FUNDAMENTALS OF THE NONHOLONOMICALLY COVARIANT FORMULATION OF THE GENERAL THEORY OF RELATIVITY

M. M. Mirianashvili, V. S. Kiriya, and A. B. Kereselidze

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The physical foundations of the nonholonomic formulation of general relativity are determined, and the role of the Fock–Ivanenko coefficients in setting up and developing the tetrad formalism in general relativity is discussed. The physical and geometrical meaning of the nonholonomic transformations used in general relativity is determined.

Study of the local properties of inhomogeneous spacetime entails the use in general relativity of nonholonomic methods connected with nonholonomic transformations of spacetime intervals and the metric tensor:

$$d\overset{\circ}{x}{}^{\alpha} = a^{\alpha}_{\lambda}\, dx^{\lambda}, \quad dx^{\lambda} = \tilde{a}^{\lambda}_{\beta}\, d\overset{\circ}{x}{}^{\beta}, \tag{1}$$

$$g_{\lambda\mu} = a^{\alpha}_{\lambda} a^{\beta}_{\mu} e_{\alpha\beta}, \tag{2}$$

where $g_{\lambda\mu}$ is the Riemannian metric tensor, and $e_{\alpha\beta}$ is the Minkowski metric tensor, $e_{\alpha\beta} = (1, -1, -1, -1)$ (Greek indices take the values 0, 1, 2, 3; Latin indices, the values 1, 2, 3).

The nonholonomic formulation of general relativity in the form of a tetrad theory of the gravitational field (G field) has its origins in investigations of Einstein, Fock, and Ivanenko. Einstein [1] was the first to use a nonholonomic transformation of the type (1). Later, Fock and Ivanenko [2, 3], to study the interaction of fermions with allowance for the G field, introduced frame transformations under which $g_{\lambda\mu}$ = inv:

$$h'^{(\alpha)}_{\lambda} = \Omega^{\alpha}_{\beta} h^{(\beta)}_{\lambda}, \tag{3}$$

$$g'_{\lambda\mu} = h'^{(\alpha)}_{\lambda} h'^{(\beta)}_{\mu} e_{\alpha\beta} = h^{(\rho)}_{\lambda} h^{(\sigma)}_{\mu} e_{\rho\sigma} = g_{\lambda\mu}, \tag{4}$$

where Ω^{α}_{β} are the Fock–Ivanenko coefficients, and $h^{(\alpha)}_{\lambda}$ are the components of the orthogonal frame (tetrads), which, like (1), relate the Galilean (physical) spacetime intervals $dx^{(\alpha)}$ to the Riemannian coordinates dx^{λ} by means of

$$dx^{(\alpha)} = h^{(\alpha)}_{\lambda}\, dx^{\lambda}, \quad dx^{\lambda} = \overline{h}^{\lambda}_{(\alpha)}\, dx^{(\alpha)}, \tag{5}$$

$$dx'^{(\alpha)} = h'^{(\alpha)}_{\lambda}\, dx^{\lambda} = \Omega^{\alpha}_{\beta}\, dx^{(\beta)}. \tag{6}$$

Because of the relations (4), (5), and (6), the coefficients Ω^{α}_{β} have the properties of the Lorentz coefficients L^{α}_{λ} that determine Lorentz transformations:

$$e'_{\lambda\mu} = L^{\alpha}_{\lambda} L^{\beta}_{\mu} e_{\alpha\beta} = e_{\lambda\mu} \tag{4a}$$

$$d\overset{\circ}{x}{}^{\alpha} = L^{\alpha}_{\beta}\, d\overset{\circ}{x}{}'^{\beta}, \quad d\overset{\circ}{x}{}'^{\beta} = \tilde{L}^{\beta}_{\alpha}\, d\overset{\circ}{x}{}^{\alpha}. \tag{6a}$$

At the same time, the Ω^{α}_{β} differ from the L^{α}_{β} because $\Omega^{\alpha}_{\beta} = \Omega^{\alpha}_{\beta}(x)$, where V is the three-dimensional velocity of the frame of reference $(\overset{\circ}{x}{}')$ relative to $(\overset{\circ}{x})$.

In connection between the analogy between (4) and (6), on the one hand, and (4a) and (6a), on the other, the Fock–Ivanenko transformations (3) play the role of Lorentz transformations in general relativity and they form the basis of the tetrad formulation of general relativity, which describes phenomena by means of the classical concepts of special relativity.

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Currently, the tetrad formulation of general relativity is widely used and has been developed by many Soviet and foreign scientists, (Ivanenko, Rodichev, Fedorov, Levashov, Ivanitskaya, Mitskevich, Møller, Synge, Shmutser, Treder, and others).

