16. M. S. Anan'in, S. I. Dymnikov, M. A. Leikand, et al., "Simplified model for calculation of the nonlinear characteristics of thin-layered elastomer elastic elements," Vopr. Dinam. Prochn., 44, 8-15 (1984).
17. K. F. Chernykh and L. V. Milyakova, "Thin rubber-metal elements," Vestn. Leningr. Univ., No. 19, 88-96 (1981).

VIBRATIONS OF AN INFINITE PLATE, IN CONTACT WITH A FLUID, WHEN EXCITED ALONG A STRAIGHT LINE BY A MOTOR OF FINITE POWER

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The study of the energetics of vibrations of elastic structures in contact with a fluid and subjected to dynamic forces is a rather complex and multifaceted problem that has been solved with the use of several simplifying assumptions. In particular, the force is usually assigned in the form of an explicit function of time. Thus, it would be interesting to undertake a comprehensive examination of a system consisting of an exciter, an elastic structure, and a fluid in the case when the vibrations in the hydroelastic subsystem influence the formation of the exciting force. In this case, the entire combined system will be autonomous [3].

The present article is devoted to an analysis of the vibrations of an infinite plate. One surface of the plate is in contact with an acoustic medium, while an eccentric exciter of finite power is placed on the other surface. A distinguishing feature of the approach being taken here toward examining the interactions in such a system is the wave form of representation of the solution for the hydroelastic subsystem, since its infinite extent precludes the use of an expansion in the corresponding natural modes and frequencies. Thus, resonance methods of investigation cannot be employed.

We will examine an elastic plate of thickness h and density ρ_0 (E is the elastic modulus; ν is the Poisson's ratio). The middle surface of the plate coincides with the plane $z = 0$ (Fig. 1). Let the half-space $z < 0$ be filled by a fluid of the density ρ. The speed of sound in the fluid is c. We assume that the vibrations of the plate are excited by a force exerted along the straight line $x = 0$. This force is generated by a rotating shaft with an eccentric. The eccentric is located on the shaft in such a way that the "linear" value of the vertical component of the force is equal to $ma \cdot d^2/dt^2\,(1 - \cos\Theta)$ (where m is the linear mass of the eccentric; a is the eccentricity; Θ is the angle of roation reckoned from the vertical). The shaft is rotated by an electric motor of limited power [3], i.e., the power of the motor is commensurate with the required load (which oscillates in the fluid along with the plate). If the moment of inertia of the shaft per unit length is equal to I and the driving moment is equal to $M(\dot{\Theta})$, then the equation of motion of the shaft can be written as follows with allowance for the effect of the bending vibrations of the plate w(x, t)

$$I\ddot{\Theta} = M(\dot{\Theta}) + ma\left[g + \frac{\partial^2 w(0,t)}{\partial t^2}\right]\sin\Theta, \tag{1}$$

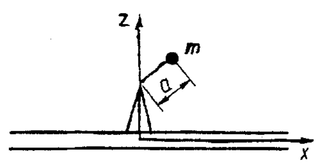

Fig. 1

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where g is acceleration due to gravity.

The equations of the bending vibrations of the plate will be represented in the form [5]

$$D\frac{\partial^4 w(x,t)}{\partial x^4} = -\rho_0 h\frac{\partial^2 w(x,t)}{\partial t^2} - ma\delta(x)\frac{d^2}{dt^2}\cos\Theta + p(x,0,t). \tag{2}$$

Here, $D = Eh^3/12(1-\nu^2)$; $\delta(x)$ is the Dirac delta function; $p(x, 0, t)$ is the sound pressure in the fluid. For the latter, we have the wave equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{3}$$

at $z < 0$.

We will also take into account the boundary condition at $z = 0$, i.e., the equality of the velocities of the plate $\partial w/\partial t$ and the fluid v_z

$$\frac{\partial w(x,t)}{\partial t} = v_z\Big|_{z=0}. \tag{4}$$

Using a relation for the acoustic field in the form $\partial p(x, z, t)/\partial z = -\rho\,\partial v_z/\partial t$, we obtain the following equation from condition (4)

$$-\rho\frac{\partial^2 w(x,t)}{\partial t^2} = \frac{\partial p(x,z,t)}{\partial z}\Big|_{z=0}. \tag{5}$$

