AXIALLY SYMMETRIC ELASTIC WAVES IN A LAMINATED COMPRESSIBLE COMPOSITE MATERIAL WITH INITIAL STRESSES

A. N. Guz', N. A. Sitenok,
and A. P. Zhuk

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The propagation of waves in laminated composite materials which have a periodic structure has been investigated both in the case with no initial stresses [1, 10, 11, 12, and others] and in the case with initial stresses [3, 4, 6, 9]. Investigations taking account of initial stresses have dealt with plane waves propagated along the layers and across the layers. In the present paper we study the problem of the effect of initial stresses on the phase velocities of axially symmetric waves. For this purpose we considered only propagated waves. We used a three-dimensional linearized theory of elasticity for bodies with initial stresses [2]. It should be noted that in the case of laminated bodies without initial stresses such an approach has also been used by other authors [1, 10, 11].

1. Statement of the Problem. We consider a laminated composite material consisting of two isotropic linearly elastic alternating layers whose elastic potentials are arbitrary twice continuously differentiable functions. All quantities associated with each layer will be indicated by superscripts in parentheses. It is assumed that axially symmetric elastic perturbations are propagated in the deformed body.

We shall distinguish three states of the body: the natural unstressed state; the initial deformed state, all of the quantities in which will be indicated with a superscript zero; and the state at a given instant of time, all quantities in which are equal to the sum of the corresponding quantities for the initial state and their perturbations caused by the wave field. With each layer in the natural state we associate a Lagrangian cylindrical coordinate system $(r^{(j)}, \theta^{(j)}, x_3^{(j)})$ and a Cartesian coordinate system $(x_1^{(j)}, x_2^{(j)}, x_3^{(j)})$.

Since the elongations of the layers differ from each other, the coordinate system in each layer will also be different, and therefore in the initial state we introduce a cylindrical coordinate system (r, θ, z) common to the two layers. We shall assume that the axis $0z$ is perpendicular to the layers and the coordinate plane $z = 0$ coincides with the interface plane between the layers.

In the discussion that follows, we shall consider the initial stressed-deformed state of the body, characterized by the parameters

$$\sigma_{11}^{*0(j)} = \sigma_{22}^{*0(j)} \neq 0; \qquad \sigma_{33}^{*0(j)} = 0; \qquad \lambda_1^{(j)} = \lambda_2^{(j)} \neq 0; \qquad \lambda_3^{(j)} \neq 0;$$
$$u_r^{0(j)} = (\lambda_1^{(j)} - 1)\, r^{(j)}; \qquad u_\theta^{0(j)} = 0; \qquad u_3^{0(j)} = (\lambda_3^{(j)} - 1)\, x_3^{(j)} \quad (j = 1,\ 2). \tag{1.1}$$

Then

$$r = \lambda_1^{(j)} r^{(j)}; \qquad \theta = \theta^{(j)}; \qquad z = \lambda_3^{(j)} x_3^{(j)} \quad (j = 1,\ 2). \tag{1.2}$$

Consequently, the thickness of the layers in the natural state, $h^{(j)}$, and in the initial state, $\tilde{h}^{(j)}$, will be related by the formulas

$$\tilde{h}^{(j)} = \lambda_3^{(j)} h^{(j)} \quad (j = 1,\ 2). \tag{1.3}$$

The equations of motion of a compressible body have the form [3]

$$L_{m\alpha}^{(j)} u_\alpha^{(j)} = 0 \quad (i, m, \alpha, \beta = 1, 2, 3;\ j = 1, 2); \tag{1.4}$$
$$L_{m\alpha}^{(j)} = \omega_{im\alpha\beta}^{(j)} \frac{\partial^2}{\partial x_i^{(j)} \partial x_\beta^{(j)}} - \rho^{(j)} \delta_{m\alpha} \frac{\partial^2}{\partial \tau^2}.$$

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In the cylindrical coordinate system, in the case of the axially symmetric problem ($u_\theta^{(j)} = 0$), the general solution of the equations of motion can be represented in terms of one function $\chi^{(j)}$ [2, 5]

$$u_r^{(j)} = -\frac{\partial^2 \chi^{(j)}}{\partial r^{(j)} \partial x_3^{(j)}}; \qquad u_\theta^{(j)} = 0;$$

$$u_3^{(j)} = \frac{\rho^{(j)} C_{lx_1}^{(j)2}}{\lambda_1^{(j)3} \lambda_3^{(j)} (a_{13}^{(j)} + \mu_{13}^{(j)})} \left(\Delta^{(j)} + \frac{C_{S_1 x_3}^{(j)2}}{C_{lx_1}^{(j)2}} \frac{\lambda_1^{(j)2}}{\lambda_3^{(j)2}} \frac{\partial^2}{\partial x_3^{(j)2}} - \frac{\lambda_1^{(j)2}}{C_{lx_1}^{(j)2}} \frac{\partial^2}{\partial \tau^2} \right) \chi^{(j)}; \tag{1.5}$$

$$\Delta^{(j)} = \frac{\partial^2}{\partial r^{(j)2}} + \frac{1}{r^{(j)}} \frac{\partial}{\partial r^{(j)}} \quad (j = 1, 2).$$

