SYNCHROTRON RADIATION FROM A CHARGED, CURRENT-CARRYING RING

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The problem of synchrotron radiation from a charged, current-carrying ring is considered. An expression is found for the spectral and angular distribution of radiation intensity including polarization effects. Some special cases of the general formula are examined: a limiting transition to the generalized Schott formula, a formula for the intensity of synchrotron radiation from a magnetic dipole oriented along the tangent to the trajectory, and a transition to the formulas for the intensity of Vavilov—Cherenkov radiation from a charged, current-carrying, annular cloud by taking sums of Bessel functions.

The foundations of the classical theory of radiation by electrons in constant magnetic fields were laid by G. A. Schott [1]. It was shown in [2] that because of the intense electromagnetic radiation from relativistic electrons in a magnetic field which bends their path, there must be a limit to the attainable energy in a circular betatron accelerator. Later, synchrotron radiation was studied in [3-13], with special interest in quantum effects in accelerators and particle-storage devices [7-13].

The recent appearance in the scientific literature of papers [14] dealing with the circular motion of charged particles in extended clouds and current-carrying objects is reason to formulate the problem of synchrotron radiation from charged, current-carrying rings and a number of related problems.

We shall consider a uniformly charged, infinitely thin, current-carrying ring of radius a, whose center describes a circle of radius R in the xy plane. The plane of the ring coincides with the plane containing the z axis and the radius vector of the center of the ring. The total charge of the ring is e. We write the electric charge density of the ring in the form

$$\rho(x, y, z, t) = \frac{e}{2\pi\xi}\{\delta(x-(R+\xi)\cos\omega_0 t)\,\delta(y-(R+\xi)\sin\omega_0 t) + \delta(x-(R-\xi)\cos\omega_0 t)\,\delta(y-(R-\xi)\sin\omega_0 t)\}\,\eta(a-z)\,\eta(z+a). \tag{1}$$

The electric current density is then to be written in the form

$$j = \begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \frac{e\omega_0}{2\pi}\left\{\begin{pmatrix} -\sin\omega_0 t \\ \cos\omega_0 t \\ 0 \end{pmatrix}\frac{R+\xi}{\xi}\delta(x-(R+\xi)\cos\omega_0 t)\times\delta(y-(R+\xi)\sin\omega_0 t) + \begin{pmatrix} -\sin\omega_0 t \\ \cos\omega_0 t \\ 0 \end{pmatrix}\frac{R-\xi}{\xi}\delta(x-(R-\xi)\cos\omega_0 t)\,\delta(y-(R-\xi)\sin\omega_0 t)\right\}$$
$$+\frac{e\Omega}{2\pi}\left\{\begin{pmatrix} \frac{z}{\xi}\cos\omega_0 t \\ \frac{z}{\xi}\sin\omega_0 t \\ -1 \end{pmatrix}\delta(x-(R+\xi)\cos\omega_0 t)\,\delta(y-(R+\xi)\sin\omega_0 t)\right.$$

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$$+\begin{pmatrix} \frac{z}{\xi}\cos\omega_0 t \\ \frac{z}{\xi}\sin\omega_0 t \\ 1 \end{pmatrix}\delta(x-(R-\xi)\cos\omega_0 t)\,\delta(y-(R-\xi)\sin\omega_0 t)\Bigg\} \times \eta(a-z)\,\eta(z+a). \quad (2)$$

Here ω_0 is the constant angular velocity of rotation of the center of the ring about the z axis; Ω is the constant angular velocity of rotation of the electric charge distributed over the ring; $\xi = \sqrt{a^2-z^2}$, $\eta(u)$ is the Heaviside step function. Here we are not considering the problem of the stability of the current-carrying, charged ring and we neglect the effect of radiation on the kinematics of the ring.

To calculate the radiation intensity from this source we use the formula

$$W = -\int (\boldsymbol{Ej})\,d\boldsymbol{r}, \quad (3)$$

which expresses the damping force acting on the source due to the electromagnetic field produced by it. For more generality we shall regard the space as filled with a transparent isotropic substance with $\varepsilon = \varepsilon(\omega)$ and $\mu = \mu(\omega)$; $n = n(\omega) = \sqrt{\varepsilon\mu}$ is the refractive index of the medium.

