Mathematical Geology, Vol. 22, No. 8, 1990

Computer-Aided Geometric Design of Geologic Surfaces and Bodies[1]

S. Auerbach[2,3] and H. Schaeben[2,4]

Bivariate quadratic simplicial B-splines are employed to obtain a C^1-smooth surface from scattered positional or directional geological data given over a two-dimensional domain. Vertices are generated according to the areal distribution of data sites, and polylines are defined along real geological features. The vertices and the polylines provide a constrained Delaunay triangulation of the domain. Note that the vertices do not generally coincide with the data sites. Six linearly independent simplex B-splines are associated with each triangle. Their defining knots and finite supports are automatically deduced from the vertices. Specific knot configurations result in discontinuities of the surface or its directional derivatives. Coefficients of a simplex spline representation are visualized as geometric points controlling the shape of the surface. This approach calls for geologic modeling and interaction of the geologist up front to define vertices and polylines, and to move control points initially given by an algorithm. Thus, simplex splines associated with irregular triangles seem to be well-suited to approximate and allow further geometrically modeling of geologic surfaces, including discontinuities, from scattered data. Applications to mathematical test as well as to real geological data are given as examples.

KEY WORDS: scattered data, approximation, simplicial B-splines, constrained triangulation, interactive geometric modeling.

> Give me four parameters and I'll draw you an elephant; give me five, and I'll make its tail wag.
>
> Enrico Fermi

INTRODUCTION

There is a mathematical problem that reads: given ν points in $3d$ Euclidian space, find a curve or surface that interpolates or approximates the curve or surface the data is assumed to be sampled of and that satisfies some constraints usually concerning the smoothness of the fit. This problem may be solved math-

---

[1]Manuscript received 16 October 1989; accepted 10 January 1990.
[2]Department of Geology, University of Bonn, F.R.G.
[3]Present address: BASYS Software GmbH, Peterstraße 2-4, 5100 Aachen, F.R.G.
[4]Present address: Laboratory of Metallurgy of Polycrystalline Materials, University of Metz, 57054 Metz, France, and BASYS Software GmbH, Peterstraße 2-4, 5100 Aachen, F.R.G.

0882-8121/90/1100-0957$06.00/1

ematically in one way or another according to its specific formulation provided that the data is sufficient. And there is a problem in mapping geology: To design a model of a family of surfaces which are not independent of each other but reflect the geologic processes that originated them. This problem does not have an immediate mathematical counterpart.

When conventionally designed by hand, geological maps and cross sections provide static two-dimensional images of the geologist's spatial model—his/her concepts of where rock bodies are, how they are arranged, what they consist of, and how they originated and developed through time. The model is an interpretation which should be consistent with available data and with expectations based on background knowledge of the environment which formed and deformed the rock bodies (Loudon, 1986).

In the past, much effort has been spent on writing software packages for graphic data presentation (e.g., contour mapping packages) which the geologist can interpret to prepare his/her map. However, these presentations (regardless of the sophisticated algorithms used) never match the geologist's background knowledge completely. Even worse, these presentation graphics do not provide a truly three-dimensional model nor do they provide a helpful means for the actual designing process. But this is exactly what is needed: A digital representation of the spatial model that enables the geologist to express his/her interpretations and codify his/her background knowledge within the computer model, most favorably interactively using three-dimensional graphic display and efficient means to locally control and modify the geometry of the model.

As early as 1984, Rüber and Siehl (1984) suggested the application of CAD-like methods in geology with special emphasis on the computer-aided geometric design (CAGD) of digital geological maps (Braun et al., 1983). However, CAGD methods available at that time were not suited to geological requirements for basic reasons given by Gold (1980) and Tipper (1979). After an introduction to the geometric aspects of geology by the "Cloos School" at Bonn, our contribution to the project of "computer-aided design of geological maps" within the priority program "digital geoscientific maps" was to develop and adapt the mathematical models and numerical methods appropriate for geoscientific purposes.

Thus, methods of CAGD have been introduced into geology (Schaeben, 1988; Schaeben and Auerbach, 1989a,b). Concerning the geometry, this model is internally consistent due to the truly three-dimensional designing process. Once the geological spatial model has been established, geological maps or cross sections can be easily generated or updated by combining it with the corresponding digital terrain model.

What shall be modeled is a geologic solid consisting of distinct units, to be distinguished by geological parameters. Thus, the geologic solid is essentially structured by surfaces that separate these distinct geologic units (e.g., the joined surface of two distinct layers in sedimentary rocks). Since a direct approach of modeling the geologic solid or equivalently the family of surfaces as a whole, although desirable (e.g., when interior points of a layer shall be endowed with material properties, say, hydraulic conductivity, for modeling groundwater flow or solute transport) appears to be too complex for the time being, it seems reasonable to model its internal surfaces successively. Due to the ways of sampling geological information as compared to the task at stake, the purpose of modeling a geologic surface gives rise to a rather pathological mathematical problem which some consider a nonmathematical problem all together.

The problem shall become evident if we think of the configuration and the content of the available numerical (i.e., quantitative or quantifiable) data vs. likely features of geologic surfaces. The sparseness and erratic regional distribution of data by comparison with the features to be observed amounts to a rather technical problem; the geological background information which includes:

- previous knowledge (e.g., about some structural features of different geologic surfaces—faults, fractures as opposed to folds);
- basic knowledge (e.g., concerning the major tectonic elements);
- basic ideas (e.g., concerning the chronological order of geologic processes that contributed to the situation to be observed today);
- fundamental ideas concerning geodynamical processes in general (i.e., paradigms like plate tectonics);

and which is all at the geologist's hand when designing a map but cannot be quantified immediately if at all poses the essential problem of mathematically modeling geologic surfaces: How to deal with soft information. And common features of geologic surfaces as multiplicity requiring an appropriate parameterization, discontinuities (usually of known location), and constraints on the set of surfaces constituting the geologic solid do not relax the problem but indeed show the necessity to use the soft information in any semi-automatic modeling procedure.

Using advanced methods of interpolation/approximation (cf. Gmelig Meyling, 1986), and extending it for purposes of interactive geometric design (Auerbach et al., 1990), we summarize the procedure proposed here as follows:

1. Find the best initial representation of an individual geologic surface, even though we are usually not able to provide all the geological knowledge as input data for the specific algorithm used. This representation will fail to completely meet the geologist's idea.
2. Carefully do some further design work on the initial surface approximation that will preserve the initial geometry where it is required (i.e., use interactive means for local changes where needed to meet all of the geologist's objectives).

3. Apply this two-stage designing successively to all geologic surfaces involved in the spatial model of the geologic solid.

It is especially the second stage where CAGD methods are utilized (i.e., where the control points are manipulated by the expert geologist until the surface meets all his/her ideas). The final set of control points may be stored as a representation of the accepted surface model. Now we may proceed to the next surface in the solid. If constraints on neighboring surfaces are fairly simple (e.g., almost parallel—constant thickness of layer), or a simple trend in the thickness that can be reasonably expressed with a simple bivariate function, then this function may be applied to the first set of control points to yield an initial guess of the second surface immediately. Otherwise, this two-stage procedure should be applied successively to all surfaces, taking care that none of the constraints of geological consistency are violated. In some unfortunate instances this may result in starting all over again with the first surface to achieve a consistent model. Thus, the process described here is one of interactively learning and increasing understanding. It will still require a lot of interaction by the expert geologist. But we did not explicitly aim at automatization in geological cartography but at interactive computer-aided geologic design, the acronym (CAGD) of which coincides with computer-aided geometric design, which should not really bother anybody because that is the genuine character of our approach, and in particular of the second stage.

There are three essentials to our approach. The first is a constrained triangulation procedure which generates triangles as uniform as possible given the vertices and some predefined edges reflecting the user's a priori knowledge of the neighborhood relation within the data. Furthermore, triangulation allows adjusting to the areal distribution of the locations of the data in the domain. To avoid any possible misunderstanding right from the beginning we would like to emphasize here that the vertices of the triangles generally do not coincide with the locations where the data are given. This fact constitutes one of the basic differences to other schemes of interpolation/approximation based on triangulation which take the locations of the data for vertices of the triangles, and should be kept in mind throughout.

The second essential is the use of basis functions for approximation and modeling with finite support, or more specifically simplicial splines. In fact, the vertices of the triangles initially define the knots of the splines. Here we would like to recall that with splines the knots play the essential role much as the polynomial degree does with polynomials. Splines may be thought of as piecewise polynomials, pieced together obeying given constraints concerning the smoothness in the joints the locations of which are defined by the knots. The distinct advantage of simplicial splines associated with irregular triangles over more conventional tensor-product B-splines implying a grid partition of the domain is their flexibility in approximation (e.g., the use of varying knot density and local control over the smoothness properties).

The third essential is interactive geometric as opposed to numeric control of the computed surface. As with all CAGD applications of Bernstein–Bezier polynomials or B-splines, the fundamental idea is to visualize the coefficients of their linear combination geometrically as control points and exploiting their favorable features. These control points are not in the surface, but the spline-surface mimics the overall behavior of the two-dimensional polygon, which consists of the control points. More specifically, these control points allow one to locally manipulate the surface that was initially determined to approximate the data. Moving a single control point in any direction causes a local change of the surface to smoothly follow this movement.

Thus, this formalism may provide the means needed to design geologic maps semi-automatically, because the expert geologist has an easy-to-use geometric approach to interactively insert his/her additional geological knowledge. Normally, this was hard to quantify in the computer-aided designing process, which now becomes an interactive, progressive understanding of the geologic situation to be studied.

Introducing semi-automatic methods into geological mapping should be met by necessary changes in collecting geologic data and in training mapping geologists because these methods require:

- reasonable sampling, storing, and retrieval of hard and soft data as long as they are geologically significant in the particular situation;
- new technical skills of the mapping geologist, (e.g., working in dialogue with a graphics workstation).

Summarizing, and most important of all, geologic knowledge must be used to guide the mathematical model; it may require a better mathematical education of geologists to fully appreciate this aspect.

MULTIVARIATE SIMPLEX SPLINES

The theory of multivariate polynomial, simplicial B-splines utilized here has essentially been developed by Dahmen and Micchelli (1982a,b, 1983a,b), and Höllig (1982, 1986).