System of coupled equations (1)-(3) and (5) describes the complex process of the transfer of energy from the electric motor to the acoustic field of the fluid and the oscillating plate. Using the Laplace transform for time and the Fourier transform for the coordinate x, we obtain general expressions for the deflection and pressure in the form

$$w(x,t) = \frac{1}{(2\pi)^2 i}\int_{\sigma-i\infty}^{\sigma+i\infty}\int_{-\infty}^{\infty} u(k,s)\,e^{ikx}e^{st}\,dk\,ds;$$
$$p(x,z,t) = \frac{1}{(2\pi)^2 i}\int_{\sigma-i\infty}^{\sigma+i\infty}\int_{-\infty}^{\infty} \xi(k,s)\,e^{ikx}e^{z\sqrt{k^2+s^2c^{-2}}}e^{st}\,dk\,ds. \tag{6}$$

Satisfaction of boundary condition (5) makes it possible to establish the relationship between the transforms. Specifically

$$\xi(k,s) = -\frac{\rho s^2 u(k,s)}{\sqrt{k^2+\frac{s^2}{c^2}}}. \tag{7}$$

Substituting (6) into (2) and using Eq. (7), we express the function u(k, s) through the transform $\Phi(s)$ of the function $ma\cdot d^2/dt^2\cos\Theta$

$$u(k,s) = \frac{\Phi(s)}{\Delta(k,s)}, \tag{8}$$

where

$$\Phi(s) = -ma\int_0^{\infty}\frac{d^2}{dt^2}\cos\Theta e^{st}\,dt;$$
$$\Delta(k,s) = Dk^4 + \rho_0 h s^2 + \frac{\rho s^2}{\sqrt{k^2+s^2c^{-2}}}. \tag{9}$$

Thus, we write the solution (6) in the form

$$w(x,t)=\frac{1}{(2\pi)^2 i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\int\limits_{-\infty}^{\infty}\frac{\Phi(s)}{\Delta(k,s)}e^{ikx}e^{st}dkds;$$

$$p(x,z,t)=-\frac{1}{(2\pi)^2 i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\int\limits_{-\infty}^{\infty}\frac{\rho s^2\Phi(s)e^{ikx}}{\Delta(k,s)\sqrt{k^2+\dfrac{s^2}{c^2}}}e^{z\sqrt{k^2+s^2c^{-2}}}e^{st}dkds. \tag{10}$$

Here, $\Phi(s)$ can be established only after finding the law of change in the angle $\Theta(t)$, i.e., after solving the equation of rotation of the shaft of the motor (1). With allowance for (10), Eq. (1) is changed to the form

$$I\ddot{\Theta}=M(\dot{\Theta})+ma\left\{g-\frac{ma}{(2\pi)^2 i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\int\limits_{-\infty}^{\infty}\frac{s^2e^{st}}{\Delta(k,s)}\left[\int\limits_0^{\infty}e^{-st}\frac{d^2}{dt^2}\cos\Theta dt\right]dkds\right\}\sin\Theta. \tag{11}$$

The equation that is obtained is nonlinear relative to the variable Θ, making it expedient to use the Poincaré method. Here, we introduce the small positive parameter $\varepsilon = mag/I\Omega_x^2$, where Ω_x is the speed of rotation of the shaft on the stationary plate. Here,

$$M(\Omega_x)=0. \tag{12}$$

We will restrict ourselves to steady-state regimes, when the moment $M(\dot{\Theta})$ corresponds to the static characteristic of the energy source [3]; let $M(\dot{\Theta})/I = \varepsilon M_1(\dot{\Theta})$. Thus, for regimes close to the steady state, we can write Eq. (11) in the form

$$\ddot{\Theta}=\varepsilon M_1(\dot{\Theta})+\varepsilon\Omega_x^2\sin\Theta+\frac{\varepsilon q\sin\Theta}{(2\pi)^2 i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\int\limits_{-\infty}^{\infty}\frac{s^2e^{st}}{\Delta(k,s)}\left[\int\limits_0^{\infty}\frac{d^2}{dt^2}(1-\cos\Theta)e^{-st}dt\right]dkds. \tag{13}$$