The components of the surface load for $x_3^{(j)} = \text{const}$, referred to the dimensions of the body in the natural state, have the form [8]

$$P_r^{*(j)} = \frac{\rho^{(j)} C_{S_1 x_3}^{(j)2}}{\lambda_3^{(j)2}} u_{r,x_3}^{(j)} + \lambda_1^{(j)} \lambda_3^{(j)} \mu_{13}^{(j)} u_{3,r}^{(j)}; \qquad P_\theta^{*(j)} = 0;$$

$$P_z^{*(j)} = \lambda_1^{(j)} \lambda_3^{(j)} a_{13}^{(j)} \left(u_{r,r}^{(j)} + \frac{u_r}{r^{(j)}} \right) + \frac{\rho^{(j)} C_{lx_3}^{(j)2}}{\lambda_3^{(j)2}} u_{3,x_3}^{(j)} \quad (j = 1, 2). \tag{1.6}$$

The functions $\chi^{(j)}$ are determined from the equations

$$\left[\left(\Delta^{(j)} + \frac{C_{S_1 x_3}^{(j)2}}{C_{lx_1}^{(j)2}} \frac{\lambda_1^{(j)2}}{\lambda_3^{(j)2}} \frac{\partial^2}{\partial x_3^{(j)2}} - \frac{\lambda_1^{(j)2}}{C_{lx_1}^{(j)2}} \frac{\partial^2}{\partial \tau^2} \right) \left(\Delta^{(j)} + \frac{C_{lx_3}^{(j)2}}{C_{S_3 x_1}^{(j)2}} \frac{\lambda_1^{(j)2}}{\lambda_3^{(j)2}} \frac{\partial^2}{\partial x_3^{(j)2}} - \frac{\lambda_1^{(j)2}}{C_{S_3 x_1}^{(j)2}} \frac{\partial^2}{\partial \tau^2} \right) - \frac{\lambda_1^{(j)6} \lambda_3^{(j)2} (a_{13}^{(j)} + \mu_{13}^{(j)})^2}{\rho^{(j)2} C_{lx_1}^{(j)2} C_{S_3 x_1}^{(j)2}} \Delta^{(j)} \frac{\partial^2}{\partial x_3^{(j)2}} \right] \chi^{(j)} = 0 \quad (j = 1, 2), \tag{1.7}$$

where, following the notations of [7], $C^{(j)}{}_{lx_i}$ and $C^{(j)}{}_{S_m x_i}$ are the velocities of the expansion waves and the shear waves in the deformed body, polarized in the plane $Ox_m x_i$ and propagated along the axis Ox_i ($i, m = 1, 2, 3$; $m \neq i$).

In all expressions, we pass to the cylindrical coordinate system in the initial state, (r, θ, z), which enables us to express the problem in a convenient form. To do this, taking account of the formulas (1.2), we write the expressions (1.5) for the displacements and the equations (1.7) for finding the functions $\chi^{(j)}$ in the following form:

$$u_r^{(j)} = -\lambda_1^{(j)} \lambda_3^{(j)} \frac{\partial^2 \chi^{(j)}}{\partial r \partial z}; \qquad u_\theta^{(j)} = 0;$$

$$u_z^{(j)} = \frac{\lambda_1^{(j)} \tilde{\rho}^{(j)} C_{lx_1}^{(j)2}}{a_{13}^{(j)} + \mu_{13}^{(j)}} \left(\Delta + \frac{C_{S_1 x_3}^{(j)2}}{C_{lx_1}^{(j)2}} \frac{\partial^2}{\partial z^2} - \frac{1}{C_{lx_1}^{(j)2}} \frac{\partial^2}{\partial \tau^2} \right) \chi^{(j)} \quad (j = 1, 2); \tag{1.8}$$

$$\left[\left(\Delta + \frac{C_{S_1 x_3}^{(j)2}}{C_{lx_1}^{(j)2}} \frac{\partial^2}{\partial z^2} - \frac{1}{C_{lx_1}^{(j)2}} \frac{\partial^2}{\partial \tau^2} \right) \left(\Delta + \frac{C_{lx_3}^{(j)2}}{C_{S_3 x_1}^{(j)2}} \frac{\partial^2}{\partial z^2} - \frac{1}{C_{S_3 x_1}^{(j)2}} \frac{\partial^2}{\partial \tau^2} \right) - \frac{\lambda_3^{(j)2} (a_{13}^{(j)} + \mu_{13}^{(j)})^2}{\tilde{\rho}^{(j)2} C_{lx_1}^{(j)2} C_{S_3 x_1}^{(j)2}} \Delta \frac{\partial^2}{\partial z^2} \right] \chi^{(j)} = 0 \quad (j = 1, 2); \tag{1.9}$$

here $\Delta = \partial^2/\partial r^2 + 1/r\, \partial/\partial r$; $\tilde{\rho}^{(j)} = \rho^{(j)}/\lambda_1^{(j)2} \lambda_3^{(j)}$; $\tilde{\rho}^{(j)}$ is the density of the j-th layer in the initial state.