For any spatial dimension s and polynomial degree k, let $n = s + k$. Let $K = \{\vec{x}^{\,0}, \ldots, \vec{x}^{\,n}\}$ be a set of points in $\mathbb{R}^s$, such that the convex hull $[\vec{x}^{\,0}, \ldots, \vec{x}^{\,n}]$ has dimension s. The standard n-simplex S^n is given by

$$S^n = \left\{ (t_0, \ldots, t_n) \,\middle|\, \sum_{i=0}^{n} t_i = 1, \quad t_i \geq 0, \quad i = 0, \ldots, n \right\} \tag{1}$$

Using the Hermite–Gennochi formula for divided differences the multivariate B-spline $M(\vec{x} \mid \vec{x}^{\,0}, \ldots, \vec{x}^{\,n})$, or briefly $M(\vec{x} \mid K)$, is implicitly defined by the identity

$$\int_{\mathbb{R}^s} f(\vec{x}) M(\vec{x} \mid \vec{x}^{\,0}, \ldots, \vec{x}^{\,n})\, dx_1 \ldots dx_s = n!$$

$$\cdot \int_{S^n} f(t_0 \vec{x}^{\,0} + \ldots + t_n \vec{x}^{\,n})\, dt_1 \ldots dt_n \tag{2}$$

which must hold for all $f \in C(\mathbb{R}^s)$, (Micchelli, 1979). There is a geometric interpretation of the s-variate B-spline $M(\vec{x} \mid K)$ with a set K of knots of cardinality $n + 1$. Let $\sigma = [\vec{v}^{\,0}, \ldots, \vec{v}^{\,n}]$ be any n-simplex such that

$$\mathrm{vol}_n(\sigma) > 0 \tag{3}$$

$$\vec{v}^{\,i}\big|_{\mathbb{R}^s} = \vec{x}^{\,i}, \qquad i = 0, \ldots, n \tag{4}$$

Then Schoenberg's well-known volume formula is recovered

$$M(\vec{x} \mid K) = \mathrm{vol}_k(\{v \in \sigma : \vec{v}\,|_{\mathbb{R}^s} = \vec{x}\})/\mathrm{vol}_n(\sigma), \qquad \vec{x} \in \mathbb{R}^s \tag{5}$$

which holds independently of σ. Hence, the B-spline $M(\vec{x} \mid K)$ is nonnegative, and its support is the convex hull $[K]$. The piecewise polynomial feature of $M(\vec{x} \mid \vec{x}^{\,0}, \ldots, \vec{x}^{\,n})$ is governed by the locations of the knots $\vec{x}^{\,i}$. Choosing an appropriate collection of knot configurations the corresponding set of B-splines forms a basis of a space of splines defined on $\mathbb{R}^s$ given $[K]$, (Dahmen and Micchelli, 1982b; Höllig, 1982).

As in the case of univariate B-splines there are recurrence relations to facilitate the numerical evaluation of multivariate B-splines; they read for (lower-order) B-splines if $n > s$,

$$M(\vec{x} \mid \vec{x}^{\,0}, \ldots, \vec{x}^{\,n}) = \frac{n}{n-s} \sum_{i=0}^{n} \lambda_i M(\vec{x} \mid \vec{x}^{\,0}, \ldots, \vec{x}^{\,i-1},$$

$$\vec{x}^{\,i+1}, \ldots, \vec{x}^{\,n}) \tag{6}$$

where $\Sigma_{i=0}^{n} \lambda_i = 1$ and $\Sigma_{i=0}^{n} \lambda_i \vec{x}^{\,i} = \vec{x}$, or if $n = s$

$$M(\vec{x} \mid \vec{x}^{\,0}, \ldots, \vec{x}^{\,s}) = \begin{cases} 1/\mathrm{vol}_s([\vec{x}^{\,0}, \ldots, \vec{x}^{\,s}]) \\ \qquad \text{if } \vec{x} \in \mathrm{int}\,([\vec{x}^{\,0}, \ldots, \vec{x}^{\,s}]) \\ 0 \quad \text{if } \vec{x} \notin [\vec{x}^{\,0}, \ldots, \vec{x}^{\,s}] \end{cases} \tag{7}$$

and for directional derivatives if $n > s$

$$D_{\vec{z}} M(\vec{x} \mid \vec{x}^{\,0}, \ldots, \vec{x}^{\,n}) = n \sum_{i=0}^{n} \mu_i M(\vec{x} \mid \vec{x}^{\,0}, \ldots, \vec{x}^{\,i-1}, \vec{x}^{\,i+1}, \ldots, \vec{x}^{\,n}) \qquad (8)$$

where $\Sigma_{i=0}^{n} \mu_i = 0$, $\Sigma_{i=0}^{n} \mu_i \vec{x}^{\,i} = \vec{z}$, and $\vec{z} \in \mathbb{R}^s$.

Bivariate Quadratic Simplex B-Splines Associated with Irregular Triangles

In order to obtain a C^1-smooth surface we employ piecewise quadratic simplicial B-splines in two variables over $\Omega \subset \mathbb{R}^2$.

By virtue of Schoenberg's volume–projection formula, each bivariate quadratic simplicial B-spline may be thought of as being defined by the vertices of a 4-simplex projected onto $\mathbb{R}^2$. Now suppose $\mathcal{P}$ is a standard simplex in $\mathbb{R}^2$, and let $\mathfrak{J}$ be a triangulation of the cylinder set $\Omega \times \mathcal{P} \subset \mathbb{R}^4$. Then $\mathfrak{J}$ is a collection of 4-simplices, each of which defines a bivariate quadratic spline by its vertices projected onto $\mathbb{R}^2$, thus a set of knot configurations $\mathcal{C}(\mathfrak{J})$ can be associated with $\mathfrak{J}$. Hence the triangulation $\mathfrak{J}$ of $\Omega \times \mathcal{P} \subset \mathbb{R}^4$ gives rise to the spline space $\mathcal{S}(\mathfrak{J}) = \operatorname{span}\{M(x|K) : K \in \mathcal{C}(\mathfrak{J})\}$. Choosing a canonical triangulation $\mathfrak{J}^*$ of $\Omega \times \mathcal{P}$, the set $\{M(x|K) : K \in \mathcal{C}(\mathfrak{J}^*)\}$ provides a basis of $\mathcal{S}(\mathfrak{J}^*)$ (cf. Dahmen and Micchelli, 1982a).

Reversing the view of projection from $\Omega \times \mathcal{P} \subset \mathbb{R}^4$ onto $\mathbb{R}^2$, we consider now the domain $\Omega \subset \mathbb{R}^2$ and any suitable triangulation Δ of Ω. Suppose Δ consists of T triangles with V vertices. We denote the vertices of Δ by $\vec{x}^{\,i}$, $i = 1, \ldots, V$ and the triangles by $[\vec{x}^{\,i}, \vec{x}^{\,j}, \vec{x}^{\,k}]$.

With every vertex $\vec{x}^{\,i,0} = \vec{x}^{\,i}$ we associate two additional points $\vec{x}^{\,i,1}$, $\vec{x}^{\,i,2}$ located near by. These additional points have to be defined to satisfy some reasonable constraints, see below. They may be thought of as being pulled apart from $\vec{x}^{\,i,0}$. The vertices of the triangles and their two associates will become the knots defining the bivariate quadratic B-spline.

With the combinatorial recipe by Dahmen and Micchelli (1982a) we choose six configurations of five knots each for each triangle $t \in \Delta$ corresponding to the canonical triangulation $\mathfrak{J}^*$ of $t \times \mathcal{P}$, and thus obtain six linearly independent bivariate quadratic B-splines for every triangle $t \in \Delta$. Now let us study an arbitrary individual triangle $t = [\vec{x}^{\,i}, \vec{x}^{\,j}, \vec{x}^{\,k}]$, $i < j < k$, see Fig. 1. The six B-splines belonging to this triangle are, respectively: $M(\vec{x} \mid K_{t,m})$, $m = 1, \ldots, 6$, $\vec{x} = (x_1, x_2) \in \mathbb{R}^2$, with

$$K_{t,1} = \{\vec{x}^{\,i,0}, \vec{x}^{\,j,0}, \vec{x}^{\,k,0}, \vec{x}^{\,k,1}, \vec{x}^{\,k,2}\}$$

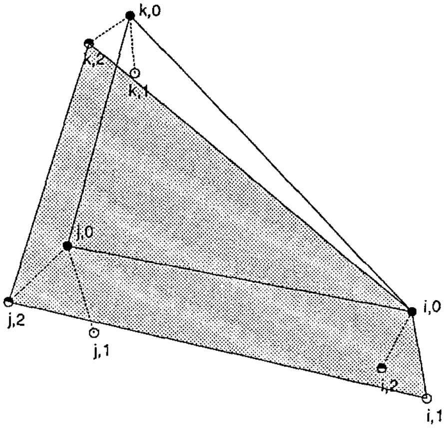


Fig. 1. Triangle $[\vec{x}^{\,i}, \vec{x}^{\,j}, \vec{x}^{\,k}] = t \in \Delta$, with vertices $\vec{x}^{\,i}$, $\vec{x}^{\,j}$, $\vec{x}^{\,k}$ labeled such that $i < j < k$, and associated points $\vec{x}^{\,i,1}$, $\vec{x}^{\,j,1}$, $\vec{x}^{\,k,1}$, $\vec{x}^{\,i,2}$, $\vec{x}^{\,k,2}$ generating knot sets $K_{t,m}$, $m = 1, \ldots, 6$. The convex hull $K_{t,6}$ is shown shaded. Displaying the vertices and associated points, a shaded circle is for index 0, open circle for index 1, and half shaded for index two.

$$
\begin{aligned}
K_{t,2} &= \{\vec{x}^{\,i,0}, \vec{x}^{\,j,0}, \vec{x}^{\,j,1}, \vec{x}^{\,k,1}, \vec{x}^{\,k,2}\} \\
K_{t,3} &= \{\vec{x}^{\,i,0}, \vec{x}^{\,j,0}, \vec{x}^{\,j,1}, \vec{x}^{\,j,2}, \vec{x}^{\,k,2}\} \\
K_{t,4} &= \{\vec{x}^{\,i,0}, \vec{x}^{\,i,1}, \vec{x}^{\,j,1}, \vec{x}^{\,k,1}, \vec{x}^{\,k,2}\} \\
K_{t,5} &= \{\vec{x}^{\,i,0}, \vec{x}^{\,i,1}, \vec{x}^{\,j,1}, \vec{x}^{\,j,2}, \vec{x}^{\,k,2}\} \\
K_{t,6} &= \{\vec{x}^{\,i,0}, \vec{x}^{\,i,1}, \vec{x}^{\,i,2}, \vec{x}^{\,j,2}, \vec{x}^{\,k,2}\}
\end{aligned}
\tag{9}
$$

These six knot configurations (i.e., their indices) can be represented as nondescending paths p_r, $r = 1, \ldots, 6$, along grid lines from $(0, 0)$ to $(2, 2)$ in the lattice of (local) knot numbers $i \in \{0, 1, 2\}$ and their second superscripts ("face") $q \in \{0, 1, 2\}$, as depicted in Fig. 2 (cf. Dahmen and Micchelli, 1982a; Höllig, 1982).

Summarizing the properties of B-splines, $M(\vec{x} \mid K_{t,m})$ is:

1. supported on the convex hull of its defining five knots;
2. nonnegative on its support;
3. piecewise quadratic with respect to the "cut-regions" defined as sets in the support set $[K]$ which are bounded but not intersected by any 1-simplex spanned by 2 points from $\{\vec{x}^{\,0}, \ldots, \vec{x}^{\,4}\}$, i.e., by the line segments joining the elements of K (see Fig. 3),
4. C^1-smooth provided that the five knots are in general position (i.e., every three knots span a nondegenerate triangle).

The entire triangular mesh Δ produces $6T$ B-splines, which are linearly independent and span all quadratics over Ω; for the proof the reader is referred to Dahmen and Micchelli (1982b).

