RENORMALIZATION GROUP IN UNRENORMALIZABLE FIELD THEORY

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The functional equations of a renormalization group are formulated for theories for dimensional coupling constants. The structure of the general solutions of the equation is determined for an invariant charge. Particular attention is paid to models with a negative mass dimensionality of the coupling constants (i.e., to models which are unrenormalizable in ordinary perturbation theory). The correspondence between the general solutions in the UV region with ordinary perturbation theory leads to nonanalyticity in terms of the coupling constant. An additional assumption that the number of invariant charges is finite leads to limitations on the parameters of the Bogolyubov R operation. The possibility of scale invariance at small distances is discussed. As an illustration of these hypotheses, an exactly solvable unrenormalizable nonrelativistic model is analyzed.

1. Renormalization Group in Theories with a Dimensional Coupling Constant

A renormalization group can be formulated as a group of finite multiplicative Dyson transformations for Green's functions supplemented by a "compensating" transformation of the coupling constant (constants) [1]. This approach facilitates an understanding of the circumstance that the existence of a renormalization group is not related to either perturbation theory or the dimensionless nature of the coupling constants and the structure of the UV divergences. Incidentally, this circumstance was familiar long ago. It is reflected in the successful application of the renormalization-group apparatus to the problem of summing the IR singularities in spinor electrodynamics [1, 2] and to the nonrelativistic problem of Coulomb screening in an electron gas [3]. To emphasize this point we prefer a slightly nonstandard derivation of the equations, which corresponds better to their fundamental nature.

For simplicity we restrict the analysis to the example of interacting scalar fields. One of the parameters of this theory, on which the n-particle amplitudes $T_n(p_1, \ldots, p_{n-1})$ depend, is the mass m which arises as the position of the pole of the Green's function:

$$\Delta(p^2) = d(p^2)(p^2 - m^2)^{-1}. \quad (1.1)$$

The amplitudes T_n depend not only on this parameter but also on a parameter related to the normalization of the single-particle state and on parameters representing the strength of the interaction (the interaction constants). The first of these is fixed by specifying the value of the function d at some point $p^2 = \lambda$

$$z_\lambda = d(p^2 = \lambda). \quad (1.2)$$

Here it is clear that the matrix elements (the observable quantities) must not depend on the normalization of the asymptotic states, so that we can write

$$T_n = z_\lambda^{-n/2} T_n(p_i, m, \lambda). \quad (1.3)$$

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In particular,

$$d(p^2) = z_\lambda d(p^2, m^2, \lambda), \quad d(\lambda, m^2, \lambda) = 1. \tag{1.4}$$

As a rule, the interaction constants are chosen to be the magnitudes of some amplitudes for certain values of their invariant arguments. We assume that there is only one such parameter and we define it, e.g., in terms of the value of the matrix element related to the four-point function $T_4 \equiv \Gamma$ at the point

$$p_i^2 = -\frac{1}{3} p_i p_j = \lambda, \quad \text{i.e.,}$$

$$\Gamma(p_i p_j) = z_\lambda^2 \Gamma(p_i p_j, m^2, \lambda, g_\lambda), \tag{1.5}$$

where

$$\Gamma\left(p_i^2 = -\frac{1}{3} p_i p_j = \lambda, m^2, \lambda, g_\lambda\right) = g_\lambda. \tag{1.6}$$

It is quite clear that instead of the determination point λ, the normalization constants z_λ, and the interaction constants g_λ, we could have chosen any other point λ_1 from the region in which d and Γ are real, and we could have chosen the other constants

$$z_{\lambda_1} = d(p^2 = \lambda_1), \quad g_{\lambda_1} = z_{\lambda_1}^2 \Gamma\left(p_i^2 = -\frac{1}{3} p_i p_j = \lambda_1\right).$$

This arbitrariness must not be reflected in the amplitudes; i.e., we must have

$$z_\lambda^{-n/2} T_n(p_i, m^2, \lambda, g_\lambda) = z_{\lambda_1}^{-n/2} T_n(p_i, m^2, \lambda_1, g_{\lambda_1}). \tag{1.7}$$

In particular,

$$z_\lambda^{-2} \Gamma(p_i p_j, m^2, \lambda, g_\lambda) = z_{\lambda_1}^{-2} \Gamma(p_i p_j, m^2, \lambda_1, g_{\lambda_1}), \tag{1.8}$$

$$z_\lambda d(p^2, m^2, \lambda, g_\lambda) = z_{\lambda_1} d(p^2, m^2, \lambda_1, g_{\lambda_1}). \tag{1.9}$$