We introduce the components of the surface load for $z = \text{const}$, referred to the dimensions of the body in the initial state. Then from (1.6), taking account of (1.2), we obtain

$$\tilde{P}_r^{(j)} = \tilde{\rho}^{(j)} C_{S_1 x_3}^{(j)2} u_{r,z}^{(j)} + \lambda_3^{(j)} \mu_{13}^{(j)} u_{z,r}^{(j)}; \qquad \tilde{P}_\theta^{(j)} = 0;$$

$$\tilde{P}_z^{(j)} = \tilde{\rho}^{(j)} C_{lx_3}^{(j)2} u_{z,z}^{(j)} + \lambda_3^{(j)} a_{13}^{(j)} \left(u_{r,r}^{(j)} + \frac{u_r^{(j)}}{r} \right) \quad (j = 1, 2). \tag{1.10}$$

The coordinate system (r, θ, z) is so chosen that the layer with superscript 1 occupies on the axis $0z$ a region $0 \leqslant z \leqslant \tilde{h}^{(1)}$, and the layer with superscript 2 occupies a region $-\tilde{h}^{(2)} \leqslant z \leqslant 0$. On the plane of contact between the two layers the continuity conditions for the displacements and the components of the surface load must be satisfied. Then for z = 0 we find

$$u_r^{(1)}(0) = u_r^{(2)}(0); \quad u_z^{(1)}(0) = u_z^{(2)}(0); \quad \tilde{P}_r^{(1)}(0) = \tilde{P}_r^{(2)}(0); \quad \tilde{P}_z^{(1)}(0) = \tilde{P}_z^{(2)}(0). \tag{1.11}$$

By Floquit's theorem [1], for $z = \tilde{h}^{(1)}$ and $z = -\tilde{h}^{(2)}$ the following periodicity conditions must be satisfied:

$$\begin{aligned} &u_r^{(1)}(\tilde{h}^{(1)}) = u_r^{(2)}(-\tilde{h}^{(2)}); \quad u_z^{(1)}(\tilde{h}^{(1)}) = u_z^{(2)}(-\tilde{h}^{(2)}); \\ &\tilde{P}_r^{(1)}(\tilde{h}^{(1)}) = \tilde{P}_r^{(2)}(-\tilde{h}^{(2)}); \quad \tilde{P}_z^{(1)}(\tilde{h}^{(1)}) = \tilde{P}_z^{(2)}(-\tilde{h}^{(2)}). \end{aligned} \tag{1.12}$$

The relations given above enable us to solve the problem of the propagation of axially symmetric perturbations in a laminated material with two periodically alternating layers.

2. Dispersion Relations. The solution of the equations (1.9) can be represented in the form of an axially symmetric wave which goes off to infinity and has an amplitude which varies over the thickness of the layers [6]

$$\chi^{(j)} = \hat{u}^{(j)}(z) H_0^{(1)}(kr) e^{-i\omega\tau} \quad (j = 1, 2). \tag{2.1}$$

Here $H_0^{(1)}(y)$ is the Hankel function; k is the wave number; ω is the frequency.

For the unknown functions $\hat{u}^{(j)}(z)$ we find from (1.9) that

$$\hat{u}^{(j)}(z) = A_1^{(j)} e^{i\alpha_1^{(j)} z} + A_2^{(j)} e^{-i\alpha_1^{(j)} z} + A_3^{(j)} e^{i\alpha_2^{(j)} z} + A_4^{(j)} e^{-i\alpha_2^{(j)} z} \quad (j = 1, 2), \tag{2.2}$$

where $A_m^{(j)}$ (m = 1, 2, 3, 4; j = 1, 2) are constants of integration; $\alpha_n^{(j)}$ (n, j = 1, 2) are the positive roots of the equations

$$\begin{aligned} &\alpha^{(j)^4} - k^2 \left[\frac{\lambda_3^{(j)^2} (a_{13}^{(j)} + \mu_{13}^{(j)})^2}{\tilde{\rho}^{(j)^2} C_{lx_3}^{(j)^2} C_{S_1x_3}^{(j)^2}} + \frac{V^2 - C_{lx_1}^{(j)^2}}{C_{S_1x_3}^{(j)^2}} + \frac{V^2 - C_{S_3x_1}^{(j)^2}}{C_{lx_3}^{(j)^2}} \right] \alpha^{(j)^2} + \\ &+ k^4 \frac{(V^2 - C_{lx_1}^{(j)^2})(V^2 - C_{S_3x_1}^{(j)^2})}{C_{lx_3}^{(j)} C_{S_1x_3}^{(j)^2}} = 0 \quad (j = 1, 2), \end{aligned} \tag{2.3}$$

and $V = \omega/k$ is the phase velocity of the axially symmetric wave defined in the coordinate system (r, θ, z).