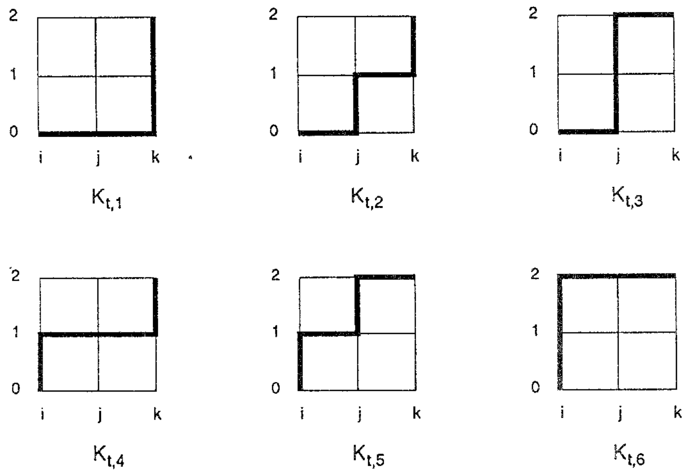


Fig. 2. Nondescending paths p_r along grid lines from $(0, 0)$ to $(2, 2)$ in the lattice of labels i, j, k of knots and labels $q \in \{0, 1, 2\}$ of faces describing the indices of the knot sets $K_{t,m}$, $m = 1, \ldots, 6$.

We can now define a surface over Ω by the bivariate function

$$s(\vec{x}) = \sum_{t=1}^{T} \sum_{m=1}^{6} c_{t,m} M(\vec{x} \mid K_{t,m}) \tag{10}$$

which is a linear combination of the splines $M(\vec{x} \mid K_{t,m})$ with coefficients $c_{t,m}$.

In the following we summarize what appears to us as a most suitable way (i.e., geometric, instructive, and visually appealing) to locally control and manipulate the shape of the surface given by Eq. (10).

To this end, we must first assure that the basis functions $M(\vec{x} \mid K_{t,m})$ are normalized such that they form a partition of unity for all $\vec{x} \in \Omega$, i.e.,

$$\sum_{t=1}^{T} \sum_{m=1}^{6} B(\vec{x} \mid K_{t,m}) = 1 \tag{11}$$

for all $\vec{x} \in \Omega$, with

$$B(\vec{x} \mid K_{t,m}) = d_{t,m} M(\vec{x} \mid K_{t,m}) \tag{12}$$

There is a simple numerical way to compute the coefficients $d_{t,m}$ by solving an appropriate linear system of the form $M\vec{d} = \vec{1}$ for each individual triangle t consecutively (Auerbach et al., 1990).

More specifically, in Auerbach et al. (1990) it has been shown how to choose six different points $\vec{y}^{\,t,k}$, $k = 1, \ldots, 6$ in B_t which do not lie on the same conic section, and where B_t denotes the subset of triangle t where only

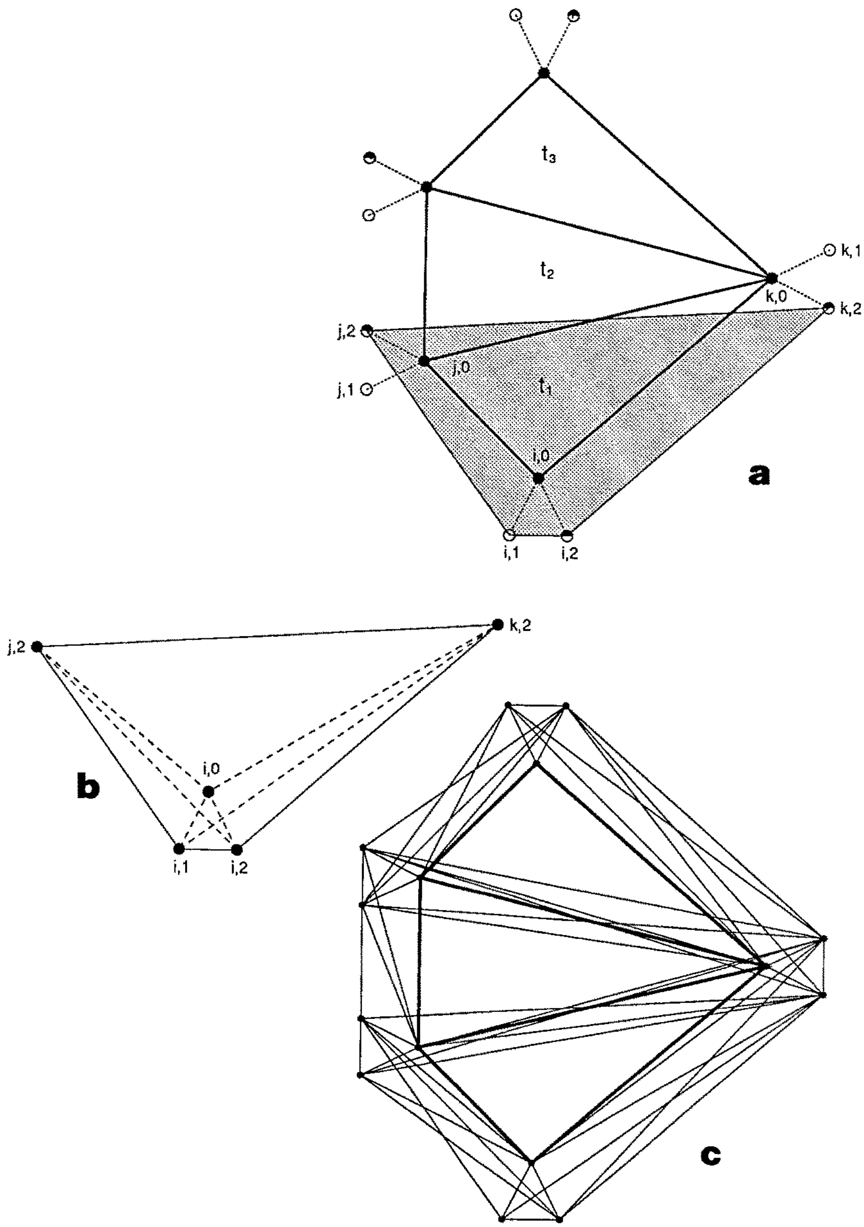


Fig. 3. (a) Triangulated vertices $\vec{x}^{\,i}$ and their two associated points $\vec{x}^{\,i,q}$, $q = 1, 2$ and support $[K_{t_1,6}]$ of one individual B-spline associated with triangle t_1. (b) "Cut-regions" of the support $K_{t_1,6}$ with respect to which its bivariate quadratic, simplicial B-spline is a piecewise quadratic polynomial. (c) "Cut-regions" of the set $\Omega = t_1 \cup t_2 \cup t_3$.

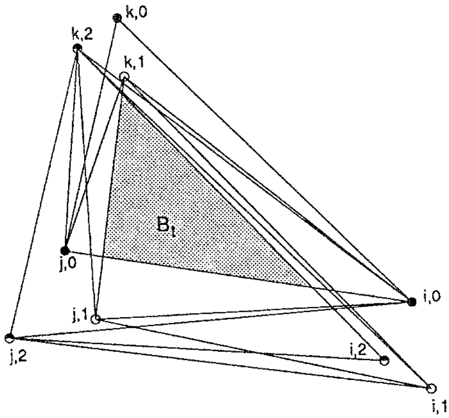


Fig. 4. Subset $B_t \subset t \in \Delta$, where only the six B-splines associated with t are different from zero defined by the intersection of the 10 triangles $[\vec{x}^{\,i,q_0}, \vec{x}^{\,j,q_1}, \vec{x}^{\,k,q_2}]$ associated with triangle $t = [\vec{x}^{\,i}, \vec{x}^{\,j}, \vec{x}^{\,k}]$, with $i < j < k$, via $Q = \{(q_0, q_1, q_2) \in \mathbb{Z}^3 \mid 0 \leq q_0 \leq q_1 \leq q_2 \leq 2\}$.

the six B-splines associated with t are different from zero (see Fig. 4). Next we solve for the coefficients $d_{t,m}$ from the linear system

$$\sum_{m=1}^{6} d_{t,m} M(\vec{y}^{\,t,k} \mid K_{t,m}) = 1, \qquad k = 1, \ldots, 6 \tag{13}$$

for each individual triangle t consecutively. Equation (13) may be read as application of the first of Marsden's identities, see Dahmen (1979), Goodman and Lee (1981).

In much the same fashion a numerical procedure was suggested by Auerbach et al. (1990) to determine geometrically meaningful control points, that is to provide abscissae for the coefficients $b_{t,m}$ to bear a geometric interpretation.

To this end, we take the same six different points $\vec{y}^{\,t,k}$, $k = 1, \ldots, 6$ in B_t and set

$$\sum_{m=1}^{6} x_{t,m} B(\vec{y}^{\,t,k} \mid K_{t,m}) = y_1^{t,k}, \qquad k = 1, \ldots, 6 \tag{14}$$

and

$$\sum_{m=1}^{6} y_{t,m} B(\vec{y}^{\,t,k} \mid K_{t,m}) = y_2^{t,k}, \qquad k = 1, \ldots, 6 \tag{15}$$

Then we solve for the coefficients $x_{t,m}$, resp. $y_{t,m}$, from the two linear systems Eq. (14), resp. (15).

Equation (14), resp. Eq. (15), may in turn be read as applications of the second of Marsden's identities, see Dahmen (1979) and Goodman and Lee (1981).

The coefficients $x_{t,m}$, resp. $y_{t,m}$, of the linear systems (14), resp. (15), provide the abscissae of the control points $\vec{b}^{\,t,m} = (x_{t,m}, y_{t,m}, b_{t,m})$ with $b_{t,m}$ being the coefficients of the normalized B-splines $B(\vec{x} \mid K_{t,m})$.

We can now rewrite more appropriately

$$s(\vec{x}) = \sum_{t=1}^{T} \sum_{m=1}^{6} b_{t,m} B(\vec{x} \mid K_{t,m}) \tag{16}$$

respectively,

$$(\vec{x}, s(\vec{x})) = \sum_{t=1}^{T} \sum_{m=1}^{6} \vec{b}^{\,t,m} B(\vec{x} \mid K_{t,m}) \tag{17}$$

where Eq. (17), complementarily to eq. (10), may be read as a convex combination of control points $\vec{b}^{\,t,m} \in \mathbb{R}^3$ weighted by $B(\vec{x} \mid K_{t,m})$.

There are now four important properties of the normalized B-splines $B(\vec{x} \mid K_{t,m})$ which allow manipulation of the surface $(\vec{x}, s(\vec{x}))$ by its corresponding control points $\vec{b}^{\,t,m}$, resp. their z-coordinates $b_{t,m}$; these are

1. *translation invariance*

$$(\vec{x}, \bar{s}(\vec{x})) = \sum_{t=1}^{T} \sum_{m=1}^{6} (\vec{b}^{\,t,m} + \vec{\delta}) B(\vec{x} \mid K_{t,m})$$

$$= (\vec{x}, s(\vec{x})) + \vec{\delta} \tag{18}$$

2. *convex hull property*

$$(\vec{x}, s(\vec{x})) \in \left[\left\{ \vec{b}^{\,t,m} \middle| \vec{x} \in [K_{t,m}] \right\}\right] \tag{19}$$

for all $\vec{x} \in \Omega$. However, in general the set

$$\left\{(\vec{x}, s(\vec{x})) \middle| x \in t\right\} \not\subset \left[\left\{ \vec{b}^{\,t,m} \middle| m = 1, \ldots, 6\right\}\right] \tag{20}$$

3. *reproduction of linear functions*

   If the control points $\vec{b}^{\,t,m}$ are in a given plane F, then $(\vec{x}, s(\vec{x}))$ represents this plane exactly.