We now resolve the Fourier transform of the field.

$$\boldsymbol{E}(\boldsymbol{\kappa}, \omega) = \frac{1}{(2\pi)^4}\iint \boldsymbol{E}(\boldsymbol{r}, t)\, e^{-i\boldsymbol{\kappa r}+i\omega t}\, d\boldsymbol{r}\, dt$$

along the vectors $\boldsymbol{a}_g$, $\boldsymbol{a}_f$, $\varkappa = (\kappa/\kappa)$ [15]:

$$\boldsymbol{a}_g = \cos\alpha\, \boldsymbol{l}_\mu + e^{i\gamma}\sin\alpha\, \boldsymbol{l}_\varepsilon, \quad \boldsymbol{a}_f = \sin\alpha\, \boldsymbol{l}_\mu - e^{i\gamma}\cos\alpha\, \boldsymbol{l}_\varepsilon, \quad (4)$$

$$\boldsymbol{E}(\boldsymbol{\kappa}, \omega) = E_g\boldsymbol{a}_g + E_f\boldsymbol{a}_f + E_\varkappa \boldsymbol{\varkappa}. \quad (5)$$

Here α and γ are real parameters, specification of which determines the character of the expansion in Eq. (5); $\varkappa = \{\sin\Theta\cos\varphi;\ \sin\Theta\sin\varphi;\ \cos\Theta\}$ is the unit vector along κ; $(\boldsymbol{a}_g\boldsymbol{a}_f^*) = \delta_{gf}$

$$\boldsymbol{l}_\mu = \left\{\frac{-\varkappa_2}{(\varkappa_1^2+\varkappa_2^2)^{1/2}};\ \frac{\varkappa_1}{(\varkappa_1^2+\varkappa_2^2)^{1/2}};\ 0\right\} = \{-\sin\varphi;\ \cos\varphi;\ 0\}, \quad (6)$$

$$\boldsymbol{l}_\varepsilon = \frac{\{-\varkappa_1\varkappa_3;\ -\varkappa_2\varkappa_3;\ (\varkappa_1^2+\varkappa_2^2)\}}{(\varkappa_1^2+\varkappa_2^2)^{1/2}} = \{-\cos\Theta\cos\varphi;\ -\cos\Theta\sin\varphi;\ \sin\Theta\}. \quad (7)$$

We do the calculations in Eq. (3) taking the laws of radiation into account. Finally, we obtain the following expression for the intensity of synchrotron radiation by a charged, current-carrying ring including polarization effects:

$$W = W_g + W_f, \quad (8)$$

$$\begin{pmatrix} W_g \\ W_f \end{pmatrix} = \left(\frac{e}{\pi c}\right)^2 \int_0^\infty\int_0^\pi \sum_{\nu=-\infty}^{+\infty} \frac{\mu\omega^2}{c'}\left\{\begin{pmatrix}\cos^2\alpha \\ \sin^2\alpha\end{pmatrix}A^2 + \begin{pmatrix}\sin^2\alpha \\ \cos^2\alpha\end{pmatrix}B^2 + \begin{pmatrix}+1 \\ -1\end{pmatrix}\sin 2\alpha\sin\gamma\, AB\right\}\delta(\omega-\nu\omega_0)\,d\omega\sin\Theta\, d\Theta, \quad (9)$$

$$A = \omega_0\int_0^a \left[\frac{R+\xi}{\xi}J_\nu'(u(R+\xi)) + \frac{R-\xi}{\xi}J_\nu'(u(R-\xi))\right]\times\cos(vz)\,dz + \Omega\nu\int_0^a \frac{z}{\xi}\left[\frac{1}{u(R+\xi)}J_\nu(u(R+\xi))\right.$$

$$\left. + \frac{1}{u(R-\xi)}J_\nu(u(R-\xi))\right]\sin(vz)\,dz, \quad (10)$$

$$B = \frac{\omega_0\nu\cos\Theta}{u}\int_0^a \frac{1}{\xi}\left[J_\nu(u(R+\xi)) + J_\nu(u(R-\xi))\right]\times\cos(vz)\,dz + \Omega\cos\Theta\int_0^a \frac{z}{\xi}\left[J_\nu'(u(R+\xi))\right.$$

$$\left. + J_\nu'(u(R-\xi))\right]\sin(vz)\,dz + \Omega\sin\Theta\int_0^a \left[J_\nu(u(R+\xi)) - J_\nu(u(R-\xi))\right]\cos(vz)\,dz, \quad (11)$$

$$u = \frac{\omega}{c'}\sin\Theta,\quad v = \frac{\omega}{c'}\cos\Theta,\quad c' = \frac{c}{n(\omega)};$$

where J_ν and J_ν' are the Bessel function and its derivative with respect to the argument.

The expression under the integral in Eqs. (8) and (9) is the spectral and angular (over the angle Θ) distribution of the radiation intensity taking polarization into account.

We can see that the emission at frequency ω in the angular interval (Θ, Θ + dΘ) is completely elliptically polarized. The components of the polarization vector, $P(P_1, P_2, P_3)$ [16], are determined by the formulas of [17].

It follows from Eqs. (9) that the emission spectrum is discrete.

It is of interest to note that in the absence of a charge current around the ring ($\Omega = 0$), the emission at angles $\Theta = 0, \pi$ is circularly polarized, as in the case of synchrotron radiation from a point charge [7]. In the opposite case ($\Omega \neq 0$) the emission at these angles is completely elliptically polarized. At these angles the emission is at the fundamental frequency, $\omega = \omega_0$.