It is not difficult to see that, with the substitution $\lambda \to \lambda_1$, transformations (1.7)-(1.9) form a group with an invariant charge,

$$\bar{g}(p^2, m^2, \lambda, g_\lambda) = d^2(p^2, m^2, \lambda, g_\lambda)\, \Gamma\left(p_i^2 = -\frac{1}{3} p_i p_j = p^2, m^2, \lambda, g_\lambda\right) \tag{1.10}$$

and the usual functional equation of a renormalization group [1],

$$\bar{g}(p^2, m^2, \lambda, g_\lambda) = \bar{g}(p^2, m^2, \lambda_1, \bar{g}(\lambda_1, m^2, \lambda, g_\lambda)), \tag{1.11}$$

where the interaction constant g_λ is defined by

$$g_\lambda = \bar{g}(\lambda, m^2, \lambda, g_\lambda). \tag{1.12}$$

These arguments graphically demonstrate that the renormalization-group equations are valid for theories with zero and positive mass dimensionality of the coupling constants (e.g., $\varphi^4_{(D)}$ with $D \leq 4$) as well as for theories with a negative mass dimensionality, which are not renormalizable in the ordinary sense (e.g., $\varphi^4_{(D)}$ with $D > 4$). They reflect the independence of the theory from the choice of the point at which the coupling constants are determined and from the normalization of the asymptotic states; consequently, they are fundamental conditions in any field theory.

For simplicity we restrict the analysis to scalar particles and a single interaction parameter. However, the same arguments can be repeated for several fields and for a large number of interaction parameters, if we derive, instead of (1.11), the equations of a multicharge renormalization group.

Our purpose is to analyze the consequences of the equation of a single-charge renormalization group for unrenormalizable models in quantum field theory with coupling constants g having negative mass dimensionalities, $[g] = [m^2]^{-k}$, $k > 0$. We assume that such models exist, i.e., that for them we can define, in a consistent manner, Green's functions which depend on only a finite number of parameters of the coupling-constant type. To further simplify the analysis we assume that for at least some of these models we can use the equations of a single-charge renormalization group.*

One example for which these assumptions hold is the solvable nonrelativistic model with a fixed nucleon, of the type

$$2\pi L_{\text{int}} = g\,\delta(\boldsymbol{x})\,\varphi(\boldsymbol{x}, t)\,\dot{\varphi}(\boldsymbol{x}, t), \tag{1.13}$$

* Such a group apparently always exists as a subgroup.

treated in [4]. The analog of the invariant charge here is the amplitude for meson—nucleon scattering $f(\omega)$, divided by ω; the explicit equation for this amplitude,

$$\bar{g}(\omega, g) = \frac{f(\omega)}{\omega} = \frac{g}{1 - g\omega[c + \sqrt{\mu^2 - \omega^2}]}, \tag{1.14}$$

shows a dependence on a single additional parameter, c. The renormalized interaction constant g in (1.14) is defined by $g = \bar{g}(0, g)$. By shifting the normalization point we can convert (1.14) to a form satisfying (1.11).

Multiplying both sides of Eq. (1.11) by $(p^2)^\kappa$ and transforming to dimensionless variables, we find

$$\bar{\gamma}(x, y, \gamma) = \bar{\gamma}\left(\frac{x}{t}, \frac{y}{t}, \bar{\gamma}(t, y, \gamma)\right), \tag{1.15}$$

where $x = p^2/\lambda$, $y = m^2/\lambda$, $t = \lambda_1/\lambda$, $\gamma = g_\lambda \lambda^k$, and $\bar{\gamma} = (p^2)^k \bar{g}$.

A remarkable property of Eq. (1.15) is its universality: it has the same form for any field theory with a single coupling constant, regardless of its dimensionality. The general solution of Eq. (1.15) in the limit y = 0 (i.e., for p^2, $\lambda \gg m^2$) can be written [5]

$$\bar{\gamma}(x, \gamma) = \Phi(x^\kappa \Phi^{-1}(\gamma)), \tag{1.16}$$

where Φ is an arbitrary function.

2. Renormalization Group and Perturbation Theory

One of the most interesting questions is the correspondence of the renormalization group and perturbation theory for unrenormalizable theories. The basic drawback of perturbation theory is known to be the increase in the number of subtractions and thus the number of undetermined constants as the order of the perturbation theory increases.

