretation in the language of physical observables in the framework of the generally covariant approach. In this approach, therefore, the problem of confronting the theory of gravitational waves with the data of a physical experiment remains unsolved.

To solve the problem of the formulation of gravitational waves in terms of observables, it is expedient to consider a more general field of gravitoinertial waves. We say that a gravitoinertial field is more general than a gravitational one in the sense that the results of its description apply to not only the gravitational but also to the inertial field of the frame of reference of the observer, and our method of description does not enable one to distinguish the one from the other. An invariant distinction between gravitational and gravitoinertial wave fields is possible on the basis of an additional verification that the adopted generally covariant criterion is fulfilled.

In contrast to gravitational waves, which are determined (in the framework of generally covariant methods) independently of the choice of either the coordinate system or the frame of reference, gravitoinertial waves are determined only for a fixed (in general, arbitrary) frame of reference of observers to within "internal" coordinate transformations (which do not change this frame of reference) [1]:

All-Union Scientific-Research Institute of the Metrological Service. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Fizika, Vol.12, pp.106-113, December, 1974.

$$
\left.\begin{aligned}
\tilde{x}^0 &= \tilde{x}^0(x^0, x^1, x^2, x^3), \quad &\text{(a)}\\
\tilde{x}^i &= \tilde{x}^i(x^1, x^2, x^3), \quad \frac{\partial \tilde{x}^i}{\partial x^0} = 0 \quad &\text{(b)}
\end{aligned}\right\}. \tag{1}
$$

Thus, whereas in the method of approximations one fixes arbitrarily the class of allowed coordinates, in the given (chronometrically invariant) approach the freedom in the choice of the allowed coordinates is consistent with a unique determination of the frame of reference [1].

Besides covariance under the transformations (1), we require invariance under the transformations (1a). In other words, our definition of gravitoinertial waves [2] must be, first, invariant under the transformations (1a) (chronometrically invariant) and, second, covariant under the transformations (1b) (spatially covariant).

This definition is based on the introduction of the chronometrically invariant, spatially covariant d'Alembert wave operator

$$
{}^*\Box = h^{i\kappa}\, {}^*\nabla_i^*\, \nabla_\kappa - \frac{1}{c^2}\frac{{}^*\partial^2}{\partial t^2}, \tag{2}
$$

where h^{ik} is the chronometrically invariant metric of the three-dimensional space of the given frame of reference ($h_{ik} = -g_{ik} + (g_{0i}g_{0k}/g_{00})$), ${}^*\nabla_i$ is the symbol of chronometrically invariant, spatially covariant differentiation with respect to the coordinates of this space (see [1]), and the asterisk here and in what follows denotes chronometrically invariant operators, to distinguish them from ordinary ones.

The employed wave functions must be chronometrically invariant three-dimensional scalars, vectors, or tensors. As chronometrically invariant representatives of the curvature tensor ([2], p. 140):

$$
X^{ij} = -c^2\frac{R_{0\cdot0\cdot}^{\cdot i\cdot j}}{g_{00}}, \quad Y^{ij\kappa} = -c\frac{R_{0\cdot\cdot\cdot}^{\cdot ij\kappa}}{\sqrt{g_{00}}}, \quad Z^{i\kappa lj} = c^2 R^{i\kappa lj}, \tag{3}
$$

which can be expressed in terms of the vector F_i of the gravitoinertial force, the tensor A_{ik} of the angular velocity of rotation about the locally comoving geodesic coordinate system, the tensor D_{ik} of the rates of deformation of the three-dimensional space of the frame of reference, and the curvature tensor C_{iklj} of this space.

The chronometrically invariant criterion of gravitoinertial waves [2] consists of the requirement that the chronometrically invariant components of the world curvature tensor X^{ij}, Y^{ijk}, and Z^{iklj} satisfy equations of the form

$$
{}^*\Box f = A, \tag{4}
$$

where A is an arbitrary function of the world coordinates that does not contain derivatives of f of order higher than the first, and the left-hand side of Eq. (4) is nontrivial, i.e., the wave function must be nonstationary and spatially inhomogeneous. Depending on the choice of the wave function f, we shall distinguish the gravitoinertial waves X^{ij}, Y^{ijk}, and Z^{iklj}. However, in vacuum, X^{ij} and Z^{iklj} are not independent but are related by

$$
X^{ij} + Z^{ilj}_{\cdot\cdot\cdot l} = 0, \tag{5}
$$

so that in what follows we shall speak of two types of gravitoinertial waves: X (or Z) waves and Y waves.