The expressions for finding the displacements of the particles in the layers are found from (1.8), taking account of (2.1) and (2.2):

$$\begin{aligned} u_r^{(j)} &= iH_1^{(1)}(kr) e^{-i\omega\tau} \sum_{n=1}^{2} \gamma_n^{(j)} (A_{2n-1}^{(j)} e^{i\alpha_n^{(j)} z} - A_{2n}^{(j)} e^{-i\alpha_n^{(j)} z}); \\ u_z^{(j)} &= H_0^{(1)}(kr) e^{-i\omega\tau} \sum_{n=1}^{2} (A_{2n-1}^{(j)} e^{i\alpha_n^{(j)} z} + A_{2n}^{(j)} e^{-i\alpha_n^{(j)} z}), \end{aligned} \tag{2.4}$$

where

$$\gamma_n^{(j)} = \frac{k\lambda_3^{(j)} \alpha_n^{(j)} (a_{13}^{(j)} + \mu_{13}^{(j)})}{\tilde{\rho}^{(j)} [k^2 (V^2 - C_{lx_1}^{(j)^2}) - \alpha_n^{(j)^2} C_{S_1x_3}^{(j)^2}]} \quad (n, j = 1, 2). \tag{2.5}$$

Substituting (2.4) into (1.10), we obtain the components of the surface forces for z = const:

$$\begin{aligned} \tilde{P}_r^{(j)} &= -H_1^{(1)}(kr) e^{-i\omega\tau} \sum_{n=1}^{2} D_n^{(j)} (A_{2n-1}^{(j)} e^{i\alpha_n^{(j)} z} + A_{2n}^{(j)} e^{-i\alpha_n^{(j)} z}); \\ \tilde{P}_z^{(j)} &= iH_0^{(1)}(kr) e^{-i\omega\tau} \sum_{n=1}^{2} E_n^{(j)} (A_{2n-1}^{(j)} e^{i\alpha_n^{(j)} z} - A_{2n}^{(j)} e^{-i\alpha_n^{(j)} z}), \end{aligned} \tag{2.6}$$

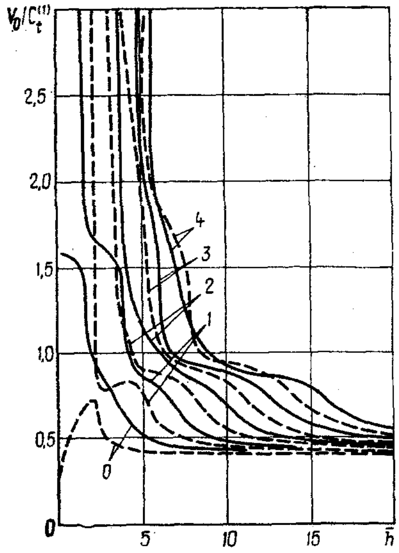

Fig. 1

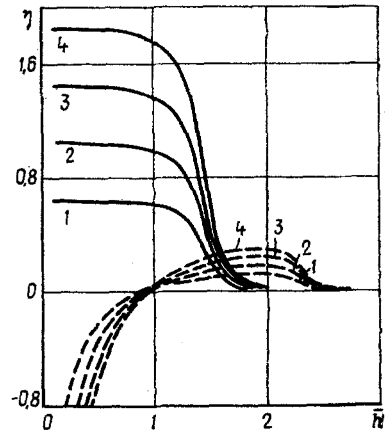

Fig. 2

where

$$D_n^{(j)} = \gamma_n^{(j)}\alpha_n^{(j)}\tilde{\rho}^{(j)}C_{S_1x_3}^{(j)^2} + k\lambda_3^{(j)}\mu_{13}^{(j)};$$
$$E_n^{(j)} = \alpha_n^{(j)}\tilde{\rho}^{(j)}C_{lx_3}^{(j)^2} + k\lambda_3^{(j)}\gamma_n^{(j)}a_{13}^{(j)} \quad (n, j = 1, 2). \tag{2.7}$$

From the continuity conditions (1.11) and the periodicity conditions (1.12), taking account of expressions (2.4) and (2.6), we obtain a system of eight homogeneous algebraic equations which enables us to determine the constants of integration. From the condition for the existence of nontrivial solutions of the system, we can write the dispersion relation corresponding to axially symmetric waves propagated in a laminated compressible medium. Since the dispersion relation is cumbersome, we shall not give it here and shall consider a number of special cases.

To do this, we represent expressions (2.4) and (2.6) in the following form:

$$u_r^{(j)} = iH_1^{(1)}(kr)\,e^{-i\omega\tau}\sum_{n=1}^{2}\gamma_n^{(j)}\left(\bar{A}_{2n-1}^{(j)}e^{Y_n^{(j)}} - \bar{A}_{2n}^{(j)}e^{-Y_n^{(j)}}\right);$$
$$u_z^{(j)} = H_0^{(1)}(kr)\,e^{-i\omega\tau}\sum_{n=1}^{2}\left(\bar{A}_{2n-1}^{(j)}e^{Y_n^{(j)}} + \bar{A}_{2n}^{(j)}e^{-Y_n^{(j)}}\right);$$
$$\tilde{P}_r^{(j)} = -H_1^{(1)}(kr)\,e^{-i\omega\tau}\sum_{n=1}^{2}D_n^{(j)}\left(\bar{A}_{2n-1}^{(j)}e^{Y_n^{(j)}} + \bar{A}_{2n}^{(j)}e^{-Y_n^{(j)}}\right);$$
$$\tilde{P}_z^{(j)} = iH_0^{(1)}(kr)\,e^{-i\omega\tau}\sum_{n=1}^{2}E_n^{(j)}\left(\bar{A}_{2n-1}^{(j)}e^{Y_n^{(j)}} - \bar{A}_{2n}^{(j)}e^{-Y_n^{(j)}}\right), \tag{2.8}$$