4. *C^1 continuity inside Ω*

   If all knots of each configuration are in general position, then for any choice of control points one obtains a C^1 continuous surface over entire Ω. This is in contrast to Bernstein–Bezier representations.

Since C^1 continuity over Ω is guaranteed only if all knots of each configuration are in general position (i.e., define nondegenerate triangles), particular knot configurations are used to intentionally generate discontinuities in the function $s(\vec{x})$ or any of its directional derivatives. Thinking of these simplicial B-splines as smoothed versions of Bernstein–Bezier forms over Ω, we find the following configurations particularly useful. A knot configuration with four collinear knots will generate a discontinuity of the corresponding spline across the line given by these four knots; and a configuration with three collinear knots will generate a discontinuity in any directional derivative across the line given by these three knots. A threefold knot will give rise to a configuration with two sets of four collinear knots (see Fig. 5). Taking all vertices $\vec{x}^{\,i}$, $i = 1, \ldots, V$, of the triangulation Δ as threefold knots, the bivariate quadratic, simplicial B-splines degenerate to the bivariate quadratic Bernstein–Bezier forms over the triangulation Δ of the data domain Ω, cf. Böhm et al. (1984), Dahmen (1986), and Farin (1986). The desirable ability to take discontinuities of the function to be approximated or modeled or any of its first derivatives into account therefore requires first of all that the lines over which the discontinuities should occur become edges of some triangles in Δ. Thus, the problem encountered here is to find a triangulation appropriate for the purposes of approximation given the vertices and some predefined edges. We termed its solution the (suboptimal) constrained Delaunay triangulation, see Auerbach (1990), Auerbach et al. (1990), Auerbach and Schaeben (1989, 1990).

The Parametric Case

For the parametric case we rewrite Eq. (17)

$$\vec{s}\,(\vec{u}\,) = \begin{pmatrix} x(\vec{u}\,) \\ y(\vec{u}\,) \\ z(\vec{u}\,) \end{pmatrix} = \sum_{t=1}^{T} \sum_{m=1}^{6} \vec{b}^{\,t,m} B(\vec{u} \mid K_{t,m}) \tag{21}$$

where $\vec{u} = (u_1, u_2) \in \Omega \subset \mathbb{R}^2$ is now a given parameterization and $K_{t,m}$ refers now to a triangulation of the parameter domain Ω. Recalling the basic properties of our $B(\vec{u} \mid K_{t,m})$, Eq. (21) is obviously a convex combination of control points $\vec{b}^{\,t,m} \in \mathbb{R}^3$.

With (x_ν, y_ν, z_ν), $\nu = 1, 2, \ldots$ denoting the positional data, in practical applications the control points $\vec{b}^{\,t,m}$are determined by solving the linear system

$$\begin{pmatrix} x_\nu \\ y_\nu \\ z_\nu \end{pmatrix} = \begin{pmatrix} x(\vec{u}_\nu) \\ y(\vec{u}_\nu) \\ z(\vec{u}_\nu) \end{pmatrix} = \sum_{t=1}^{T} \sum_{m=1}^{6} \vec{b}^{\,t,m} B(\vec{u}_\nu) \mid K_{t,m}) \tag{22}$$

or more explicitly

$$x_\nu = \sum_{t=1}^{T} \sum_{m=1}^{6} b_x^{t,m} B\big((u_{1_\nu}, u_{2_\nu}) \mid K_{t,m}\big) \tag{23}$$

$$y_\nu = \sum_{t=1}^{T} \sum_{m=1}^{6} b_y^{t,m} B\big((u_{1_\nu}, u_{2_\nu}) \mid K_{t,m}\big) \tag{24}$$

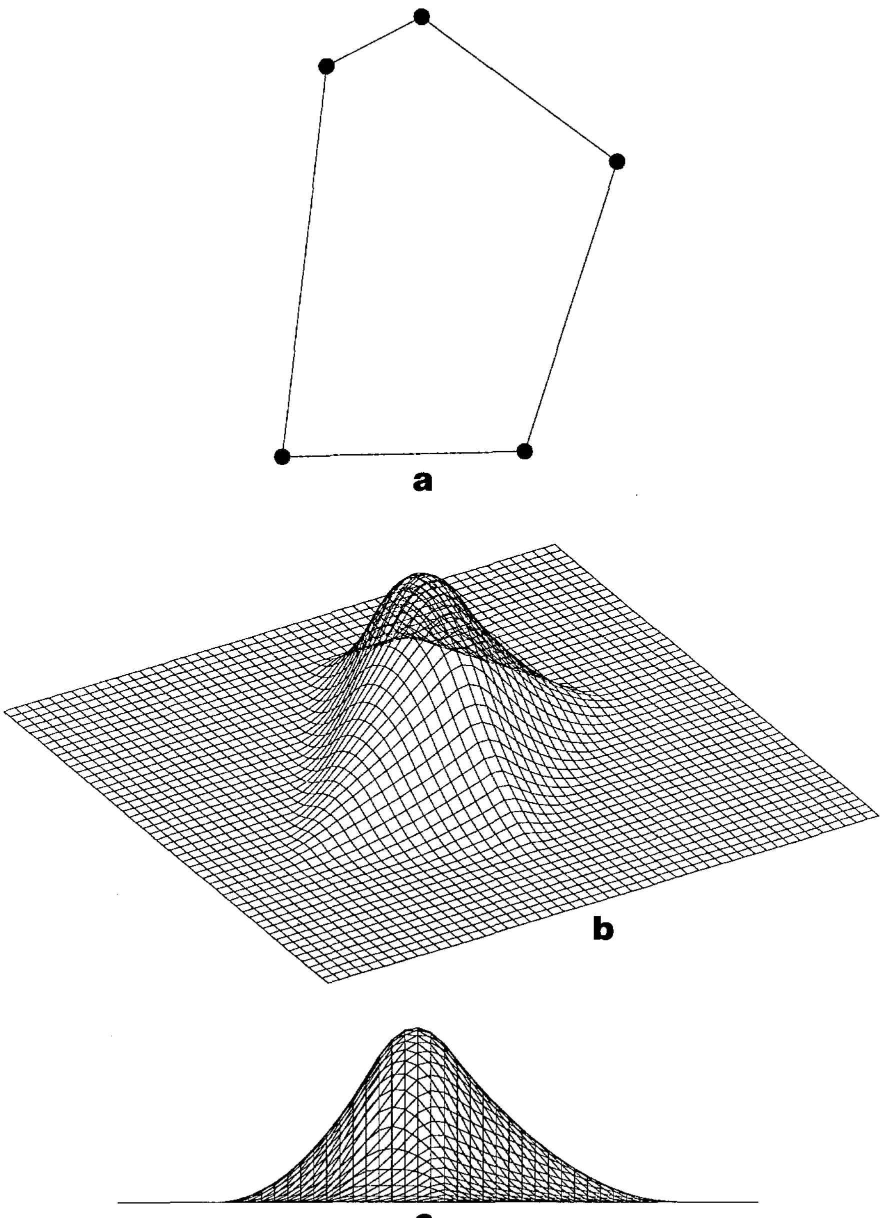


Fig. 5. (a) Set of knots in general position and its corresponding individual spline. (b) Perspective view of the spline defined by the knot configuration of a. (c) Side view of the spline defined by the knot configuration of a. (d) Knot configuration with four collinear knots generating a discontinuity in the spline over the line of collinearity. (e) Perspective view of the spline defined by the knot configuration of d. (f) Side view of the spline defined by the knot configuration of d. (g) Knot configuration with three collinear knots generating a discontinuity in the directional derivatives of the spline across the line of collinearity. (h) Perspective view of the spline defined by the knot configuration of d. (i) Side view of the spline defined by the knot configuration of d.

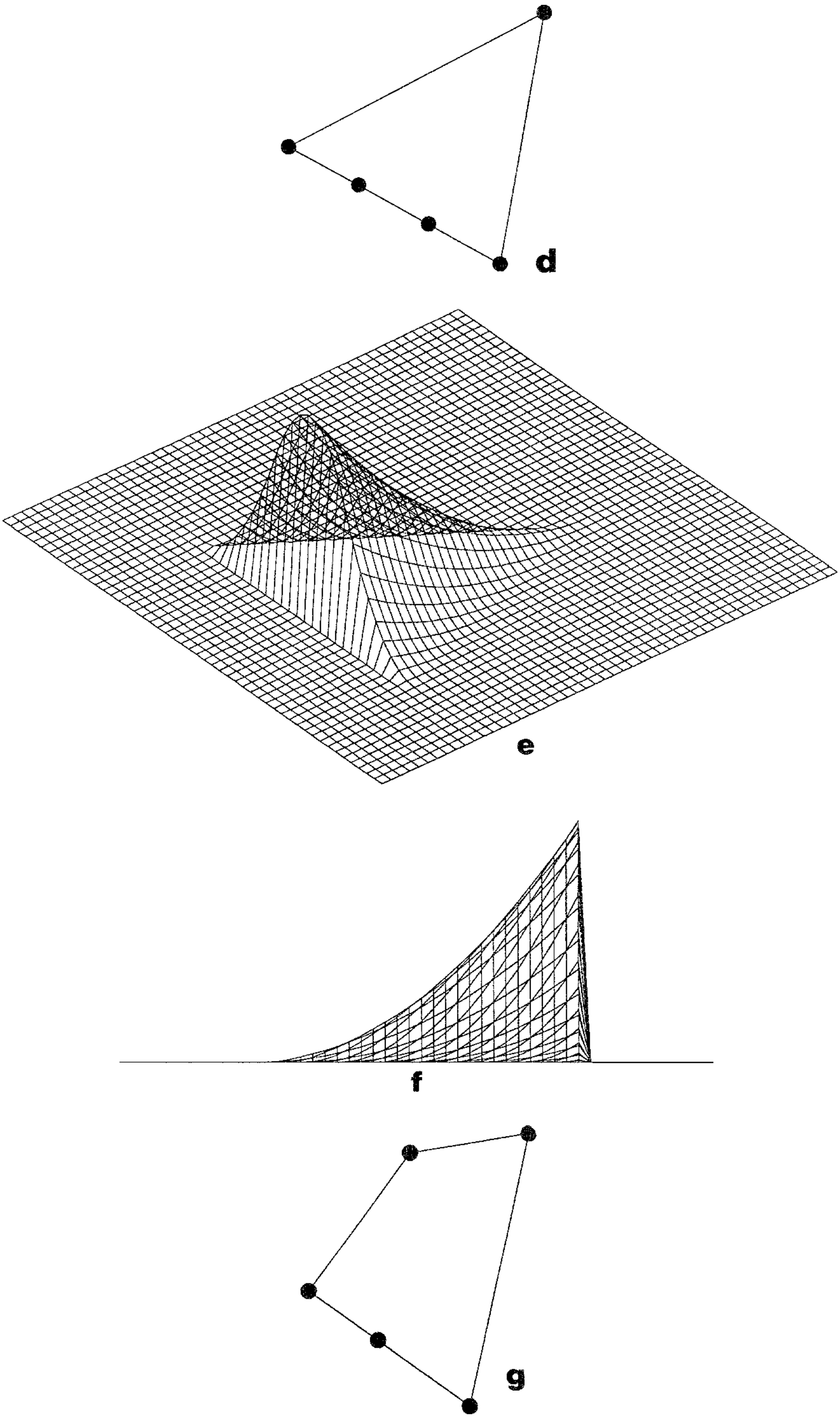
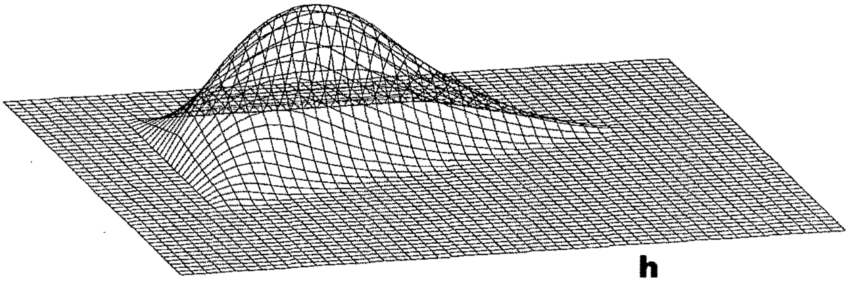
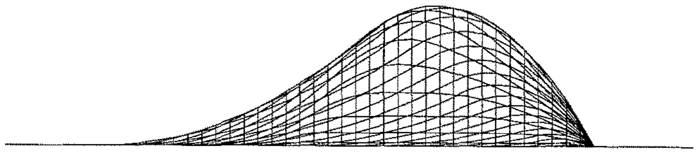