Here

$$q=\frac{ma\Omega_x^2}{g}.$$

Using the substitution of variables $\dot{\Theta} = \Omega$, we seek the solution for the regimes of interest to us in the form of expansions

$$\Omega=\Omega_0+\varepsilon\{\ldots\};\quad \Theta=\Omega_0 t+\varepsilon\{\ldots\};$$

$$w(x,t)=w_0(x,t)+\varepsilon\{\ldots\}=\frac{\Omega_0^2 ma}{2\pi}\operatorname{Re}\left[e^{-i\Omega_0 t}\int\limits_{-\infty}^{\infty}\frac{e^{ikx}}{\Delta(k,i\Omega_0)}dk\right]+\varepsilon\left\{\ldots\right\};$$

$$p(x,z,t)=\frac{\Omega_0^4\rho ma}{2\pi}\operatorname{Re}\left[e^{-i\Omega_0 t}\int\limits_{-\infty}^{\infty}\frac{e^{ikx}e^{z\sqrt{k^2-\Omega_0^2c^{-2}}}}{\Delta(k,i\Omega_0)\sqrt{k^2-\Omega_0^2c^{-2}}}dk\right]+\varepsilon\left\{\ldots\right\}. \tag{14}$$

Here, Ω_0 is the velocity associated with the steady-state regimes; Re is the real part.

Let us take a closer look at the deflection function $w_0(x, t)$

$$w_0(x,t)=\frac{\Omega_0^2 ma}{2\pi}\operatorname{Re}\left[e^{-i\Omega_0 t}\int\limits_{-\infty}^{\infty}\frac{e^{ikx}}{\Delta(k,i\Omega_0)}dk\right].$$

In accordance with the radiation conditions [1]

$$(k^2-k_0^2)^{1/2}=\begin{cases}(k^2-k_0^2)_p^{1/2} & |k|\geqslant k_0;\\ -i(k_0^2-k^2)_p^{1/2} & |k|\leqslant k_0.\end{cases} \tag{15}$$

We then write the Fourier integral in the form

$$w_0(x,t)=\frac{\Omega_0^2 ma}{2\pi D}\,\mathrm{Re}\left\{e^{-i\Omega_0 t}\left[\int\limits_{|k|\leqslant k_0}\frac{e^{ikx}}{k^4-k_1^4-\dfrac{i\rho k_1^4}{\rho_0 h\,(k_0^2-k^2)_p^{1/2}}}\,dk+\right.\right.$$

$$\left.\left.+\int\limits_{|k|\geqslant k_0}\frac{e^{ikx}}{k^4-k_1^4-\dfrac{\rho k_1^4}{\rho_0 h\,(k^2-k_0^2)_p^{1/2}}}\,dk\right]\right\}. \qquad (16)$$

Here, we used the notation: k_1 is the wave number of the free plate $k_1=\left(\Omega_0^2\frac{\rho_0 h}{D}\right)^{1/4}$; k_0 is the acoustic wave number of the fluid $k_0 = \Omega_0/c$. Meanwhile, only positive values of the root, designated as $(\;)_p^{1/2}$, should be taken in the integrands. Integration is done in (16) only for the real values of k, including those where the integrands have singularities. The first integral in (16) is a natural integral. The second integral presents a much more difficult problem. It has two intervals of integration $-\infty < k \leq -k_0$ and $k_0 \leq k < \infty$. Figure 2a shows the location of its singular points. In fact, they are determined from the equation $k^4-k_1^4=\frac{\rho k_1^4}{\rho_0 h\,(k^2-k_0^2)_p^{1/2}}$. In Fig. 2a, the left side of this equation is shown by curves 1, while the right side is shown by curves 2. The points of intersection of the curves at $k = \pm k_2$ correspond to singularities of the integrand. It follows from Fig. 2a that k_2 is greater than k_0 and k_1. Here, the singularity at the points $\pm k_2$ is of the simple pole type. As before, the definition of the second integral of (16) at $k \neq \pm k_2$ is not clear. To find it analytically, we continue the integrand on the complex plane. Here, it must be taken into account that the function $(k^2 - k_0^2)^{1/2}$ will be two-valued, while the points $\pm k_0$ will be branch points. Thus, it will be necessary to make cuts through them. In particular, cuts can be made along the lines $\mathrm{Re}(k^2 - k_0^2)^{1/2} = 0$, while, in accordance with conditions (15), integration is performed over the surface with $\mathrm{Re}(k^2 - k_0^2)^{1/2} \geq 0$.