where

$$\bar{A}_{2n-1}^{(j)} = A_{2n-1}^{(j)}e^{z-Y_n^{(j)}}; \qquad \bar{A}_{2n}^{(j)} = A_{2n}^{(j)}e^{Y_n^{(j)}-z};$$
$$Y_n^{(j)} = i\alpha_n^{(j)}\left(z + (-1)^j\frac{\tilde{h}^{(j)}}{2}\right) \quad (n, j = 1, 2).$$

Case A. Suppose that in an elastic body there is propagated an axially symmetric wave in which the displacements $u_r^{(j)}$ are odd (antisymmetric) and the displacements $u_z^{(j)}$ are even (symmetric) with respect to the middle surface of the layers. Such a wave is referred to, using the terminology of [1], as transverse in the mean, since the displacements averaged over the period of the structure will be perpendicular to the layers. In accordance with the proposition, in (2.8) we must take $\bar{A}^{(j)}_{2n-1} = \bar{A}^{(j)}_{2n}$ $(n, j = 1, 2)$. In this case the

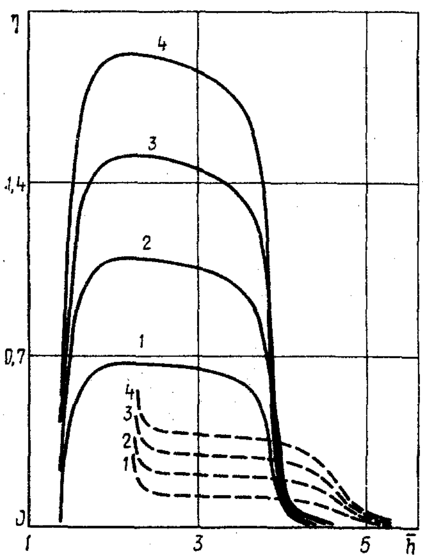

Fig. 3

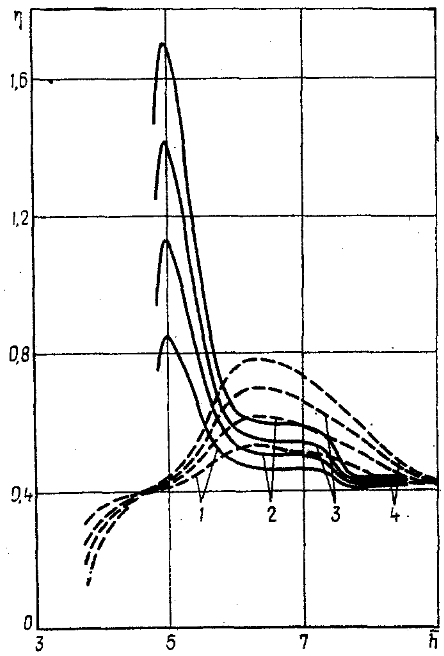

Fig. 4

continuity conditions (1.11) and the periodicity conditions (1.12) coincide, and the system for determining the constants of integration takes the form

$$\sum_{j=1}^{2}\sum_{n=1}^{2}\gamma_n^{(j)}\bar{A}_{2n-1}^{(j)}\,\mathrm{sh}\,(iz_n^{(j)})=0;\qquad \sum_{j=1}^{2}(-1)^{j+1}\sum_{n=1}^{2}\bar{A}_{2n-1}^{(j)}\,\mathrm{ch}\,(iz_n^{(j)})=0;$$
$$\sum_{j=1}^{2}(-1)^{j+1}\sum_{n=1}^{2}D_n^{(j)}\bar{A}_{2n-1}^{(j)}\,\mathrm{ch}\,(iz_n^{(j)})=0;\qquad \sum_{j=1}^{2}\sum_{n=1}^{2}E_n^{(j)}\bar{A}_{2n-1}^{(j)}\,\mathrm{sh}\,(iz_n^{(j)})=0. \tag{2.9}$$

From the condition that system (2.9) has a nontrivial solution, we obtain the dispersion relation

$$\begin{vmatrix} D_2^{(1)}-D_1^{(1)} & -(D_1^{(2)}-D_1^{(1)}) & -(D_2^{(2)}-D_1^{(1)}) \\ \gamma_2^{(1)}T_2^{(1)}-\gamma_1^{(1)}T_1^{(1)} & \gamma_1^{(2)}T_1^{(2)}+\gamma_1^{(1)}T_1^{(1)} & \gamma_2^{(2)}T_2^{(2)}+\gamma_1^{(1)}T_1^{(1)} \\ E_2^{(1)}T_2^{(1)}-E_1^{(1)}T_1^{(1)} & E_1^{(2)}T_1^{(2)}+E_1^{(1)}T_1^{(1)} & E_2^{(2)}T_2^{(2)}+E_1^{(1)}T_1^{(1)} \end{vmatrix}=0, \tag{2.10}$$

where

$$T_n^{(j)}=\mathrm{tg}\,(z_n^{(j)});\qquad z_n^{(j)}=\alpha_n^{(j)}\frac{\tilde{h}^{(j)}}{2}\quad (n,\,j=1,\,2).$$