In the tetrad formulation it is assumed that the primary and basic characteristic of the G field is not the metric tensor $g_{\lambda\mu}$ but the tetrad coefficients $h_\lambda^{(\alpha)}$. According to Møller, "the gravitational field is not simply a metric field but essentially a tetrad field" [4]. According to the proposal of Ivanenko, it is natural to take "the more primitive tetrad and not the metric components as basis and formulate general relativity accordingly."*

In this connection, the $h_\lambda{}^{(\alpha)}$ are determined from the tetrad equations of the G field, gauge conditions being invoked for unique determination of the $h_\lambda{}^{(\alpha)}$ (Rodichev, Møller, Schwinger). The tetrad equations of the G field and the gauge conditions lead accordingly to the principle of relativity and to the determination of $h_\lambda{}^{(\alpha)}$ in the form of functions of the coordinates, which makes it possible to describe the "fine structure" of the G field, namely: the interaction of electrons, protons, and other fermions with allowance for the G field (Fock–Ivanenko).

In the framework of tetrad theory, considerable importance attaches to the concept of a noninertial frame of reference, which was proposed by Rodichev in his papers. He defined a frame of reference as a collection of tetrads, whose position and orientation geometrically reflect the motion of a test body in a given force field. The tetrad field determines the frame of reference in the presence of a physical basis (for example, sun, stars, gyroscopes, pendulum signals, etc.), but after it has been calibrated [6]. Following Rodichev, Treder defines a frame of reference as follows: "a frame of reference is a field of four-frames" [5]. However, this definition of a frame of reference has a formal, mathematical nature unless it is related in the sense of Rodichev to a physical basis.

A new approach to the nonholonomic formulation of general relativity was sketched and implemented in investigations by a group at Tbilisi (Mirianashvili, Kiriya, Gobedzhishvili, Kereselidze, Gogsadze, Saliya, Vepkhvadze, Lezhava, Gargamadze, Erkomaishvili)†. The investigations of this group on the nonholonomic formulation of general relativity arose in connection with the generalization in [7] by Mirianashvili and Gobedzhishvili of the method proposed by Lenz in an unpublished paper [see Sommerfeld's book Electrodynamics, Moscow (1958), p. 49] in which the Schwarzchild solutions of the equations of the G field were obtained by means of incomplete Lorentz transformations (contraction of lengths and time intervals).

The nonholonomic methods in general relativity were developed further in this group by Kiriya [8], who replaced the incomplete Lorentz transformations by special nonholonomic transformations of the form (1) subject to the condition that in the absence of a G field they go over into the Lorentz transformations of special relativity connecting two moving frames of reference (the Lorentz condition). In later publications of the Tbilisi group, nonholonomic methods were used in conjunction with nonholonomic mathematical analysis to solve a number of problems in general relativity. Of these, we mention [9-15].

The fundamental role in the formalism of nonholonomic transformations is played by the nonholonomic transformations (1) that satisfy the Lorentz condition. The coefficients of these transformations are determined from the ten equations (2), which contain the 16 unknowns a_λ^α. The presence of free a_λ^α enables one to subordinate them to the Lorentz condition as well as invoking gauge conditions. The Lorentz conditions introduce into a_λ^α arbitrary velocities V and Ω of the translational and rotational motion of the tetrad, which leads to the expressions $a_\lambda^\alpha = a_\lambda^\alpha(g_{\mu\nu}, V, \Omega)$. The 16 functions a_λ^α now depend on the 16 quantities $g_{\lambda\mu}$, V, Ω, of which the parameters V and Ω replace the gauge conditions of the tetrad formulation of general relativity.

Because of the Lorentz condition and the presence of velocities in a_λ^α, the nonholonomic transformations (1) play the role of Lorentz transformations in general relativity. Therefore, the nonholonomic transformations (1) determined in the above manner are a generalization of Lorentz transformations to a dependence on the gravitational potentials and the accelerated motion of the frame of reference.

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* See Ivanenko's foreword to Treder's book [5].

† The work of this group on the application of nonholonomic methods in general relativity has been published in the following conference proceedings: GR-II (Tbilisi, 1965), GR5 (Tbilisi, 1968), GR-III (Erevan, 1972) and also in the collections Modern Problems of Gravitation (Tbilisi, 1967), Problems of Gravitation (Tbilisi, 1974), and in other scientific publications.

This nature of the nonholonomic transformations (1) leads to their interpretation as formulas for going over from an inertial frame of reference to a noninertial frame of reference and vice versa (IFR $\rightleftharpoons$ NFR) in the absence or presence of a G field.