We want to take the integral (16) over the real axis k and express it through an integral over a closed contour in the complex plane by using the Cauchy theorem. Thus, we need to determine all of the singular points k in the complex plane. We designate $\gamma = (k^2 - k_0^2)^{1/2}$. Then the singularities can be found if we find γ as the root of the equation [7]

$$\gamma^5+2k_0^2\gamma^3+(k_0^4-k_1^4)\,\gamma-\frac{\rho k_1^4}{\rho_0 h}=0, \qquad (17)$$

obtained from the condition $\Delta(k, i\Omega_0) = 0$. Stewart [8] showed that this equation has either one positive root and two pairs of conjugate roots for a frequency Ω_0 above the coincidence frequency $\omega_C = c^2(\rho_0 h/D)^{1/2}$ or one positive real root, two negative real roots, and one pair of complex conjugate roots for Ω_0 below the coincidence frequency. Since Eq. (2) yields larger errors even at $\Omega_0 = 0.7\omega_C$ [6], we will consider only the second case. In accordance with the Routh–Hurwitz conditions, a pair of complex roots will lie in the first and fourth quadrants and have positive $\mathrm{Re}\gamma$. Each γ corresponds to two values $k = \pm(\gamma^2 + k_0^2)^{1/2}$. Figure 2b shows the location of the roots of Eq. (17). Here, γ_3 and γ_4 correspond to $\pm k_3$ and $\pm k_4$, not satisfying radiation conditions (15) and not lying on the chosen plane of integration. Thus, the singularities are located at the points $\pm k_2$, $\pm k_5$, $\pm k_5^*$ (where * denotes the taking of the conjugate value).

Calculating (15) for positive x, we need to close the contour of integration in the upper half-plane [1]. Similar to the case of negative x, the contour is already closed in the lower half-plane k. Figure 2c shows the contour of integration. Meanwhile, to determine the path around the poles $\pm k_2$ and the branch points $\pm k_0$, we need to introduce an infinitesimal damping into the plate and analyze the displacement of the poles. The Cauchy theorem gives the following expression for the displacements $w_0(x, t)$

$$w_0(x,t)=\frac{\Omega_0^2 ma}{2\pi}\,\mathrm{Re}\left\{e^{-i\Omega_0 t}2\pi i\left[\frac{e^{ik_2x}}{\Delta'(k_2, i\Omega_0)}+\frac{e^{ik_5x}}{\Delta'(k_5, i\Omega_0)}+\right.\right.$$

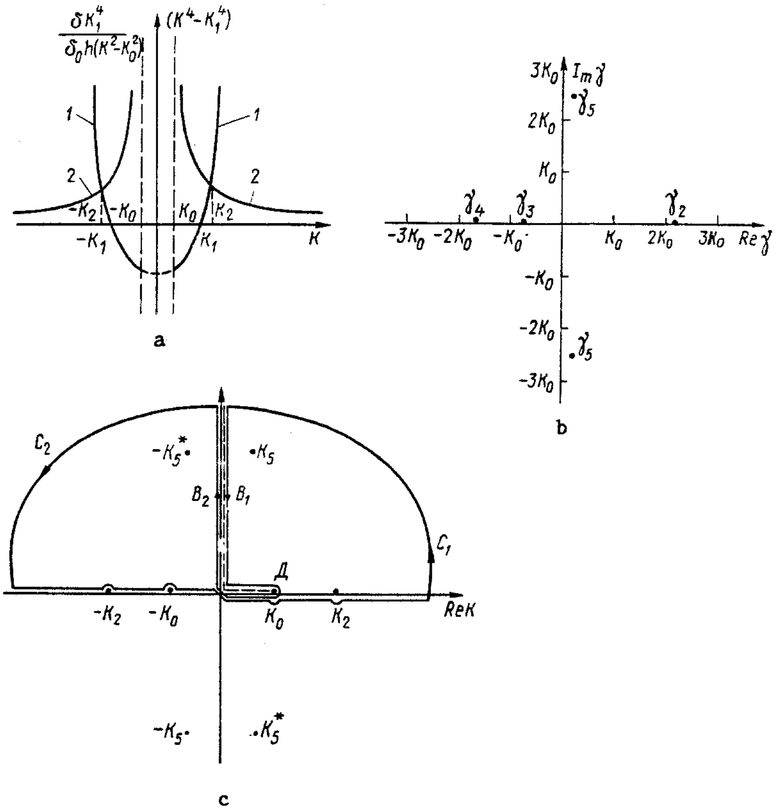

Fig. 2

$$+\frac{e^{-ik_5^{*}x}}{\Delta'(-k_5^{*}, i\Omega_0)}\Bigg] - e^{-i\Omega_0 t}\int\limits_{C_1+B_1+D_1+B_2+C_2}\frac{e^{ikx}}{\Delta(k, i\Omega_0)}\,dk\Bigg\}; \quad x\geqslant 0. \tag{18}$$

The first three terms in (18) are the residues at the poles. In accordance with the Jordan lemma, the integrals over C_1 and C_2 are equal to zero. The integral over D_1 around the point k_0 approaches zero as the radius of the path approaches zero. The integrals over B_1 and B_2 correspond to the integrals over the edges of the cut. It is easily shown that on the Riemann on which we integrate $\mathrm{Re}(k^2 - k_0^2)^{1/2} \geq 0$, we should regard $B_1(k^2 - k_0^2)^{1/2}$ as $i(k_0^2 - k^2)_p^{1/2}$ along B_1 and as $-i(k_0^2 - k^2)_p^{1/2}$ along B_2.