Case B. Now suppose that in the axially symmetric wave the displacements $u_r^{(j)}$ are even and the displacements $u_z^{(j)}$ are odd with respect to the middle surface of the layers. In this case the wave is called longitudinal in the mean [1], since the displacements averaged over the period of the structure are directed along the layers. In (2.8) we set $\bar{A}^{(j)}{}_{2n-1} = -\bar{A}^{(j)}{}_{2n}$ (n, j = 1, 2); then the continuity and periodicity conditions (1.11) and (1.12) coincide. Consequently the system for determining the constants of integration takes the form

$$\sum_{j=1}^{2}(-1)^{j+1}\sum_{n=1}^{2}\bar{A}_{2n-1}^{(j)}\gamma_n^{(j)}\,\mathrm{ch}\,(iz_n^{(j)})=0;\qquad \sum_{j=1}^{2}\sum_{n=1}^{2}\bar{A}_{2n-1}^{(j)}\,\mathrm{sh}\,(iz_n^{(j)})=0;$$
$$\sum_{j=1}^{2}\sum_{n=1}^{2}\bar{A}_{2n-1}^{(j)}D_n^{(j)}\,\mathrm{sh}\,(iz_n^{(j)})=0;\qquad \sum_{j=1}^{2}(-1)^{j+1}\sum_{n=1}^{2}\bar{A}_{2n-1}^{(j)}E_n^{(j)}\,\mathrm{ch}\,(jz_n^{(j)})=0, \tag{2.11}$$

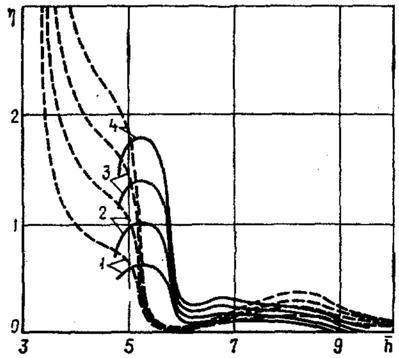

Fig. 5

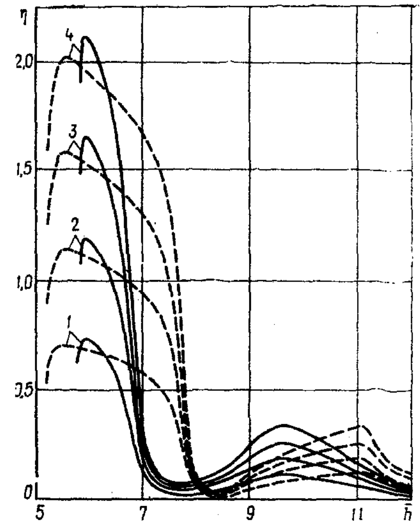

Fig. 6

and the dispersion relation can be written in the form

$$\begin{vmatrix} D_2^{(1)} - D_1^{(1)} & -(D_1^{(2)} - D_1^{(1)}) & -(D_2^{(2)} - D_1^{(1)}) \\ \gamma_2^{(1)} C_2^{(1)} - \gamma_1^{(1)} C_1^{(1)} & \gamma_1^{(2)} C_1^{(2)} + \gamma_1^{(1)} C_1^{(1)} & \gamma_2^{(2)} C_2^{(2)} + \gamma_1^{(1)} C_1^{(1)} \\ E_2^{(1)} C_2^{(1)} - E_1^{(1)} C_1^{(1)} & E_1^{(2)} C_1^{(2)} + E_1^{(1)} C_1^{(1)} & E_2^{(2)} C_2^{(2)} + E_1^{(1)} C_1^{(1)} \end{vmatrix} = 0, \tag{2.12}$$

where

$$C_n^{(j)} = \operatorname{ctg}(z_n^{(j)}) \quad (n, j = 1, 2).$$

Equations (2.10) and (2.12) obtained above coincide with the dispersion equations for plane waves [9], and if there are no initial stresses, they pass into the known equations for plane waves propagated in periodically laminated elastic bodies [1, 11].

3. Example. We use the relations obtained above to investigate the effect of initial stresses on the phase velocity of axially symmetric waves propagated along the layers in an elastic body with two alternating layers. The elastic potential of the materials of the layers is taken in the Murnaghan form

$$\Phi^{0(j)} = \frac{1}{2} \lambda^{(j)} A_1^{(j)2} + \mu^{(j)} A_2^{(j)} + \frac{a^{(j)}}{3} A_1^{(j)3} + b^{(j)} A_1^{(j)} A_2^{(j)} + \frac{c^{(j)}}{3} A_3^{(j)}, \tag{3.1}$$

where $\lambda^{(j)}$, $\mu^{(j)}$, $a^{(j)}$, $b^{(j)}$, $c^{(j)}$ are elastic constants; $A_m^{(j)}$ ($m = 1, 2, 3$; $j = 1, 2$) are algebraic invariants.