Let $\Omega = 0$ and $a \to 0$. Then, making some elementary transformations in Eqs. (10) and (11), we find the generalized Schott formula for the spectral and angular intensity distribution of radiation from a point charge uniformly gyrating in a circle of radius R about the z axis:

$$\begin{pmatrix} W_g \\ W_f \end{pmatrix} = \left(\frac{e}{c}\right)^2 \int\limits_0^\infty \int\limits_0^\pi \sum_{\nu=-\infty}^{+\infty} \frac{\mu\omega^2}{c'} \left\langle \begin{pmatrix} \cos^2\alpha \\ \sin^2\alpha \end{pmatrix} v^2 J_\nu'^2 + \begin{pmatrix} \sin^2\alpha \\ \cos^2\alpha \end{pmatrix} (c' \operatorname{ctg}\Theta)^2 J_\nu^2 + \begin{pmatrix} +1 \\ -1 \end{pmatrix} \sin 2\alpha \sin\gamma \, vc' \operatorname{ctg}\Theta \times J_\nu J_\nu' \right\rangle \delta(\omega - \nu\omega_0)\, d\omega \sin\Theta\, d\Theta; \quad J_\nu = J_\nu(uR). \tag{12}$$

We now find the first nonzero terms, which contain a^2, in Eqs. (10) and (11). Then the part of Eq. (9) due to the current of electric charge around the ring gives the following results:

$$\begin{pmatrix} W_g' \\ W_f' \end{pmatrix} = \frac{\mu_\varphi^2}{c^2} \int\limits_0^\infty \int\limits_0^\pi \sum_{\nu=-\infty}^{\infty} \frac{\mu\omega^4}{c'} n^2 \sin\Theta\, d\Theta\, \delta(\omega - \nu\omega_0) \times \left\langle \begin{pmatrix} \cos^2\alpha \\ \sin^2\alpha \end{pmatrix} \left(\frac{c' \operatorname{ctg}\Theta}{\omega_0 R}\right)^2 J_\nu^2(uR) + \begin{pmatrix} \sin^2\alpha \\ \cos^2\alpha \end{pmatrix} J_\nu'^2(uR) + \begin{pmatrix} +1 \\ -1 \end{pmatrix} \sin 2\alpha \sin\gamma \frac{c' \operatorname{ctg}\Theta}{\omega_0 R} J_\nu(uR) J_\nu'(uR) \right\rangle, \tag{13}$$

where $\mu_\varphi = (e\Omega a^2/2c)$ is the magnetic moment of the annular current. Summing over polarizations in Eq. (13) we find the result corresponding to [18] and, further, letting $R \to 0$, we obtain the known formula of Landau and Lifshitz.

It is of interest to find the limiting value of Eqs. (9) as $R \to \infty$ for $v = \beta c = \omega_0 R \neq 0$.

We have to use the method of adding Bessel functions described in [19, 20], and utilize the sums numbered 8.530 and the integrals numbered 3.876, 6.672, and 6.677 in [21]. We find that

$$\lim_{\substack{R\to\infty \\ v\neq 0}} \begin{pmatrix} W_g \\ W_f \end{pmatrix} = \begin{cases} \begin{pmatrix} W_g^{\text{V-C}} \\ W_f^{\text{V-C}} \end{pmatrix} & n\beta > 1, \\ 0 & n\beta < 1. \end{cases} \tag{14}$$

$$\begin{pmatrix} W_g^{\text{V-C}} \\ W_f^{\text{V-C}} \end{pmatrix} = \begin{pmatrix} \cos^2\alpha \\ \sin^2\alpha \end{pmatrix} \left\{ \frac{e^2}{c} \int\limits_0^\infty \mu\beta_0 (\beta\beta_0^{-1} - 1) J_0^2 \,\omega\, d\omega + \frac{e^2}{c} \left(\frac{\Omega a}{c}\right)^2 \int\limits_0^\infty \mu\beta^{-1} (\beta\beta_0^{-1} + 1)^{-1} J_1^2 \,\omega\, d\omega \right\}$$

$$+ \begin{pmatrix} \sin^2\alpha \\ \cos^2\alpha \end{pmatrix} \left\{ \frac{e^2}{c} \int\limits_0^\infty \mu\beta_0 (1 - \beta^{-1}\beta_0) J_0^2 \,\omega\, d\omega + \frac{e^2}{c} \left(\frac{\Omega a}{c}\right)^2 \int\limits_0^\infty \mu\beta_0^{-1} (\beta\beta_0^{-1} + 1)^{-1} J_1^2 \,\omega\, d\omega \right\}$$

$$+ \begin{pmatrix} +1 \\ -1 \end{pmatrix} \sin 2\alpha \sin\gamma \frac{e^2}{c} \cdot \frac{\Omega a}{c} \int\limits_0^\infty \mu\beta^{-1} \beta_0 (\beta^2 \beta_0^{-2} - 1)^{1/2} J_0 J_1 \,\omega\, d\omega.$$