Fig. 5. Continued.



h



i

Fig. 5. Continued.

$$z_\nu = \sum_{t=1}^{T} \sum_{m=1}^{6} b_z^{t,m} B\big((u_{1_\nu}, u_{2_\nu}) \big| K_{t,m}\big) \tag{25}$$

While Eq. (25) obviously corresponds to Eq. (26) of the functional case, Eq. (23), resp. Eq. (24), replaces Eq. (14), resp. (15) of the functional case. Thus, in the parametric case we do not have to "internally" determine the abscissae for some coefficients to act as geometric control points as in the functional case, but $\vec{b}^{\,t,m}$ are control points of the surface given by Eq. (21). Practically speaking, in the parametric case we solve a system of $3 \times NU$ equations, while in the functional case we solve $2 \times T$ internal 6×6 systems and one system with NU equations when NU is the total number of data points. It should be noted that the former case gives rise to a sparse structureless system, while the latter is broken and establishes two highly structured subsystems which can be solved independently of each other in T consecutive steps. The real problem of practically applying the parametric case is the fact that the user has to provide an initial appropriate parameterization [i.e., an approximate topology of the cloud of data points $(x_\nu, y_\nu, z_\nu) \in \mathbb{R}^3$]. For a given topology, we show in Fig. 6a a faulted surface as an example of an application of the parametric surface representation, Eq. (21), and in Fig. 6b the same faulted surface and its corresponding geometric control points.

DEFINITION OF KNOTS FOR PRACTICAL APPLICATIONS OF BIVARIATE QUADRATIC B-SPLINES

The knots, or more specifically the set of knot configurations associated with any triangle, are defined in three basic steps.

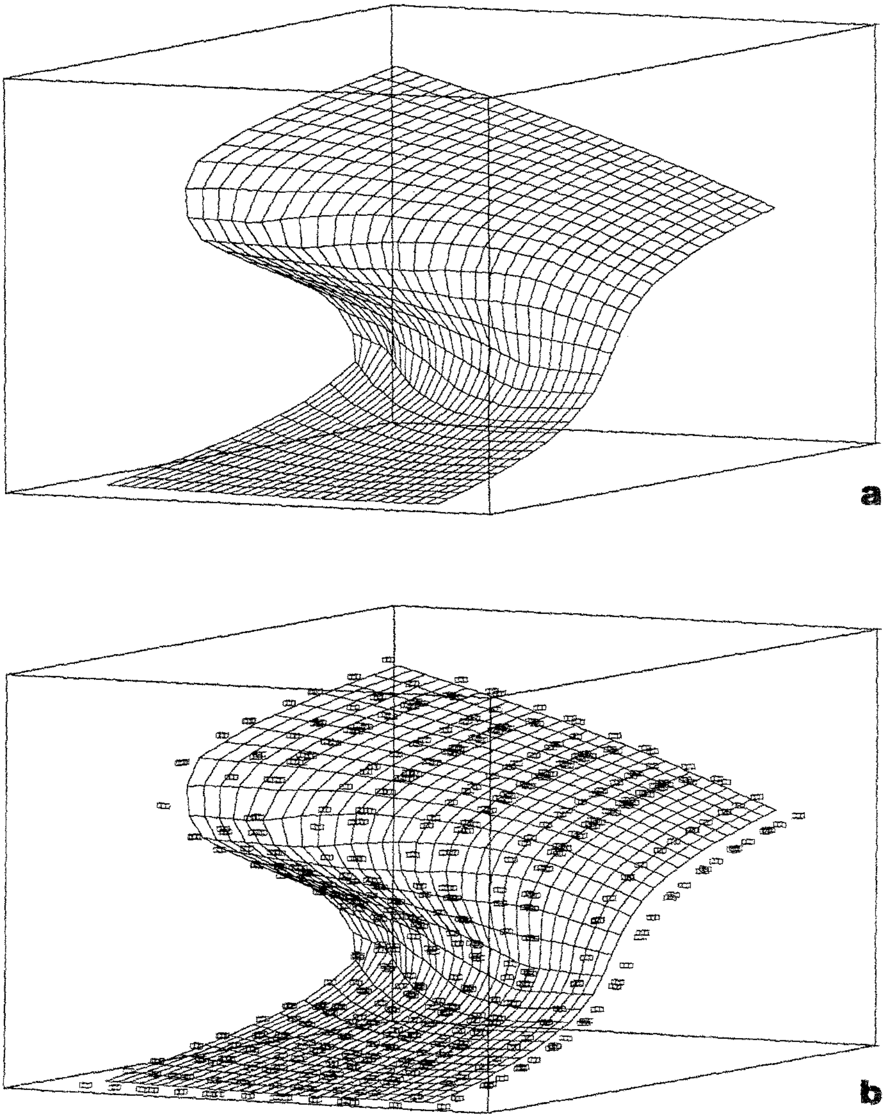


Fig. 6. (a) Perspective view of parametric representation of a faulted surface patch, (b) with geometric control points.

1. Define vertices of triangles according to:
   —varying density of data, resp. information;
   —known positions of discontinuities of the surface or any of its directional derivatives (predefined edges);
   —matching of adjacent domains of modeling and mapping.

2. Define edges to compose a Delaunay triangulation of the vertices (Dirichlet-, Thiessen tesselation, Voronoi polygon) subject to constraints—i.e., predefined edges, resulting in (suboptimal) constrained Delaunay triangulation of the vertices,
3. Define knots of splines by pulling apart threefold vertices of the triangulation subject to constraints concerning the smoothness of the surface to be approximated or modeled.

Generation of Vertices

The vertices are generated primarily according to the varying density of data, and the available a priori information concerning the surface to be approximated or modeled. Their convex hull will usually define the domain of the surface. The vertices should be placed to generally avoid triangles with less than six data of position or gradient; as a rule of thumb there should be more than 12 data per triangle.

Subsets of points of the domain, where discontinuities of the function itself or its directional derivatives occur, must become edges in the next step of triangulation. Therefore, vertices must be defined such that their corresponding polygon describes these subsets, resp. edges. Placing the vertices one should always remember that the edges to be defined in the next step will cause data to belong to different triangles.

The generation of vertices is actually the starting point of the geometrical modeling process; it is also the first instance when the user can introduce "a priori" knowledge about the surface to be approximated and modeled. Therefore, we recommend that the vertices are to be defined interactively with a graphics screen whenever their number is sufficiently small to be reasonably handled in this fashion. For large numbers of vertices to be generated, procedures developed for finite element applications (e.g., Cavendish, 1974) can be adapted.

Constrained Delaunay Triangulation of the Data Domain

The basic idea is to adapt Lawson's triangulation procedure (Lawson, 1972, 1977; Sibson, 1978) to honor natural neighborhood relations known a priori by the user because it would generally be lost otherwise; thus it resembles features of a concept put forward by Sibson (1981), and allows the user to incorporate his/her additional a priori knowledge. In practical applications this is exactly the instance when user-controlled modeling begins.

This a priori knowledge can basically be expressed by marking given vertices that must not be joined to form an edge of any triangle—e.g., in order to avoid interpolation across the known location of a discontinuity of the function to be interpolated/approximated, or that must be joined to form an edge of some triangle, e.g., in order to preserve the natural neighborhood relation of the data with successive locations along a given line segment. In both cases, the a priori knowledge concerning the neighborhood relation of data locations can actually be incorporated effectively by defining some edges prior to an automatic procedure which must then honor these predefined edges. We may in fact use a modified version of Lawson's triangulation generalized to preserve predefined edges. This new procedure is then a constrained triangulation which generates triangles which are as uniform as possible given the vertices and some predefined edges. We termed this new procedure (suboptimal) "constrained Delaunay triangulation" (Auerbach et al., 1990). A more detailed discussion and first applications to digital elevation models were given by Auerbach and Schaeben (1989, 1990).

Construction of the constrained triangulation $\tilde{\Delta}^*$ requires modifications of the algorithm to construct the Delaunay triangulation Δ^*—see Auerbach (1990) for two different variants. Another convenient algorithmic aspect to it is that to incorporate an additional edge subsequently into a given (sub)optimal triangulation requires its local updating only.

Generation of Knots

For arbitrary triangulations Δ there are no canonical positions for the points $\vec{x}^{\,i,1}$ and $\vec{x}^{\,i,2}$, $i = 1, \ldots, V$, which ensure B-splines of optimal smoothness. However, the rules for pulling the knots apart should consider that

- numerical stability of the B-spline basis is preserved if the distances between knots $\vec{x}^{\,i,q}$, $q = 1, 2$ and $\vec{x}^{\,i,0}$ are kept restricted.
- to facilitate the evaluation of linear combinations of B-splines at any given point in $\Omega \in \mathbb{R}^2$, the number of B-spline incidences should be kept small by using triangulations Δ as regular as possible, avoiding situations where many triangles share the same vertex, and referring to a systematical numbering of knots.
- to achieve optimal smoothness it is required that all knots from all knot sets in $\mathcal{C}$ are in general position (i.e., each three knots of the same knot set must define a nondegenerate triangle), and to achieve good approximation rates for derivatives nearly coalescent or nearly collinear points in the same knot set must be avoided.

A heuristic algorithm to compute the actual locations of the knots $\vec{x}^{\,i,1}$, $\vec{x}^{\,i,2}$ associated with $\vec{x}^{\,i}$ has been formulated in Auerbach et al. (1990) which gives satisfying results concerning the criteria mentioned above. It may be understood as compromise between a fully interactive user-controlled procedure as suggested by Dahmen and Michhelli (1982a), and a completely automatic procedure solving a sequence of optimization problems as realized by Gmelig Meyling (1986). More sophisticated procedures following the general rules given above did not prove worth the additional effort in computing demands with respect to the finally computed surfaces.

To allow surfaces not vanishing on the boundary of Ω or with nonvanishing first-order directional derivatives, it is necessary to guarantee that the "partition of unity" property holds in the entire set Ω. This can be easily obtained by moving the knots $\vec{x}^{\,i,1}$, $\vec{x}^{\,i,2}$ for which $\vec{x}^{\,i,0}$ lies on the boundary outside of Ω (see Fig. 9).