The parameters $a_{mn}^{(j)}$, $\mu_{mn}^{(j)}$ and $\lambda_n^{(j)}$ [6], taking account of (3.1), can be calculated without using any more than a linear approximation [8]. The generalized stresses in such a case are equal to the physical stresses. As a result, we obtain

$$\begin{aligned} a_{11}^{(j)} &= \lambda^{(j)} + 2\mu^{(j)} + (\lambda_1^{(j)2} - 1)(2a^{(j)} + 4b^{(j)} + c^{(j)}) + (\lambda_3^{(j)2} - 1)(a^{(j)} + b^{(j)}); \\ a_{33}^{(j)} &= \lambda^{(j)} + 2\mu^{(j)} + 2(\lambda_1^{(j)2} - 1)(a^{(j)} + b^{(j)}) + (\lambda_3^{(j)2} - 1)(a^{(j)} + 3b^{(j)} + c^{(j)}); \\ a_{13}^{(j)} &= \lambda^{(j)} + (\lambda_1^{(j)2} - 1)(2a^{(j)} + b^{(j)}) + (\lambda_3^{(j)2} - 1)(a^{(j)} + b^{(j)}); \\ \mu_{13}^{(j)} &= \mu^{(j)} + (\lambda_1^{(j)2} - 1)\frac{4b^{(j)} + c^{(j)}}{4} + (\lambda_3^{(j)2} - 1)\frac{2b^{(j)} + c^{(j)}}{4}; \\ \lambda_1^{(j)2} - 1 &= \frac{(\lambda^{(j)} + 2\mu^{(j)})\sigma_{11}^{0(j)} - \lambda^{(j)}\sigma_{33}^{0(j)}}{\mu^{(j)}(3\lambda^{(j)} + 2\mu^{(j)})}; \\ \lambda_3^{(j)2} - 1 &= 2\frac{(\lambda^{(j)} + \mu^{(j)})\sigma_{33}^{0(j)} - \lambda^{(j)}\sigma_{11}^{0(j)}}{\mu^{(j)}(3\lambda^{(j)} + 2\mu^{(j)})} \quad (j = 1, 2). \end{aligned} \tag{3.2}$$

We shall confine our attention to the case in which the initial stressed state of the laminated elastic body has the form

$$\sigma_{11}^{0(1)} \neq 0; \quad \sigma_{33}^{0(1)} = \sigma_{11}^{0(2)} = \sigma_{33}^{0(2)} = 0. \tag{3.3}$$

The results of the numerical solutions of the dispersion relations (2.10) and (2.12) for the ratio of layer thicknesses $\xi = h^{(2)}/h^{(1)} = 1$ are shown in the figures. The examples of [11] were used to check that the calculation program for the phase velocities was operating correctly. Figure 1 shows curves characterizing the variation of $V_0/C_t^{(1)}$ as a function of the parameter $\bar{h} = k_t^{(1)}h^{(1)}$. The curves for the relative variation of the phase velocities $\eta = (V_0 - V)/V_0 \cdot 10^3$ of the symmetric (solid curves) and the antisymmetric (dashed curves) axially symmetric waves as functions of the parameter $\bar{h}$ for modes 0-4 are shown in Figs. 2-6, respectively. The quantities V_0 and V are, respectively, the phase velocities in the laminated body without and with initial stresses (3.3). The numbers next to the curves indicate the degree of loading $\psi = \sigma_{11}^{0(1)}/\mu^{(1)}$ (1, 2, 3, 4 correspond to 0.0004, 0.00065, 0.0009, and 0.00115).

As the material of the first layer, we took 09G2S steel, and for the second we took plastic. The elastic constants are given in [5]. It can be seen from the figures that for some ranges of frequencies the calculations should be carried out with higher precision, but these ranges do not play a large role, since there the effect of the initial stresses on the phase velocities of the waves is insignificant. For low frequencies $\bar{h}$ (the long-wave approximation) the calculation of the phase velocities of the waves can be carried out according to the theory of reduced moduli. However, the numerical results show that in this case the moduli depend on the stresses [3].

It should be noted that the authors carried out the calculation of the phase velocities of axially symmetric waves for different ratios of the layer thicknesses and different materials.

Conclusions. From an analysis of the above results for $\xi = 1$ we can conclude the following:

1. The initial stresses have a substantial effect on the phase velocities of the generated waves.

2. There are frequencies at which the relative phase velocity is independent of the initial stresses.

3. Each mode has a frequency range in which the variation of the phase velocity caused by the initial stresses is strongly dependent on the frequency.

4. As the ratio of layer thicknesses varies, there is also a variation both in the critical frequencies and in the nature of the dependence of the phase velocity on the frequency and on the initial stresses.