The interpretation of the nonholonomic transformations (1) subordinated to the Lorentz condition as formulas for the transition IFR $\rightleftharpoons$ NFR was first introduced in [8]. Then, in [11, 12] a general definition was given of the concept of a noninertial frame of reference in general relativity on the basis of the following assumptions: a) in the absence of a G field Lorentz transformations are valid locally for any accelerated motion of the frame of reference; b) in the presence of a G field the coefficients of the Lorentz transformations L^{α}_{λ} and $\tilde{L}^{\beta}_{\mu}$ must be replaced by the coefficients of the nonholonomic transformations a^{α}_{λ} and $\tilde{a}^{\lambda}_{\beta}$, these being subordinated to the Lorentz condition; c) transformations of the coordinate system $\overset{\circ}{x}{}^{\alpha}=\overset{\circ}{x}{}^{\alpha}(x^{\lambda})$ in the case of a IFR $\rightleftharpoons$ NFR take into account the change in the position and orientation of the frame of reference relative to the basis, and they must therefore have the form $\overset{\circ}{t} = t$, $\overset{\circ}{x}{}^{i} = \overset{\circ}{x}{}^{i}(t, x^{k})$; d) the motion of a distinguished frame of reference (x_N) is determined relative to the main frame of reference ($\overset{\circ}{x}$) related to the basis; e) the law $v < c$ must be observed.

These basic assumptions enable one in the general case to determine the coefficients of the transition IFR $\rightleftharpoons$ NFR explicitly, and it was found that they have the Fock—Ivanenko properties (under the transition IFR $\rightleftharpoons$ NFR, $g_{\lambda\mu}$ = inv [11, 12].

In contrast to the field definition of a noninertial frame of reference in the framework of the classical tetrad formulation of general relativity, its definition on the basis of a)-e) is a nonfield definition. In this case, one has a description of the motion of one distinguished frame of reference (x_N) independently of the external field with an arbitrarily specified velocity relative to the fixed frame of reference ($\overset{\circ}{x}$) associated with the basis. It is assumed that the motion of the frame of reference (x_N) relative to ($\overset{\circ}{x}$) can be due to the effect of a motor or a force field.

The formalism of nonholonomic transformations is based largely on two principles:

1) in an infinitesimally small region of spacetime special relativity holds (Einstein's local principle [1]);

2) nonholonomic transformations of physical variables take into account the local influence of the G field and acceleration on physical processes [11].

The first principle, on the basis of nonholonomic transformations of spacetime intervals and physical variables that satisfy the Lorentz condition, leads to an explicit mathematical connection between general and special relativity through the establishment of formulas for the transition from a locally Galilean frame of reference ($\overset{\circ}{x}$) to a non-Galilean frame of reference (x_G) in the presence of a G field. The second principle determines the local influence of the G field and acceleration on physical processes from the point of view of an external observer in whose frame of reference the G field has been eliminated (by the transition $g_{\lambda\mu} \to e_{\lambda\mu}$). This is also related to the Lorentz condition.

The main feature of the nonholonomic formulation of general relativity in this variant is the nonholonomically covariant formulation of general relativity, which can be implemented on the basis of nonholonomic mathematical analysis and Einstein's local principle. To obtain the nonholonomically covariant formulation, it is not adequate merely to express general relativity in nonholonomic coordinates. For example, the equations of the electromagnetic field were expressed by a number of authors in nonholonomic coordinates using Cartan's exterior differentiation, but the equations of the electromagnetic field generalized in this way do not satisfy Einstein's local principle since they are not connected to the ordinary Maxwell equations by means of a nonholonomic transformation. Apart from the formal expression of general relativity in nonholonomic coordinates, for the nonholonomic covariance of general relativity Einstein's local principle and the Lorentz condition must be satisfied for the a^{α}_{λ} and $\tilde{a}^{\lambda}_{\alpha}$.

Note that the classical tetrad formulation of general relativity, except for the tetrad equations of the G field, does not satisfy the requirement of nonholonomic covariance. In addition, the need for nonholonomic covariance in the tetrad formulation under the nonholonomic transformation (5) is due to the arbitrariness in the choice of the orientation of the tetrads, so that nonholonomic covariance of the tetrad formulation of general relativity has a purely geometrical meaning [6]. In contrast, in the nonholonomically covariant formulation of general relativity the requirement of nonholonomic covariance, in connection with the Lorentz condition, represents an extension of Lorentz covariance of the laws of physics in an inertial frame of reference to all laws in a noninertial frame of reference in the presence of G field. Therefore, the requirement of nonholonomic covariance in the framework of the nonholonomically covariant formulation of general relativity has the meaning of a physical law.