APPROXIMATION AND MODELING

For purposes of practical approximation and modeling, the general procedure is as follows.

- Constrained (suboptimal) triangulation of data domain.
- Provide six particular points for each triangle to:
  —normalize the corresponding six B-splines [Eq. (13)];
  —compute the abscissae of their control points [Eqs. (14) and (15)].
- Calculate and store elements of matrix by evaluating corresponding splines at the coordinates of data points, thus establishing a large linear system

$$z_\nu = \sum_{t=1}^{T} \sum_{m=1}^{6} b_{t,m} B\big((x_\nu, y_\nu) \,|\, K_{t,m}\big) \tag{26}$$

  for the unknown coefficients $b_{t,m}$, with (x_ν, y_ν, z_ν), $\nu = 1, 2, \ldots$ covering the positional data.
- Remove zero columns of matrix if there are any.
- Add equations corresponding to additional information (e.g., gradients).
- Solve the system in the sense of (weighted) least squares.
- Repair for zero columns (i.e., for gaps in the data) by estimating missing coefficients from neighboring coefficients.
- Visualize the computed surface.
- Visualize coefficients of the linear combination as control points.
- Establish different levels of local control by optionally regrouping coefficients, resp. control points, in different ways.

EXAMPLES

To give practical examples of the versatility of this approach, we first apply the procedure outlined in the previous section to a mathematical test function composed of tilted planes and trigonometric functions such that the resulting surface is mathematically represented as a given bivariate function with various discontinuities as depicted in Fig. 7a and b. We would like to emphasize here

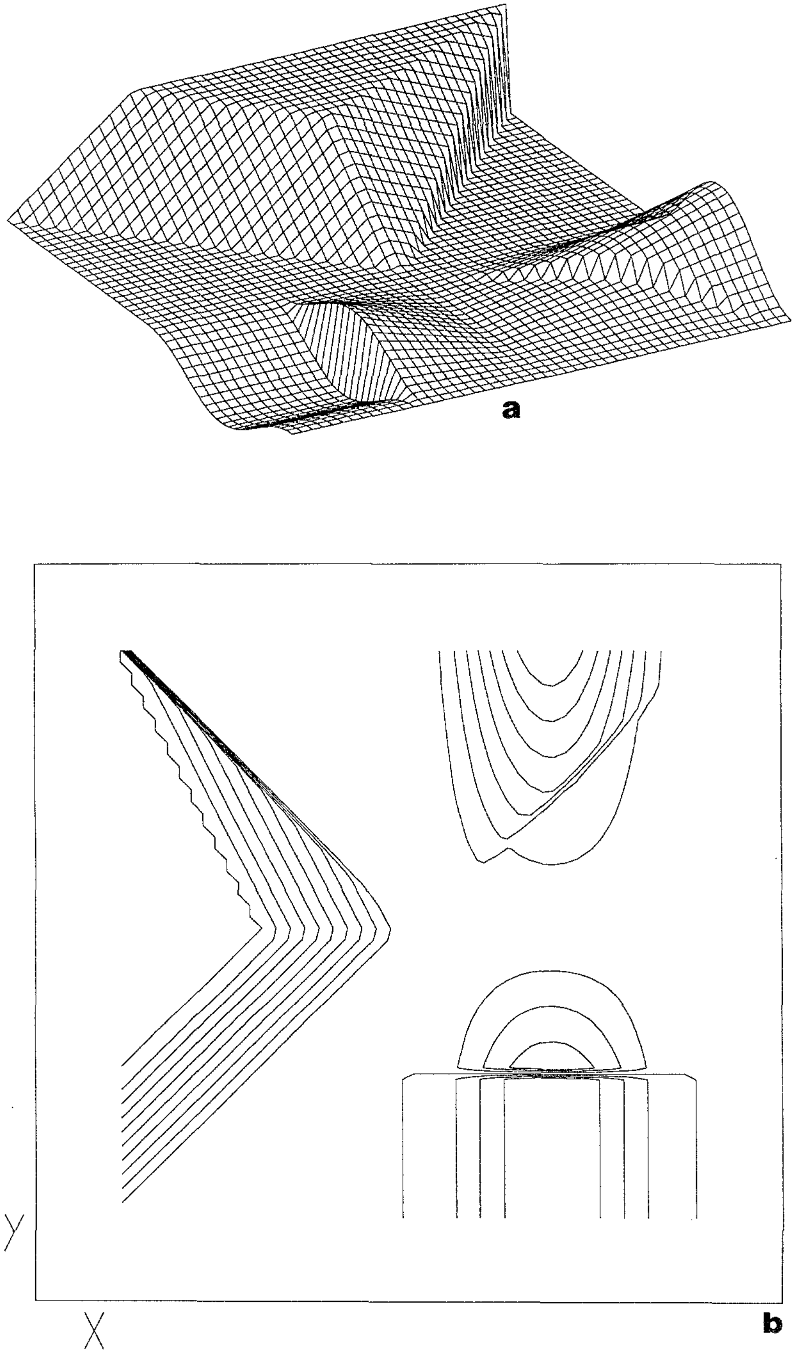


Fig. 7. Mathematical test function (a) perspective view, (b) contour lines.

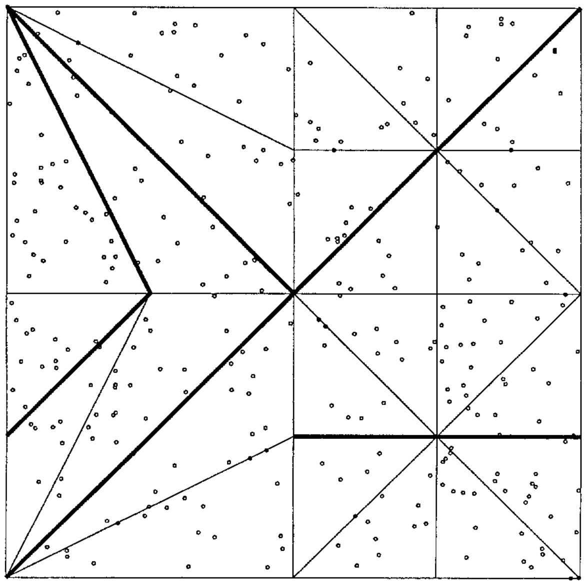


Fig. 8. Data sites (○) in the domain Ω, vertices of triangles adjusted to areal distribution of data sites, predefined edges reflecting a priori knowledge of the surface to be approximated or modeled (bold lines), and edges according to (suboptimal) constrained Delaunay triangulation honoring the predefined edges.

that displaying the contour lines often provides a more sensitive control of the features of a surface even though we used a simple standard routine to compute them from a regular quadratic grid of calculated z-values.

The resulting mathematical surface was sampled at 300 data sites in the domain Ω, which was then triangulated as regularly as possible given the user-defined vertices adjusted to the areal distribution of data sites and the predefined edges according to the known locations of the discontinuities (see Fig. 8).

In the next step, the knots were defined by pulling the vertices apart; note collinear knots along predefined edges (Fig. 9).

In Fig. 10, the subregions B_t (shaded) of each triangle t are shown; any empty or degenerated B_t subset would indicate that some knots may not be most appropriately numbered or placed, and warn that the normalization of B-splines and computation of the abscissae of control points as suggested here and coded in our routines may get numerically instable. The essential issue of both the triangulation of given vertices and some given edges and the "pulling apart" of knots is to avoid this situation. In case an empty B_t should yet occur, it can usually be repaired by small changes of the sites of a few vertices and/or local update of the triangulation and/or relabeling and/or relocating of a few knots. Empty sets B_t do not pose a general mathematical problem, at most a minor

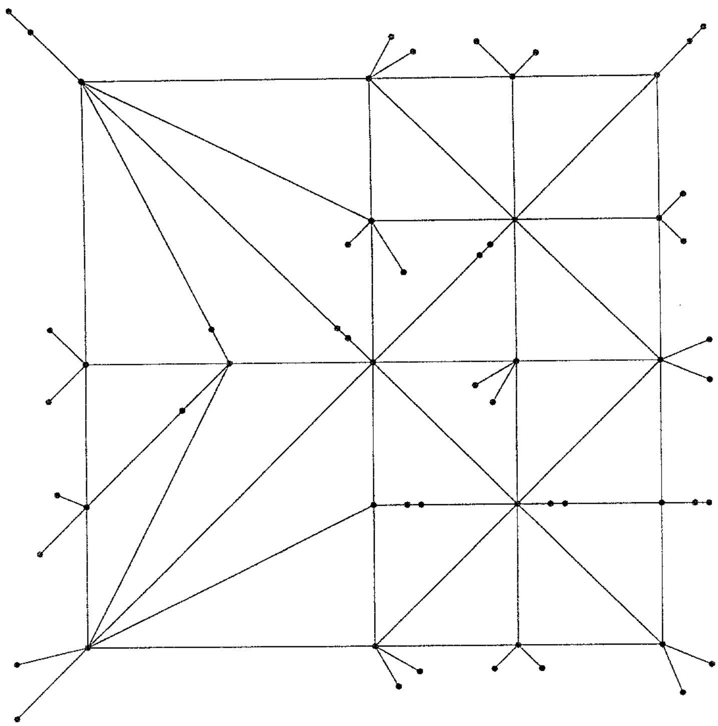


Fig. 9. Definition of knots by pulling vertices apart such that (i) partition of unity over Ω is guaranteed, (ii) all knots are in general position except for only those (iii) along predefined edges to reproduce the known discontinuities.

limitation to our numerical approach of employing simplex B-splines or to the applicability of the routines coded so far.

The surface of the linear combination of quadratic B-splines approximating the given data in least-squares sense is shown in Fig. 11a and b, revealing small deviations of the initial surface only. A three-dimensional perspective view of the geometric control points is conveyed by Fig. 11c, whereas Fig. 11d just shows the abscissae of the control points in the data domain Ω.

The second is an example with geologic data from the "Hildesheimer Wald" formation in Lower Saxony, West Germany. The Hildesheimer Wald is a salt anticline of triassic rocks on top of Zechstein salt. The base of the Detfurth sequence was sampled because it clearly separates the coarse Detfurth sandstone from the finer and bright Avicula sandstone of the Volpriehausen sequence. From the data and an initial geological interpretation, the constrained triangulation was derived for each of the three disjunct parts of the domain individually (see Fig. 12). Then the knots were appropriately located and the corresponding surface approximating the data in least-squares sense was computed; it is depicted in Fig. 13. To give an idea of the interactive modeling

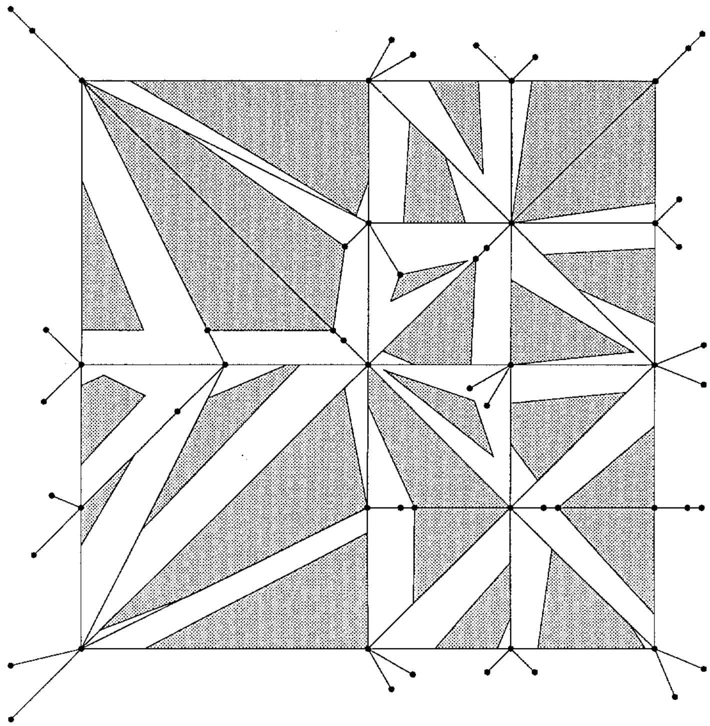


Fig. 10. Subregions B_t (shaded) of each triangle t, where only the six B-splines associated with t are different from zero and all others vanish such that $s(\vec{x})|_{B_t}$ is a bivariate quadratic polynomial.

facilities, we show the influence of moving an arbitrary geometric control point of the computed surface over the smallest of the three subdomains in Fig. 14.