LITERATURE CITED

1. L. M. Brekhovskikh, Waves in Laminated Media [in Russian], Izd. Akad. Nauk SSSR, Moscow (1957).
2. A. N. Guz', Stability of Elastic Bodies under Finite Deformations [in Russian], Naukova Dumka, Kiev (1973).
3. A. N. Guz', "On the theory of composite materials with initial stresses," in: Mechanics of Deformed Bodies and Structures [in Russian], Nauka, Moscow (1975), pp. 140-148.
4. A. N. Guz', "Linearized theory of the propagation of elastic waves in bodies with initial stresses," Prikl. Mekh., 14, No. 4, 3-32 (1978).
5. A. N. Guz', A. P. Zhuk, and F. G. Makhort, Waves in a Layer with Initial Stresses [in Russian], Naukova Dumka, Kiev (1976).
6. A. N. Guz' and Le Minh Khanh, "Wave propagation in composite laminated materials with large initial deformations," Prikl. Mekh., 12, No. 1, 3-11 (1976).
7. A. N. Guz', F. G. Makhort, and O. I. Gushcha, Introduction to Acousticoelasticity [in Russian], Naukova Dumka, Kiev (1977).
8. A. N. Guz', F. G. Makhort, O. I. Gushcha, and V. K. Lebedev, "On the theory of wave propagation in an elastic isotropic body with initial deformations," Prikl. Mekh., 6, No. 12, 42-49 (1970).
9. Le Minh Khanh, "Wave propagation along layers in laminated compressible materials with initial deformations," Prikl. Mekh., 13, No. 9, 21-26 (1977).

10. S. M. Rytov, "Acoustic properties of finely laminated media," Akust. Zh., 2, No. 1, 71-78 (1956).
11. San, Akhenbach, and German, "Harmonic waves in a laminated medium which are propagated in the direction of lamination," Proc. ASME [Russian translation], Prikl. Mekh., No. 2, 86-88 (1968).
12. N. A. Shul'ga, Fundamentals of the Mechanics of Laminated Media with Periodic Structure [in Russian], Naukova Dumka, Kiev (1981).

INVESTIGATION OF THE STRESSES OCCURRING IN THE NONSTATIONARY PERTURBATION OF AN AXISYMMETRIC CAVITY SURFACE

Yu. N. Podil'chuk and Yu. K. Rubtsov

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Besides the traditional methods [1, 2], among the number of intensively developed methods of solving nonstationary elasticity theory problems are methods of the theory of geometric optics, or ray methods. The ray series method is used [3, 4, 8, 9] to investigate different nonstationary problems of elasticity theory. However, in the case of axisymmetric wave fronts, the calculation of the large quantity of ray series terms needed to determine the time dependence of the stresses being investigated behind a nonstationary wave front evokes serious difficulties. The method of nearby characteristics [14] developed in application to plane nonstationary problems of elasticity theory [12, 13], assists in averting them. The purpose of this paper is a further development of the method of nearby characteristics in application to axisymmetric nonstationary problems of elasticity theory.

1. Let us examine the problem of a nonstationary perturbation of the surface of an axisymmetric convex cavity. To solve this problem, a system of orthogonal curvilinear coordinates must be used which is formed by the fronts being propagated from the wave cavity and the normals to it, i.e., by rays. As is known from the theory of geometric optics [4, 5], such fronts can be described by using the functions $\tau(\vec{x})$ and $\bar{\tau}(\vec{x})$, which satisfy the eiconal equations

$$(\vec{\nabla}\tau)^2 = \frac{1}{\alpha^2}; \qquad (\vec{\nabla}\bar{\tau})^2 = \frac{1}{\beta^2}, \tag{1.1}$$

where the function τ determines the expansion wave front propagation (P-waves) and the function $\bar{\tau}$ determines the shear wave front (S-wave), α and β are the respective propagation velocities of these waves.

To find τ and $\bar{\tau}$ the Cauchy problem for (1.1) must be solved, respectively, by taking into account that at the initial time $t = 0$ the surface of the P- and S-wave fronts determined by the equations $\tau - t = 0$ and $\bar{\tau} - t = 0$ agrees with the surface of the cavity. As is known [6], the solution of this problem reduces to the solution of a system of ordinary differential equations. As a result of solving this system of equations we obtain expressions completely determining the P- and S-wave fronts that are being propagated from the cavity, and the rays, or bicharacteristics, perpendicular to them [8]

$$\vec{s} = \vec{n}\xi + \vec{f}(\theta, \eta); \tag{1.2}$$

$$\tau = \frac{|\vec{s} - \vec{f}(\theta, \eta)|}{\alpha} = \frac{\xi}{\alpha}. \tag{1.3}$$

Here (1.2) determines the system of rays, and (1.3) the system of fronts for the P-waves (the corresponding expressions for the S-waves are written analogously), $\vec{n}$ is the unit normal to the cavity surface, ξ is the length of a ray measured from the cavity surface, $\vec{f}(\theta, \eta)$ is a vector function determining the cavity surface (the equation $\vec{s} = \vec{f}(\theta, \eta)$ is a parametric equation of the cavity), and θ, η are certain coordinates on the cavity surface (Fig. 1).

We introduce a coordinate system (ξ, γ) formed by rays and P-wave fronts and an analogous coordinate system for the S-waves in the x-r plane passing through the axis of symmetry,

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