For simplicity we treat the case k = 1 (the important results can be generalized to the case k ≠ 1). The perturbation-theory expansion for invariant charge is

$$\gamma(x, \gamma) = \gamma x\{1 + \gamma[Ax\ln x + c_{11}(x-1)] + \gamma^2[Bx^2\ln^2 x + Ex^2\ln x + c_{22}(x-1)^2 + c_{21}(x-1)] + \gamma^3[\dots] + \dots\}. \tag{2.1}$$

We now assume that $\bar{\gamma}$ can be expanded in accordance with

$$\bar{\gamma}(x, \gamma) = \gamma x[1 + \gamma\varphi_1(x) + \gamma^2\varphi_2(x) + \dots], \tag{2.2}$$

where Φ can be expanded in accordance with

$$\Phi(z) = z + z^2\Phi_1 + z^3\Phi_2 + \dots \tag{2.3}$$

Substituting (2.2) and (2.3) into (1.16) we find expressions for φ_i which are polynomials in x and which explicitly do not correspond to perturbation theory. For example, we find

$$\varphi_1 = (x-1)\Phi_1. \tag{2.4}$$

What is going on? Can it be that the hypothesis regarding the possibility of expanding γ is incorrect (cf., e.g., [6])? If we assume for Φ_i some slight nonanalyticity with respect to z, we find, instead of (2.4),

$$\varphi_1(x, \gamma) = x\Phi_1(x\gamma) - \Phi_1(\gamma), \tag{2.5}$$

and we find

$$\varphi_2(x, \gamma) = \varphi_1^2(x, \gamma) + \varphi_1(x, \gamma)\frac{d\Phi_1(x\gamma)}{d\ln x\gamma} + x^2\Phi_2(x\gamma) - \Phi_2(\gamma). \tag{2.6}$$

Comparison of (2.5) and (2.6) with (2.1) shows that

$$\Phi_1(x\gamma) = a_1 \ln\frac{x\gamma}{\beta} \quad \text{or} \quad c_{11} = a_1\ln\gamma/\beta, \tag{2.7}$$

$$\Phi_2 = a_2\ln^2 z + b\ln z + d, \tag{2.8}$$

where a_1 = A, a_2 and b are related to A, B, E, and β. Accordingly, correspondence between the renormalization group and perturbation theory requires the appearance of a nonanalyticity with respect to the coupling constant. For cases with k ≠ 1 this is a nonanalyticity of the type $\gamma^{\lambda/\kappa}$; for values κ of which are integers, on the other hand, a nonanalyticity of the type γ is added.

A second consequence is an interrelation among the undetermined coefficients $c_{\kappa m}$. We see from (2.8), (2.7), and (2.6) that in the third-order theory only one additional arbitrary constant, d, appears, instead of the two in (2.1). In general, using the expansion

$$\bar{\gamma}(x, \gamma) = \gamma + (x-1)\psi_1(\gamma) + (x-1)^2\psi_2(\gamma) + \dots \tag{2.9}$$

and the differential equation of the renormalization group which the invariant charge obeys

$$x\frac{d\bar{\gamma}(x, \gamma)}{dx} = \psi_1(\bar{\gamma}(x, \gamma)), \tag{2.10}$$

we can easily show that, by specifying a function $\psi_1(\gamma)$, we unambiguously determine all the ψ_n (i.e., knowledge of c_{k1} gives us all the c_{ln} with $n > 1$). For example,

$$2\psi_2(\gamma) = \psi_1(\gamma)(\psi_1'(\gamma) - 1). \tag{2.11}$$

We have thus shown that perturbation theory for the coupling constants of negative dimensionality can be reduced to a correspondence with the renormalization-group equations. Furthermore, we have found that all the arbitrariness for the invariant charge can be incorporated in a polynomial of fifth degree in x (in the coefficients $c_{\kappa 1}$) and thus in a finite number of counterterms. This result is an important step toward a construction of a scheme of renormalizations of these theories. To continue in this direction we must study the problem of the divergences of the higher Green's functions.

3. Asymptotic UV Relations

An important feature of the arguments in §2 is the presence in the components $\varphi_n(x, \gamma)$ of arbitrary terms of the type

$$x^n \Phi_n(x\gamma) - \Phi_n(x), \tag{3.1}$$

which can (and in fact do) contain the leading asymptotic terms $\sim (x \ln x\gamma)^n$.

Accordingly, and in contrast with the usual renormalizable models (with dimensionless coupling constants), the principal asymptotic components φ_n are mutually independent. Accordingly, in particular, the Gell—Mann—Low equation in (2.10) turns out to be ineffective for proving the approximating properties of perturbation theory.