CONCLUSIONS

Using simplicial B-splines associated with irregular triangles for the purposes of approximation and geometric modeling, the shortcomings of tensor product B-splines are overcome which are induced by the fact that tensor product splines require a grid of knots topologically equivalent to a rectangular grid, while at the same time many of their favorable features are preserved. To be more specific, polynomial B-splines associated with irregular triangles provide a method combining all the properties required by geometric modeling of geologic surfaces and bodies:

- triangles are adjustable to the spatial distribution of data, resp. their abscissae;
- data providing positional (x, y, z) information can be processed;

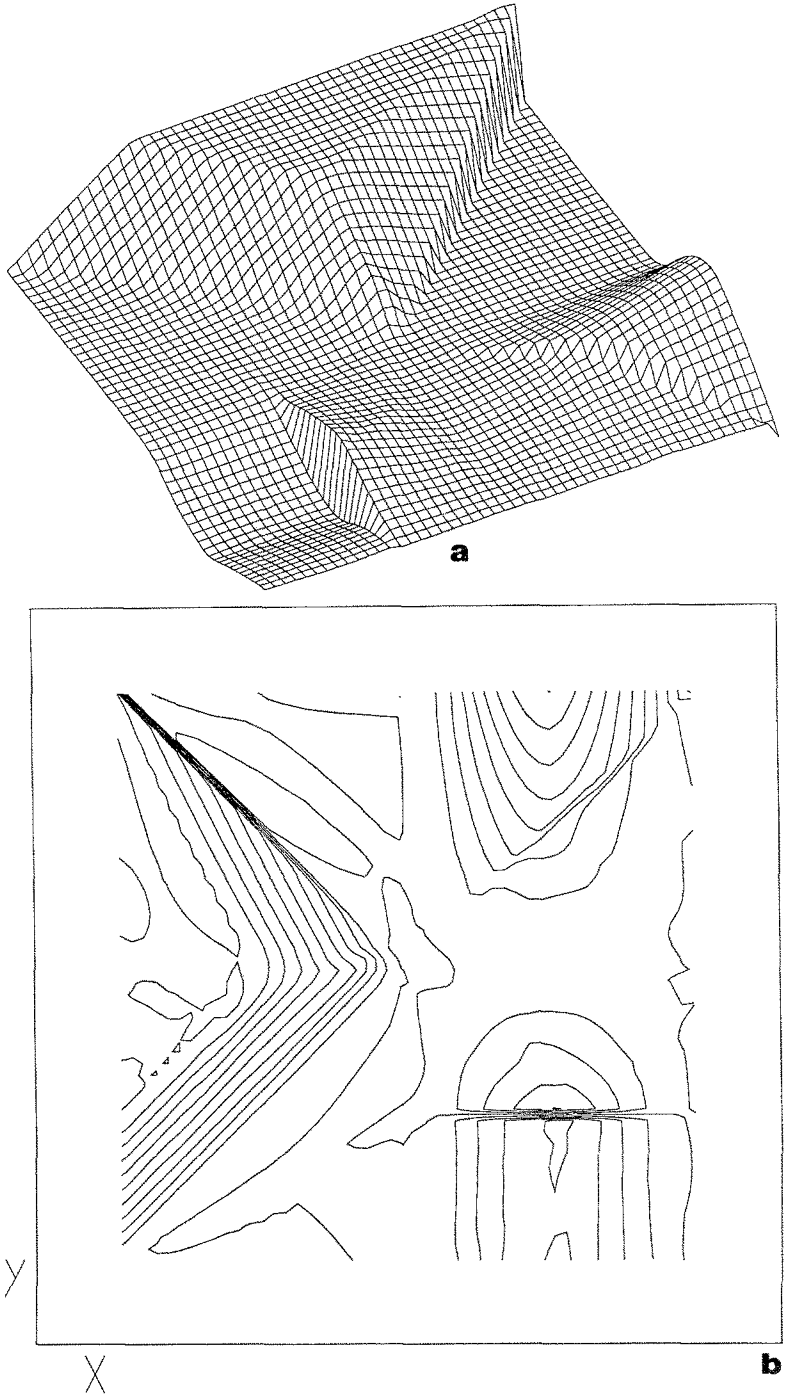
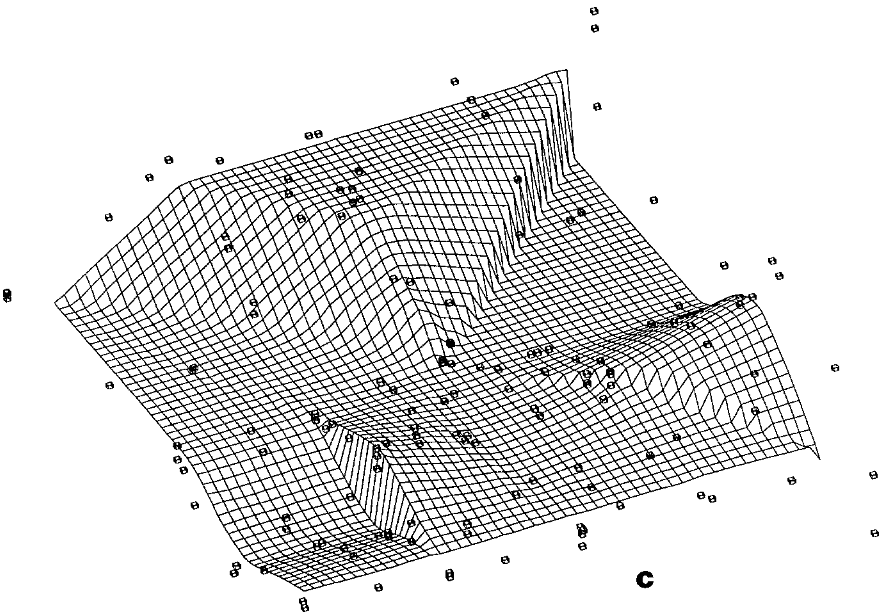
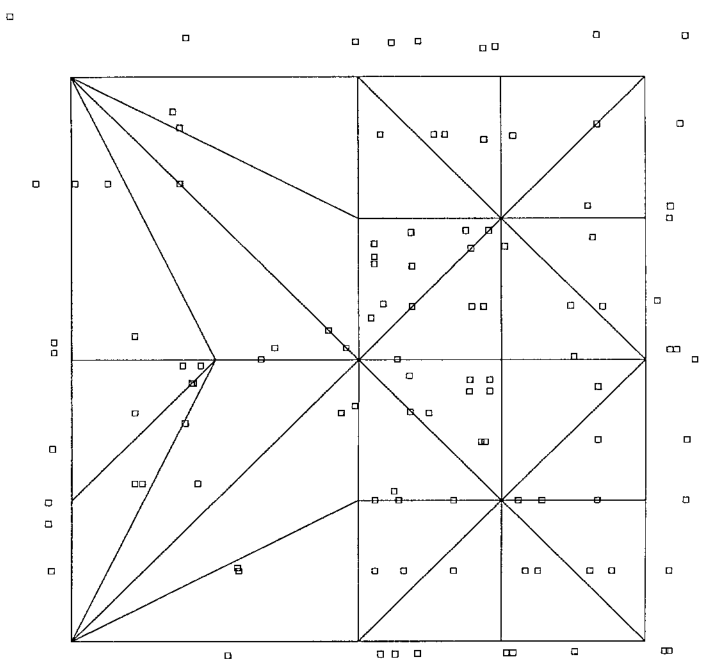


Fig. 11. Computed surface of linear combination of quadratic B-splines (a) perspective view, (b) contour lines, (c) with geometric control points, (d) abscissae of control points.



c



d

Fig. 11. Continued.

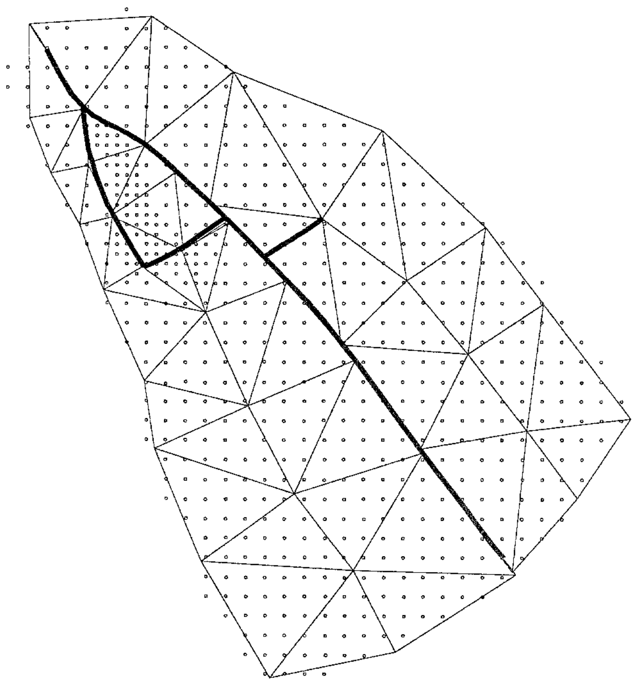


Fig. 12. Data domain of the sampled base of the "Detfurth" sequence divided into three disjunct parts according to the major faults (bold), data sites (○), and triangulation subject to predefined edges (bold).

- data providing information on any directional derivatives can be processed when available or accessible; however, they are not necessarily required;
- data providing information on the geometry to be recovered/modeled that can be formalized into discontinuities of the function itself or any of its directional derivatives can be processed;
- given a topology consistent with the data (i.e., an approximate parameterization of the surface to be recovered/modeled), surfaces can be represented which cannot be represented by functions;
- local control of the surface is provided by corresponding control points establishing the "convex hull," thus interactive computer-aided geometric modeling is possible.