Here

$$\beta_0 = (\varepsilon\mu)^{-1/2}, \quad J_0 = J_0\left(\frac{a\omega}{v}\sqrt{\beta^2\beta_0^{-2}-1}\right),$$

$$J_1 = J_1\left(\frac{a\omega}{v}\sqrt{\beta^2\beta_0^{-2}-1}\right).$$

We have obtained the formulas, including polarization effects, for the intensity of Vavilov—Cherenkov radiation from a charged, current-carrying annular cloud moving translationally with a constant velocity v, in a transparent, isotropic, dispersive medium, and uniformly rotating about its axis of symmetry with an angular velocity Ω. Here $v^2 + (\Omega a)^2 < c^2$.

Letting $a \to 0$, we find the formula for the Vavilov—Cherenkov radiation from a charged point particle in the form given in [22, 23]. Summing further over the polarizations, we obtain the results of [24, 25].

LITERATURE CITED

1. G. A. Schott, Electromagnetic Radiation, Cambridge (1912); Ann. Phys., 24, 635 (1907).
2. D. D. Ivanenko and I. Ya. Pomeranchuk, Dokl. Akad. Nauk SSSR, 44, 343 (1944).
3. L. A. Artsimovich and I. Ya. Pomeranchuk, J. Phys. USSR, 9, 267 (1945); ZhÉTF, 16, 379 (1946).
4. D. D. Ivanenko and A. A. Sokolov, Dokl. Akad. Nauk SSSR, 59, 1551 (1948).
5. J. Schwinger, Phys. Rev., 75, 1912 (1949).
6. D. D. Ivanenko and A. A. Sokolov, Classical Field Theory [in Russian], Gostekhizdat (1951).
7. A. A. Sokolov, Introduction to Quantum Electrodynamics [in Russian], Fizmatgiz, Moscow (1958).
8. A. A. Sokolov and I. M. Ternov, Dokl. Akad. Nauk SSSR, 153, 1052 (1963).
9. A. A. Sokolov and I. M. Ternov (editors), Synchrotron Radiation [in Russian], Nauka, Moscow (1966).
10. V. G. Bagrov, Doctoral Dissertation, Physics Faculty, Moscow State University (1969).
11. V. N. Baier, V. N. Katkov, and V. S. Fadin, Radiation from Relativistic Electrons [in Russian], Nauka, Moscow (1974).
12. A. A. Sokolov and I. M. Ternov, The Relativistic Electron [in Russian], Nauka, Moscow (1974).
13. A. A. Kolomenskii and A. N. Lebedev, The Theory of Circular Accelerators [in Russian], Fizmatgiz, Moscow (1962).
14. A. A. Kolomenskii, "Brief communications in physics," FIAN, No. 8, 67 (1972); in: Symposium on Collective Accelerating Methods [in Russian], Dubna (1972), p. 115.
15. A. B. Kukanov, Izv. Vuzov SSSR, Fizika, No. 8, 108 (1969).
16. L. A. Artsimovich (editor), Techniques for Determining the Fundamental Properties of Atomic Nuclei and Elementary Particles [in Russian], Fizmatgiz, Moscow (1965).
17. A. B. Kukanov, G. A. Lavrova, and B. D. Orisa, Vestnik MGU, Ser. Fiz.-Astr., No. 111 (1971).
18. A. B. Kukanov and G. A. Lavrova, Vestnik MGU, Ser. Fiz.-Astr., No. 6, 718 (1972); No. 2, 250 (1974).
19. A. B. Kukanov, Vestnik MGU, Ser. Fiz.-Astr., No. 5, 606 (1971).
20. A. B. Kukanov and A. V. Konstantinovich, Vestnik MGU, Ser. Fiz.-Astr., No. 4 (1973).
21. I. S. Gradshtein and I. M. Ryzhik, Tables of Integrals, Sums, Series, and Products [in Russian], Nauka, Moscow (1971).
22. Č. Muzikař, ZhÉTF, 39, 163 (1960); Č. Muzikař, and V. E. Pafomov, Czechosl. J. Phys., B11, 709 (1961).
23. A. B. Kukanov, Opt. i Spektr., 14, 121 (1963); A. B. Kukanov and Thai Quang, Opt. i Spektr., 15, 124 (1963).
24. I. M. Frank and I. E. Tamm, Dokl. Akad. Nauk SSSR, 14, 107 (1937).
25. D. D. Ivanenko and V. N. Tsytovich, ZhÉTF, 28, 291 (1955).