§1. The Two Types of Gravitoinertial Waves.

The General Case

In a vacuum, the Riemann world tensor $R_{\alpha\beta\gamma\delta}$ satisfies the identities [3]

$$
\Box^\sigma_\sigma R_{\mu\alpha\beta\nu} + R_{\sigma\alpha\beta\cdot}^{\cdot\cdot\cdot\delta} R_{\delta\mu\nu\cdot}^{\cdot\cdot\cdot\sigma} + 2\left(R_{\sigma\alpha\nu\delta}^{\cdot\cdot\cdot\cdot} R_{\mu\cdot\beta\cdot}^{\cdot\delta\cdot\sigma} + R_{\alpha\mu\sigma\cdot}^{\cdot\cdot\cdot\delta} R_{\nu\delta\beta\cdot}^{\cdot\cdot\cdot\sigma}\right) = 0. \tag{6}
$$

Writing out this system in a chronometrically invariant, spatially covariant form, we arrive at the three systems

$$
{}^*\Box X^{ij} = \underset{(1)}{A^{ij}}, \tag{7}
$$

$$
{}^*\Box Y^{ij\kappa} = \underset{(2)}{A^{ij\kappa}}, \tag{8}
$$

TABLE 1

<table>
<tr><th colspan="2">Chronometrically invariant characteristics of the reference space</th><th>Chronometrically invariant criterion for the observable $X^{ij}(Z^{iklj})$</th><th>Chronometrically invariant criterion for the observable Y^{ijk}</th></tr>
<tr><td colspan="2">$F_i = 0, \quad A_{i\kappa} = 0, \quad D_{i\kappa} = 0,$</td><td colspan="2" rowspan="3">Not satisfied</td></tr>
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$A_{i\kappa} \neq 0,$
$D_{i\kappa} = 0$</td><td>$\frac{{}^*\partial A_{i\kappa}}{\partial t} = 0$</td><td>Not satisfied</td><td>Satisfied</td></tr>
<tr><td>$\frac{{}^*\partial A_{i\kappa}}{\partial t} \neq 0$</td><td colspan="2" rowspan="2">Satisfied</td></tr>
<tr><td colspan="2">$D_{i\kappa} \neq 0$</td></tr>
</table>

$$ {}^*\Box Z^{i\kappa lj} = \underset{(3)}{A}{}^{i\kappa lj}, \tag{9}$$

where $\underset{(1)}{A}{}^{ij}$, $\underset{(2)}{A}{}^{ijk}$, $\underset{(3)}{A}{}^{iklj}$ are, respectively, the chronometrically invariant, spatially covariant tensors of second, third, and fourth rank, respectively, and they do not contain derivatives of higher than first order of the wave functions X^{ij}, Y^{ijk}, Z^{iklj}.

In [4], the relations (7)-(9) were used to investigate the conditions of existence of gravitoinertial waves of the observables X^{ij}, Y^{ijk}, and Z^{iklj} for all possible types of frames of reference. We represent these results in the form of Table 1.

It can be seen from this that gravitoinertial waves of both types (X and Y waves) exist only in frames of reference characterized by one of two conditions:

$$D_{i\kappa} \neq 0, \quad F_i = 0, \quad A_{i\kappa} = 0, \tag{10}$$

or

$$F_i \neq 0, \quad A_{i\kappa} \neq 0, \quad \frac{{}^*\partial A_{i\kappa}}{\partial t} \neq 0, \quad D_{i\kappa} = 0. \tag{11}$$

A frame of reference of the first type is synchronous; we shall call one of the second type (moving with acceleration and rotating without deformation) a rigid frame of reference.

In a synchronous frame of reference, the observables X^{ij}, Y^{ijk}, and Z^{iklj} have the form [4]

$$X^{ij} = -DD^{ij} + D^i_\kappa + D^{\kappa j} + C^{ij}, \quad (C^{ij} = h_{lm} C^{iljm}), \tag{12}$$

$$Y^{ij\kappa} = {}^*\nabla^j D^{i\kappa} - {}^*\nabla^i D^{j\kappa}, \tag{13}$$

$$Z^{i\kappa lj} = D^{i\kappa} D^{lj} - D^{il} D^{\kappa j} - c^2 C^{i\kappa lj}, \tag{14}$$

so that the X (Z) waves and Y waves are due solely to the presence of nonstationary deformation of the frame of reference and the inhomogeneity of this deformation.