Furthermore, it is not difficult to see that in general the renormalization group is of little assistance in the solution of such a problem. Let us assume that we have managed to sum all the leading asymptotic forms of the Feynman diagrams of the type $(x\gamma \ln x\gamma)^n$ and we have managed to find for the invariant charge the expression

$$\bar{\gamma}_{PT}(x, \gamma) = \Psi(x\gamma). \tag{3.2}$$

Then using the general solution (1.14) we find, instead of (3.2),

$$\bar{\gamma}_{RG}(x, \gamma) = \Psi(x\Psi^{-1}(\gamma)). \tag{3.3}$$

Since we have $\Psi^{-1}(\gamma) \simeq \gamma + O(\gamma^2)$ at small γ, Eq. (3.3) is essentially the same as (3.2).

Nevertheless, Eq. (3.2) is of interest in the regions

$$\text{a) } x\gamma \ln x\gamma \lesssim 1, \quad \text{b) } \ln x\gamma \gg 1, \quad \text{c) } x\gamma \lll 1. \tag{3.4}$$

For weak interactions this permits us to take a big step toward the unitary limit. However, we cannot go beyond the unitary limit in the region

$$x\gamma \gg 1\,. \tag{3.5}$$

4. Discussion

This analysis is based on the hypothesized existence of a single-charge renormalization group. However, the results do not all depend on this hypothesis in equal measure. The conclusion regarding the nonanalytic nature of the dependence on the interaction constants is not related to this hypothesis. It results simply from the circumstance that in the UV limit, p^2, $\lambda \gg m^2$, for which the dependence on the masses drops out; the only dimensional parameter which is permitted by the renormalization group for eliminating the dimensionality of the nonanalytic (e.g., logarithmic) dependence on the momentum is the interaction constant. Consequently, this conclusion must remain valid even for a multicharge renormalization group.

With regard to the coupling between the subtraction parameters [of the type in (2.1)], we note that the form of this coupling for a multicharge renormalization group can turn out to be different, since in the derivation we explicitly used the differential equation for a single-charge renormalization group.

However, the very fact of coupling apparently is retained for renormalization groups with any number of charges. Nor is the conclusion in §3 regarding the impossibility of going beyond the unitary limit affected, since the lack of coupling between the leading terms of various orders is due solely to the dimensionality of the coupling constant.

One of the possibilities for reaching region (3.5) is based on the hypothesis of scale invariance at small distances [7], which requires that, in the limit $x \to \infty$, we have

$$\bar{\gamma} \to \text{const}, \; \Gamma \to x^{-\kappa-\varepsilon}, \; \varepsilon \geqslant 0. \tag{4.1}$$

In particular, this behavior follows from the model of (1.13), (1.14).

We would like to emphasize the following circumstances:

A) Simply the existence of a renormalization group for unrenormalizable theories permits us to hope that the hypothesis of scale invariance will not contradict the general principles of quantum field theory (in contrast with the semiclassical case). Furthermore, this group makes it possible to approach the higher Green's functions, since it has been shown [8] that each term in the skeletal expansion for the higher functions turns out to be finite, by virtue of (4.1). The self-consistency of this approach, of course, remains one of the basic problems.

Furthermore, the renormalization group implies that the range of applicability of this hypothesis is

$$p_i p_j \approx p_i^2 \gg m^2, \; g^{-1/\kappa}. \tag{4.2}$$

B) Region (3.7) is a far nonphysical region, and from the practical standpoint the study of scale invariance would be an idle game if there did not exist methods permitting us to determine the consequences of this hypothesis for physical processes. Here we are thinking of the Wilson expansion [9] and the more regular method proposed by Efremov et al. [10]. This latter method leads to several interesting qualitative properties of high-energy processes [10] (a power-law decrease in the elastic and inelastic cross sections with increasing transverse momentum, a modified Regge picture of the scattering in the diffraction region, etc.), which can be checked experimentally.

In this connection it would be interesting to apply the apparatus used above to the unrenormalizable quark theory of the type $L_{\text{int}} = g(\bar{\psi}0\,\psi)(\bar{\psi}0\,\psi)$ (see, e.g., [11]).

In conclusion we wish to reemphasize the general nature of the renormalization group and the validity of its equations not only for theories with a polynomial Lagrangian interaction but also for nonpolynomial, e.g., the chirally invariant, Lagrangians which depend on a single interaction constant. A characteristic property of the renormalization group for such theories is a coupling even between the invariant charges constructed from Green's functions of various orders, as in the Yang–Mills theory.

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