It follows from the above that the Lorentz conditions play a fundamental role in the formalism of the nonholonomically covariant formulation of general relativity. Among all possible nonholonomic transformations, the Lorentz conditions select those in which coordinate effects are eliminated and purely gravitational situations are taken into account. These conditions eliminate the occurrence of geometrical G fields corresponding in the sense of Laue to coordinates and enable one to establish when the nonholonomic transformations describe a change of gauge or a transition to a different frame of reference.

The physical and geometrical meaning of the nonholonomic transformations, in addition to the above principles and conditions, is also determined by the relation between nonholonomic transformations and the torsion and curvature of metric space and the gravitational force.

On the one hand, the torsion and curvature tensors are covariant under nonholonomic transformations [10, 11]:

$$\overset{\circ}{K}{}^{\alpha}_{\beta\gamma} = a^{\alpha}_{\lambda}\tilde{a}^{\mu}_{\beta}\tilde{a}^{\nu}_{\gamma}K^{\lambda}_{\mu\nu}, \quad \overset{\circ}{R}{}^{\alpha}_{\beta\gamma\delta} = a^{\alpha}_{\lambda}\tilde{a}^{\mu}_{\beta}\tilde{a}^{\nu}_{\gamma}\tilde{a}^{\sigma}_{\delta}R^{\lambda}_{\mu\nu\sigma}, \tag{7}$$

where $\overset{\circ}{K}{}^{\alpha}_{\beta\gamma}$ and $\overset{\circ}{R}{}^{\alpha}_{\beta\gamma\delta}$ are referred to the local frame of reference $(\overset{\circ}{x})$ of the tangent space. It follows from (7) that if $(\overset{\circ}{x})$ is a locally Galilean frame of reference and $K^{\lambda}_{\mu\nu} \neq 0$ and $R^{\lambda}_{\mu\nu\sigma} \neq 0$, then $\overset{\circ}{K}{}^{\alpha}_{\beta\gamma} \neq 0$ and $\overset{\circ}{R}{}^{\alpha}_{\beta\gamma\delta} \neq 0$ also. However, this holds when $d\overset{\circ}{s} = ds$.* This means that the nonholonomic transformation (1) for $d\overset{\circ}{s} = ds$ does not carry a metric space with torsion and curvature into a flat space. On the other hand, the gravitational four-force in nonholonomic coordinates has the form [11]

$$\overset{\circ}{F}{}^{\alpha}_{G} = G^{\alpha}_{\beta\gamma}\overset{\circ}{U}{}^{\beta}\overset{\circ}{U}{}^{\gamma}, \tag{8}$$

where $\overset{\circ}{U}$ is the four-velocity in $(\overset{\circ}{x})$, and $G^{\alpha}_{\beta\gamma}$ are the connection coefficients, which transform in accordance with

$$G^{\alpha}_{\beta\gamma} = a^{\alpha}_{\lambda}\tilde{a}^{\mu}_{\beta}\tilde{a}^{\nu}_{\gamma}\Gamma^{\lambda}_{\mu\nu} + a^{\alpha}_{\lambda}\tilde{a}^{\nu}_{\gamma}\tilde{a}^{\lambda}_{\beta,\nu}. \tag{9}$$

It also follows from (8) and (9) that $\overset{\circ}{F}{}^{\alpha}_{G} \neq 0$ in $(\overset{\circ}{x})$ even when $\Gamma^{\lambda}_{\mu\nu} = 0$.

Thus, neither curvature nor the gravitational force can be eliminated by means of a nonholonomic transformation in the locally Galilean frame of reference $(\overset{\circ}{x})$. Therefore, Einstein's local principle cannot be understood as a transition by means of a nonholonomic transformation to a locally Galilean frame of reference $(\overset{\circ}{x})$ in which the laws of special relativity hold because of the elimination of the force G field, curvature, etc. Instead, Einstein's local principle must be understood nontrivially, as the validity of the laws of special relativity in an infinitesimally small region of spacetime when locally Galilean coordinates are introduced in it, in the presence of the G field as an external force field, and the curvature in the infinitesimally small region of space introduced in this way does not affect locally the physical process. This is comprehensible since the transition to an infinitesimally small region does not by itself mean the elimination of the G field and curvature of spacetime, and physical processes are related by force effects.

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* The case $d\overset{\circ}{s} \neq ds$ is investigated in [11], in which it is shown that when $d\overset{\circ}{s} = ds$ nonholonomic transformations go beyond the framework of Euclidean and Riemannian geometry.

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