Thus,

$$\int\limits_{B_1+B_2}\frac{e^{ikx}}{\Delta(k, i\Omega_0)}\,dk = -2i\int\limits_0^{k_0} P(k)\,e^{ikx}dk + 2i\int\limits_0^{i\infty} P(k)\,e^{ikx}dk, \tag{19}$$

where

$$P(k) = \frac{\rho k_1^4 (k_0^2 - k^2)_p^{1/2}}{D\rho_0 h\left[(k^4 - k_1^4)^2 (k_0^2 - k^2) + \left(\frac{\rho k_1^4}{\rho_0 h}\right)^2\right]}$$

Since the sum of the residues at the points k_5 and $-k_5^*$ is equal to $-4\pi\,\mathrm{Im}\frac{e^{k_5 x}}{\Delta'(k_5, i\Omega_0)}$ (i.e., $\Delta'(k_5, i\Omega_0) = D\left[4k_5^3 + \frac{\rho k_1^4}{\rho_0 h}k_5(k_5^2 - k_0^2)_p^{-3/2}\right]$), then the deflection $w_0(x, t)$ has the form

$$w_0(x, t) = \frac{\Omega_0^2 ma}{2\pi}\,\mathrm{Re}\left\{e^{-i\Omega_0 t}\left[2\pi i\frac{e^{ik_2 x}}{\Delta'(k_2, i\Omega_0)} - \right.\right.$$

$$\left.\left. - 4\pi\,\mathrm{Im}\frac{e^{ik_5 x}}{\Delta'(k_5, i\Omega_0)} + 2i\int\limits_0^{k_0} P(k)\,e^{ikx}dk - 2i\int\limits_0^{i\infty} P(k)\,e^{ikx}dk\right]\right\} = w_1(x)\cos\Omega_0 t + w_2(x)\sin\Omega_0 t.$$

$$w_1(x) = -\Omega_0^2 ma\left[\frac{\sin k_2 x}{\Delta'(k_2, i\Omega_0)} + 2\mathrm{Im}\frac{e^{ik_5 x}}{\Delta'(k_5, i\Omega_0)} - \right.$$

$$\left. - \frac{1}{\pi}\int_0^\infty P(i\eta)e^{-\eta x}d\eta\right]; \quad w_2(x) = \Omega_0^2 ma\left[\frac{\cos k_2 x}{\Delta'(k_2, i\Omega_0)} + \frac{1}{\pi}\int_0^{k_0} P(k)e^{ikx}dk\right]. \tag{20}$$

It follows from the above solution that the deflection of the plate $w_0(x, t)$ is described by a frequency function Ω_0 of complex form. To find the values of Ω_0 realized in steady-state regimes, we need to solve the equation obtained from (1) by the substitution of Eqs. (14) and (20) and subsequent averaging over the period $2\pi/\Omega_0$. Specifically, we need to solve the equation

$$\varepsilon M_1(\Omega_0) - \varepsilon\frac{q\Omega_0^2}{2ma} w_2(0) = 0. \tag{21}$$

Equation (21) is a ratio of moments: the moment of the energy source $\varepsilon M_1(\Omega_0)$ and the vibrational moment $-\varepsilon g\Omega_0^2/2ma\cdot w_2(0)$ created on the shaft of the motor by the hydroelastic subsystem. Since the vibrational moment is a complex nonlinear function, Eq. (21) can only be solved numerically. For a physical generalization of the numerical solution, we introduce the dimensionless quantities [2]

$$X = \frac{\Omega_0}{\omega_c}; \quad y = \frac{k}{k_0}; \quad y_2 = \frac{k_2}{k_0}; \quad b = \frac{\rho D^{1/2}}{c(\rho_0 h)^{3/2}}. \tag{22}$$