If the given gravitational field admits a rigid frame of reference, then in it the observables X^{ij}, Y^{ijk}, Z^{iklj} take the form

$$X^{ij} = -3A^{i\cdot}_{\cdot\kappa} A^{j\kappa} + C^{ij}, \tag{15}$$

$$Y^{ij\kappa} = {}^*\nabla^j A^{i\kappa} - {}^*\nabla^i A^{j\kappa} + \frac{2}{c^2} A^{jl} F^\kappa, \tag{16}$$

$$Z^{i\kappa lj} = A^{i\kappa} A^{lj} - A^{il} A^{\kappa j} + 2A^{ij} A^{\kappa l} - c^2 C^{i\kappa lj}. \tag{17}$$

It follows that the X (Z) waves are due to the nonstationarity of the angular velocity tensor A^{ik}, and the Y waves to nonstationarity of the tensor A^{ik} and the simultaneous action of the acceleration and rotation ($A^{ji}F^k$).

Note an interesting special case. Suppose the acceleration field F_i is irrotational: $^*\nabla_{[k}F_{i]} = 0$. Then by virtue of the identities [1]

$$\frac{{}^*\partial A_{i\kappa}}{\partial t} + {}^*\nabla_{[\kappa} F_{i]} = 0 \tag{18}$$

the tensor A_{ik} is stationary, so that X (Z) waves are absent but the Y waves nevertheless exist even when there is stationary rotation since in this case

$$\frac{{}^*\partial Y^{ij\kappa}}{\partial t} = \frac{2A^{ji}}{c^2}\frac{{}^*\partial F^{\kappa}}{\partial t}. \tag{19}$$

§2. Two Types of Gravitoinertial Waves. Linear Approximation

Let us consider the wave equation (4) for the case of weak waves. Suppose the gravitational field is described by the metric

$$\left.\begin{aligned} & g_{\alpha\beta} = \delta_{\alpha\beta} + a_{\alpha\beta}, \quad \delta_{\alpha\beta} = \{+1, -1, -1, -1\}, \\ & a_{\alpha\beta}(x^0, x^1, x^2, x^3) \ll 1. \end{aligned}\right\} \tag{20}$$

Then the allowed coordinate transformations are of the type

$$\tilde{x}^{\alpha} = x^{\alpha} + \xi^{\alpha}, \tag{21}$$

where $\xi^{\alpha} \ll 1$, and $\partial\xi^i/\partial x^0 = 0$. The conditions of the infinitesimally small transformations (21), which conserve the splitting of the metric (20), do not change the chosen frame of reference either.

We introduce the notation

$$\begin{aligned} & g_{00} = 1 + a_0, \quad g_{0i} = a_i, \quad g_{i\kappa} = \delta_{i\kappa} + a_{i\kappa}, \\ & \delta_{i\kappa} = \{-1, -1, -1\}. \end{aligned} \tag{22}$$

For this metric, all three chronometrically invariant mechanical characteristics of the space of the given frame of reference are nonzero:

$$F_i = \frac{c}{1 + a_0}\left(\frac{\partial a_i}{\partial t} - \frac{c}{2}\frac{\partial a_0}{\partial x^i}\right), \tag{23}$$

$$A_{i\kappa} = \frac{c}{\sqrt{1 + a_0}}\left(\frac{\partial a_i}{\partial x^{\kappa}} - \frac{\partial a_{\kappa}}{\partial x^i}\right), \tag{24}$$

$$D_{i\kappa} = -\frac{1}{2\sqrt{1+a_0}}\frac{\partial a_{i\kappa}}{\partial t} \tag{25}$$

together with one geometric chracteristic:

$$C_{i\kappa lj} = \frac{\partial^2 a_{\kappa j}}{\partial x^i \partial x^l} + \frac{\partial^2 a_{il}}{\partial x^{\kappa} \partial x^j} - \frac{\partial^2 a_{\kappa l}}{\partial x^i \partial x^j} - \frac{\partial^2 a_{ij}}{\partial x^{\kappa} \partial x^l}. \tag{26}$$