ACKNOWLEDGMENTS

This communication summarizes the results accomplished when its authors were with Prof. Dr. A. Siehl at the Geologisches Institut der Universität Bonn

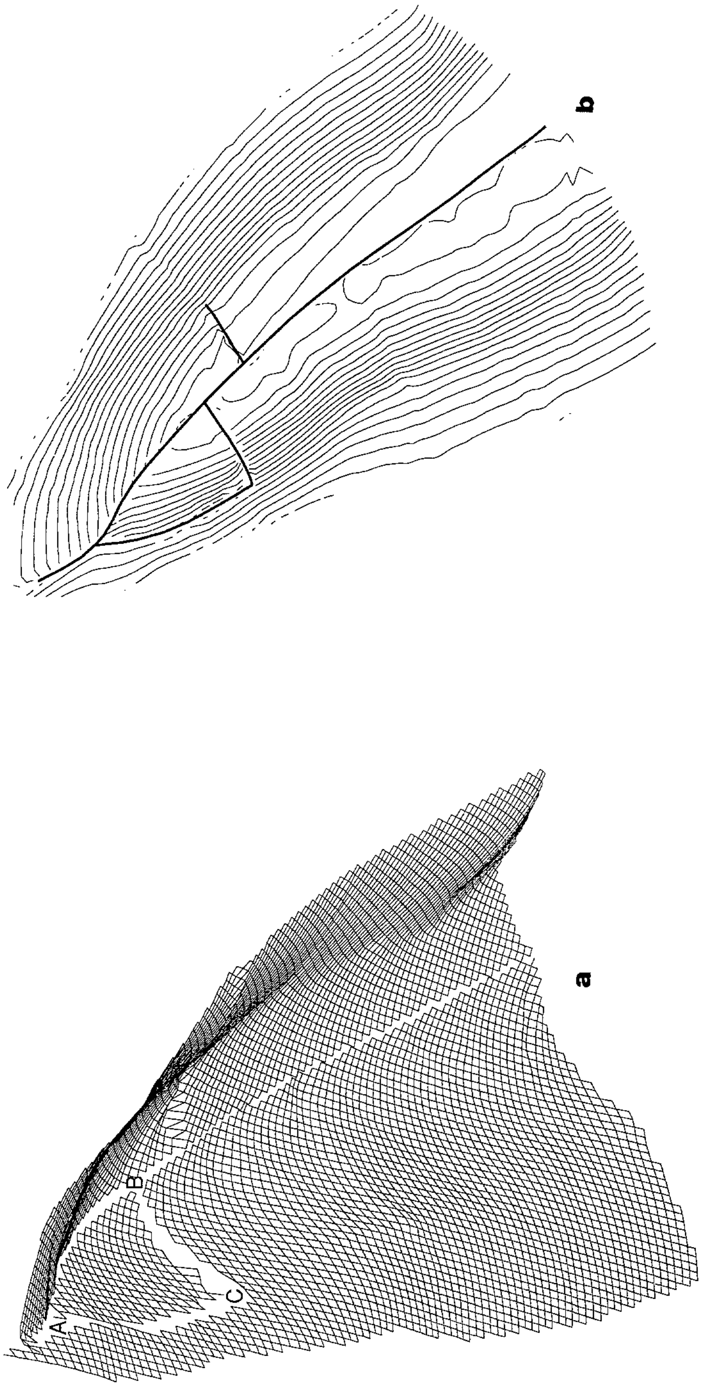


Fig. 13. Computed surface separating Detfurth sandstone from Avicula sandstone of the Volpriehausen sequence (a) perspective view, (b) contour lines.

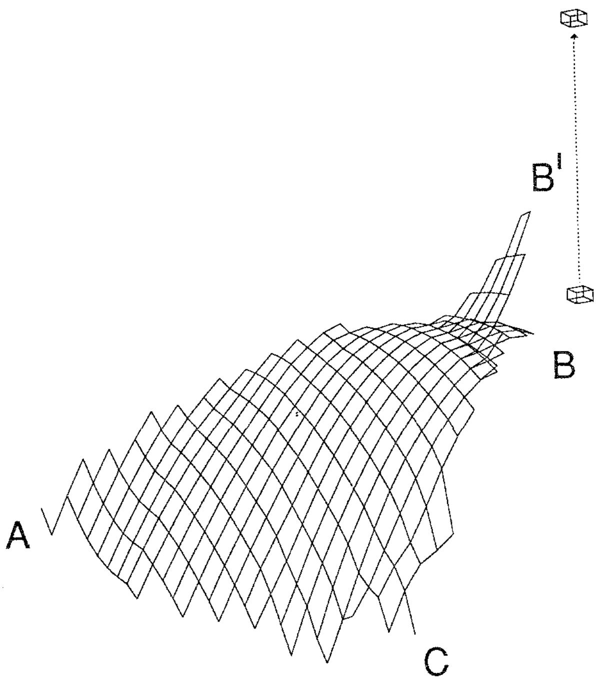


Fig. 14. Section of the computed surface over the smallest and best sampled subdomain with one of its geometric control point (◻), and local change of this surface by moving the shown control point upward.

in the project, ''computer-aided design of geological maps,'' within the priority program ''digital geoscientific maps'' of the German Science Foundation (DFG) which Prof. Dr. R. Vinken has been the scientific director of (cf. Vinken, 1986). We thank Prof. Dr. A. Siehl, Dr. O. Rüber, and all the other geologists in the group for their introduction to and continuing discussion of the geometric aspects of geology as well as all colleagues within the priority program for helpful discussions on several occasions during the last five years.

Special thanks are due to Prof. W. Böhm, Braunschweig, for his introduction to computer-aided geometric design (cagd), and Prof. W. Dahmen, W Berlin, Prof. K. Scherer, Bonn, and Prof. C. R. Traas, Twente, for many instructive discussions on splines and their practical applicability. We would also like to thank Jeffrey M. Yarus, Littleton, Colorado, for his helpful review.

The authors acknowledge financial support by DFG, Bonn.

REFERENCES

Auerbach, S., 1990, Approximation mit bivariaten B-splines über Dreiecken: Ms. Thesis, Dept. of Applied Mathematics, Univ. of Bonn, FRG, (unpublished).

Auerbach, S., Neamtu, M., Gmelig Meyling, R., and Schaeben, H., 1990, Approximation and Geometric Modeling with Simplex B-Splines Associated with Irregular Triangles: CAGD (to appear).

Auerbach, S., and Schaeben, H., 1989, Computer-Aided Geometric Design in Geosciences: II. Practical Application to Digital Elevation Models and Topographic Maps: Abstract, 28th International Geological Congress, Washington, July 9–19, 1989.

Auerbach, S., and Schaeben, H., 1990, Surface Representations Reproducing Given Digitized Contour Lines: Math. Geol., v. 22, p. 723–742.

Böhm, W., Farin, G., and Kahmann, J., 1984, A Survey of Curve and Surface Methods in cagd: CAGD, v. 1, p. 1–60.

Braun, R., Friedinger, P., Rüber, O., Schroer, R., Siehl, A., and Stamm, R., 1983, Die kartenbezogene Auswertung geologischer Dateien im Dialog—Fallstudie Loccum: Geol. Jb., A 70, p. 179–210.

Cavendish, J. C., 1974, Automatic Triangulation of Arbitrary Planar Domains for the Finite Element Method: Intern. J. Numer. Meth. Eng., v. 8, p. 679–696.

Dahmen, W., 1979, Polynomials as Linear Combinations of Multivariate B-Splines: Math. Z., v. 169, p. 93–98.

Dahmen, W., 1986, Bernstein-Bezier Representation of Polynomial Surfaces: ACM SIGGRAPH 86, Dallas, TX, Aug. 18–22, 1986.

Dahmen, W., and Micchelli, C. A., 1982a, Multivariate Splines—A New Constructive Approach, *in* Barnhill, R. E., and Böhm, W. (Eds.), Surfaces in Computer Aided Geometric Design: Proceedings of the conference held at the Mathematical Research Institute at Oberwolfach, W-Germany, April 25–30, 1982, North-Holland Publishing, Amsterdam, p. 191–215.

Dahmen, W., and Micchelli, C. A., 1982b, On the Linear Independence of Multivariate B-Splines. I. Triangulations of Simploids: SIAM J. Numer. Anal., v. 19, p. 992–1012.

Dahmen, W., and Micchelli, C. A., 1983a, On the Linear Independence of Multivariate B-Splines. II. Complete Configurations: Math. Comp., v. 41, p. 143–163.

Dahmen, W., and Micchelli, C. A., 1983b, Recent Progress in Multivariate Splines, *in* Chui, C. K., Schumaker, L. L., and Ward, J. D. (Eds.), Approximation Theory IV: Academic Press, New York, p. 27–121.

Farin, G., 1986, Triangular Bernstein-Bezier patches: CAGD, v. 3, p. 83–127.

Gmelig Meyling, R. H. J., 1986, Polynomial Spline Approximation in Two Variables: Ph.D. Thesis, University of Amsterdam.

Gold, C., 1980, Geological Mapping by Computer, *in* Taylor, D. R. F., (Ed.), The Computer in Contemporary Cartography: Wiley, New York, p. 151–190.

Goodman, T. N. T., and Lee, S. L., 1981, Spline Approximation Operators of Berstein-Schoenberg Type in One and Two Variables: J. App. Theory, v. 33, p. 248–263.

Höllig, K., 1982, Multivariate splines: SIAM J. Num. Anal., v. 19, p. 1013–1031.

Höllig, K., 1986, Multivariate Splines: Lecture Notes: AMS Short Course Series, New Orleans.

Lawson, C. L., 1972, Generation of a Triangular Grid with Application to Contour Plotting: Jet Propulsion Laboratory, Internal Report 299.

Lawson, C. L., 1977, Software for C1 Surface Interpolation, *in* Rice, J. R. (Ed.), Mathematical Software III: Proc., University of Wisconsin, Madison, March 28–30, 1977, Academic Press, New York, p. 161–194.

Loudon, T. V., 1986, Spatial Models: British Geological Survey.

Micchelli, C. A., 1979, On a Numerically Efficient Method for Computing Multivariate B-Spline, *in* Schempp, W., and Zeller, K. (eds.), Multivariate Approximation Theory: Birkhäuser, Basel, p. 211–248.

Rüber, O., and Siehl, A., 1984, Digitale Raummodelle geologischer Flächenverbände: DFG proposal (unpublished).

Schaeben, H., 1988, Improving the Geological Significance of Computed Surfaces with cagd Methods: Geol. Jb., A104, p. 263–280.
Schaeben, H., and Auerbach, A., 1989a, Computer Aided Geometric Design in Geosciences: I. Methodology: Polynomial Splines on Triangles: Abstract, 28th IGC, Washington, 1989.
Schaeben, H., and Auerbach, A., 1989b, Computer Aided Geometric Design in Geosciences: III. Practical Application to Three-Dimensional Geologic Modeling and Digital Geological Maps: Abstract, 28th IGC, Washington, 1989.
Sibson, R., 1978, Locally Equiangular Triangulations: Computer J., v. 21, p. 243–245.
Sibson, R., 1981, A brief Description of Natural Neighbourhood Interpolation, *in* Barnett, V. D. (ed.), Graphical Methods for Multivariate Data: John Wiley & Sons, Chichester, p. 21–36.
Tipper, C. R., 1979, Surface Modeling Techniques: Number Four Series on Spatial Analysis: Kansas Geological Survey, Lawrence, Kansas.
Vinken, R., 1986, Digital Geoscientific Maps: A Priority Program of the German Society for the Advancement of Scientific Research: Math. Geol., v. 18, p. 237–246.