As a result, the equation being examined (with approximation of the function $\varepsilon M_1(\Omega_0)$ by a linear function of the form $\varepsilon M_1(\Omega_0) = \frac{N^0}{I} - \frac{N_1}{I}\Omega_0\Big)$ is written in the following form

$$N(X_x - X) = XT(X, b),$$

where

$$N = \frac{2N_1 D}{(ma)^2 c^3}; \; X_x = \frac{\Omega_x}{\omega_c}; \; T(X, b) = \frac{1}{4y_2^3 + \dfrac{by_2}{X^3(y_2^2 - 1)^{3/2}}} +$$

$$+ \frac{b}{\pi X^3}\int_0^1 \frac{(1-y^2)^{1/2}}{(y^4 - X^{-2})^2(1-y^2) + b^2 X^{-6}}\,dy. \tag{23}$$

Figure 3 shows graphs of the functions $N(X_x - X)$ and $XT(X, b)$ corresponding to the dimensionless driving and vibrational moment for $X < 0.7$ in the case of steel (curves 1 and 2; $b = 0.13$; $N = 0.057$; $X_x(^1) = 0.497$) and aluminum (curves 3 and 4; $b = 0.39$; $N = 0.02$; $X_x(^2) = 0.5$) plates of equal thickness h in contact with water and excited by electric motors having the same moment $M(\dot{\Theta})$. It should be noted that the choice of the frequency range $0 \leq X \leq 0.7$ was dictated by the range of application of the plate vibration equation (2). When $X > 0.7$, it is necessary to change over to more complex models of the process. It is evident from Fig. 3 that the vibrational moment is a monotonically increasing function of the frequency of vibration. This accounts for the fact that, with a decreasing static characteristic for the motor, Eq. (23) has only the solution $X = \Omega_0/\omega_c$. The latter in turn means that there is a unique regime of interaction of the motor and the vibrating plate. Here, $X_{1,2} < X_x(^{1,2})$. This result is not due to the consumption of energy by the plate for internal damping and by the fluid, with allowance for its viscosity [4]. Instead, it is due exclusively to transfer of the energy of the motor by uniform waves in the plate and the fluid and by the "resultant" waves [1]. The reduction in the speed of rotation of the shaft will be greater in the case of interaction of the motor with the aluminum plate, i.e., the amplitudes of the vibrations will be greater than in the case of a steel plate. As a result, an aluminum plate in contact with a fluid creates a larger vibrational moment, and the relation $X_2 < X_1$ is satisfied.

Thus, in contrast to a finite system, the interaction of the vibrations of a fluid-loaded infinite plate and an energy source of limited power does not have resonance regimes. As a result, the speed of rotation of the motor shaft does not have more than one value in

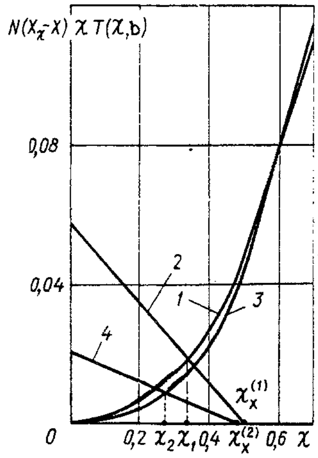

Fig. 3

steady-state regimes of interaction. In such regimes, moreover, a finite system exerts a reciprocal effect on the energy source only if the source is damped. Infinite systems create a vibrational moment without damping, i.e., the energy of the source is transported by travelling waves.

LITERATURE CITED

1. V. T. Grinchenko and V. V. Meleshko, Harmonic Vibrations and Waves in Elastic Bodies [in Russian], Naukova Dumka, Kiev (1981).
2. L. Ya. Gutin, "Acoustic radiation of an infinite plate excited by a concentrated force normal to the plate," Akust. Zh., 10, No. 4, 369-371 (1964).
3. V. O. Kononenko, Vibrating Systems with Finite Excitation [in Russian], Nauka, Moscow (1964).
4. K. V. Frolov and T. S. Krasnopol'skaya, "Forced vibrations of a circular plate in a fluid with finite excitation," Mashinovedenie, No. 6, 28-34 (1981).
5. D. Feit and Y. N. Lin, "The nearfield response of a line-driven fluid-loaded plate," J. Acoust. Soc. Am., 78(2), August, 763-766 (1985).
6. M. C. Junger and D. Feit, Sound, Structures, and Their Interaction, MIT, Cambridge, MA (1972).
7. W. A. Strawderman, Ko Sung-Hwan, and A. H. Nuttale, "The real roots of the fluid-loaded plate," J. Acoust. Soc. Am., 66(2), 579-585 (1979).
8. A. D. Stuart, "Acoustic radiation from submerged plates," J. Acoust. Soc. Am., 60, 1160-1169 (1976).