The wave functions (with allowance for the field equations) take the form

$$X^{i\kappa} = c^2 C_{i\kappa} = c^2\left[\frac{\partial^2 a_{ij}}{\partial x^{\kappa}\partial x^j} + \frac{\partial^2 a_{\kappa j}}{\partial x^i \partial x^j} - \delta^{il}\left(\frac{\partial^2 a_{i\kappa}}{\partial x^j \partial x^l} + \frac{\partial^2 a_{jl}}{\partial x^l \partial x^{\kappa}}\right)\right], \tag{27}$$

$$Z^{i\kappa lj} = c^2 C_{i\kappa lj} = c^2\left(\frac{\partial^2 a_{\kappa j}}{\partial x^i \partial x^l} + \frac{\partial^2 a_{il}}{\partial x^{\kappa}\partial x^j} - \frac{\partial^2 a_{\kappa l}}{\partial x^i \partial x^j} - \frac{\partial^2 a_{ij}}{\partial x^{\kappa}\partial x^l}\right), \tag{28}$$

$$Y^{ij\kappa} = \frac{\partial}{\partial x^j}(D_{i\kappa} + A_{i\kappa}) - \frac{\partial}{\partial x^i}(D_{j\kappa} + A_{j\kappa}) = \frac{1}{\sqrt{1+a_0}}\left[c\left(\frac{\partial^2 a_i}{\partial x^j \partial x^{\kappa}} - \frac{\partial^2 a_j}{\partial x^i \partial x^{\kappa}}\right) - \frac{1}{2}\left(\frac{\partial^2 a_{j\kappa}}{\partial x^i \partial t} - \frac{\partial^2 a_{i\kappa}}{\partial x^j \partial t}\right)\right], \tag{29}$$

and the following relations hold:

$$\frac{1}{2}\frac{\partial^2}{\partial t^2}(\delta^{i\kappa} a_{i\kappa}) - c^2\frac{\partial^2 a_i}{\partial x^i \partial t} - \frac{c^2}{2}\Delta a_0 = 0, \tag{30}$$

$$\frac{1}{2}\frac{\partial^2 a_{ij}}{\partial x^j \partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^i \partial t}(\delta^{j\kappa} a_{j\kappa}) + c\left(\frac{\partial^2 a_j}{\partial x^i \partial x^j} - \delta^{j\kappa}\frac{\partial^2 a_i}{\partial x^j \partial x^{\kappa}}\right) = 0, \tag{31}$$

where $\Delta = \partial^2/\partial x^{1 2} + \partial^2/\partial x^{2 2} + \partial^2/\partial x^{3 2}$ is the Laplace operator. It can be seen from (26) and (27) that in the linear approximation the chronometrically invariant wave functions X^{ij} and Z^{iklj} are determined solely by the spatial curvature C_{iklj}, whereas Y^{ijk} is determined solely by the nonuniformities of the deformations D_{ik} and the angular velocity A_{ik}.

Let us investigate the physical nature of the X (Z) and Y waves for these two types of frame of reference (synchronous and rigid). As follows from (26)-(28), in a synchronous frame of reference there exist waves of both types: X (Z) and Y. In a rigid frame of reference, the X (Z) waves are absent because the wave functions are stationary, and the Y waves exist only under the condition that the field F_i is not irrotational: $^*\nabla_{[k}F_{i]} \neq 0$. Thus, in the linear approximation, in contrast to the general case, the observer establishes the presence of only Y waves in a rigid frame of reference, but he cannot establish the existence of X (Z) waves.

In the general case ($F_i \neq 0$, $A_{ik} \neq 0$, $D_{ik} \neq 0$), the Y waves are the result of superposition of deformation waves and waves due to the combined influence of the acceleration F_i and rotation A_{ik} of the space of the given frame of reference, while the X (Z) waves are, as before, purely deformational in nature.

Thus, the X (Z) waves and the Y waves have different physical natures.

The chronometrically invariant d'Alembert operator in the case of weak waves takes the form

$$^*\Box = \Delta - \frac{1}{c^2}\frac{^*\partial^2}{\partial t^2}, \tag{32}$$

where Δ is the ordinary three-dimensional Laplacian, and instead of the ordinary operator of differentiation with respect to the coordinate x^0 we have the chronometrically invariant operator

$$\frac{^*\partial}{\partial t} = \frac{1}{\sqrt{1+a_0}}\frac{\partial}{\partial t}. \tag{33}$$

The wave equation (4) for weak waves simplifies appreciably since, first, the chronometrically invariant Laplacian is replaced by the ordinary one, and, second, the rather cumbersome right-hand side vanishes because of the assumption of linearity; the equation becomes

$$\left(\frac{\partial^2}{\partial x^{1 2}} + \frac{\partial^2}{\partial x^{2 2}} + \frac{\partial^3}{\partial x^{3 2}} - \frac{1}{c^2}\frac{^*\partial^2}{\partial t^2}\right) f = 0. \tag{34}$$

Measurable quantities are the tensor of the three-dimensional curvature for purely deformational waves and the inhomogeneity of the tensor of the angular velocity and the tensor of the rates of deformation for the Y waves.

Note that in a freely falling frame of reference ($F_i = 0$), and in particular in a synchronous one ($F_i = 0$, $A_{ik} = 0$), the wave equation (34) takes the form of the ordinary wave equation of the special theory of relativity:

$$\Delta f - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0. \tag{35}$$

For this case, the X (Z) and Y waves have a purely deformational nature, and the corresponding wave functions (X^{ij}, Y^{ijk}, Z^{iklj}) are determined as the solutions of the wave equation (35) from the ordinary equations of mathematical physics. Thus, introducing the notation $x^1 = x$, $x^2 = y$, $x^3 = z$ and considering the Cauchy problem for Eq. (35) with the initial conditions

$$f_{tt} = c^2 \Delta f, \quad f(M, 0) = \varphi(M), \quad f_t(M, 0) = \psi(M), \quad M = M(x, y, z), \tag{36}$$

we obtain for the wave functions (observables) the integral representation

$$f(M, t) = \frac{1}{4\pi c}\left[\frac{\partial}{\partial t} t \int\int_{S_{ct}} \varphi(P)\, d\Omega + t \int\int_{S_{ct}} \psi(P)\, d\Omega_p\right],$$
$$(dS_p = (ct)^2\, d\Omega_p), \tag{37}$$

where $S_{ct} = S_{ct}^M$ is a sphere of radius ct with center at the point M.

We turn to the case of weak plane waves described by the metric [5]

$$\left.\begin{aligned} g_{\alpha\beta} &= \delta_{\alpha\beta} + \xi_{\alpha\beta}, \quad \xi_{\alpha\beta}(x^0 + x^1) \ll 1, \\ \xi_{22} &= -\xi_{33} = a, \quad \xi_{23} = \xi_{32} = b, \quad a^2 + b^2 \ll 1 \end{aligned}\right\}. \tag{38}$$

For this metric, the chronometrically invariant F_i and A_{ik} are zero, and D_{ik} has the form

$$D_{22} = -D_{33} = \frac{1}{2}\frac{\partial a}{\partial t}, \quad D_{23} = \frac{1}{2}\frac{\partial b}{\partial t}, \tag{39}$$

i.e., weak plane waves have a purely deformational nature. The wave functions take the form

$$X^{22} = -X^{33} = -\frac{1}{2}\frac{\partial^2 a}{\partial t^2}, \quad X^{23} = -\frac{1}{2}\frac{\partial^2 b}{\partial t^2}, \tag{40}$$

$$Z^{1212} = -Z^{1313} = \frac{c^2}{2}\frac{\partial^2 a}{\partial x^{1^2}}, \quad Z^{1213} = \frac{c^2}{2}\frac{\partial^2 b}{\partial x^{1^2}}, \tag{41}$$

$$Y^{213} = Y^{312} = -\frac{1}{2}\frac{\partial^2 b}{\partial x^1 \partial t}, \quad Y^{212} = -Y^{313} = -\frac{1}{2}\frac{\partial^2 a}{\partial x^1 \partial t}. \tag{42}$$

Because a and b are functions of $x^0 + x^1$, in the linear approximation the plane waves of the two types (X and Y) are identical, and the wave equations take the form

$$\frac{\partial^2}{\partial t^2}\left(\frac{1}{c^2}\frac{\partial^2 a}{\partial t^2} - \frac{\partial^2 a}{\partial x^{1^2}}\right) = 0, \tag{43}$$

$$\frac{\partial^2}{\partial t^2}\left(\frac{1}{c^2}\frac{\partial^2 b}{\partial t^2} - \frac{\partial^2 b}{\partial x^{1^2}}\right) = 0. \tag{44}$$

Einstein's equations for weak plane waves reduce to the two relations

$$\frac{1}{c^2}\frac{\partial^2 a}{\partial t^2} - \frac{\partial^2 a}{\partial x^{1^2}} = 0, \tag{45}$$

$$\frac{1}{c^2}\frac{\partial^2 b}{\partial t^2} - \frac{\partial^2 b}{\partial x^{1^2}} = 0. \tag{46}$$

Therefore, weak plane X (Y) waves always exist by virtue of Einstein's equations.

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