NUMERICAL DETERMINATION OF THE STRESS CONCENTRATION AROUND A HOLE IN A CIRCULAR CYLINDRICAL SHELL

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1. Let us represent the stress state of a shell with a hole [1, 4] as the sum of a fundamental and perturbed state. The perturbed state is described by a theory of shells with a high index of variability

$$\Delta\Delta\sigma + 8i\beta^2 \frac{\partial^2\sigma}{\partial x^2} = 0, \tag{1.1}$$

where x, y are the generator and directrix of the middle surface of a shell of radius R, Δ is the Laplace operator, $4\beta^2 = \sqrt{3(1-\nu^2)}R^{-1}h^{-1}$; h is the shell thickness, and ν is the Poisson ratio.

We write the expression of the deflection w(x, y), the rotations, stress resultants and moments in the shell [4] in terms of the stress function $\sigma(x, y)$ as

$$Ehw = 4\beta^2 R(\sigma + \bar{\sigma});\ Eh(\vartheta_x - i\vartheta_y) = 8\beta^2 R \frac{\partial}{\partial z}(\sigma + \bar{\sigma}); \tag{1.2}$$

$$T_x + T_y - \frac{8i\beta^2 R}{1+\nu}(G_x + G_y) = i\Delta\sigma;\ T_y - T_x + 2iS_{xy} = 2i\frac{\partial^2}{\partial z^2}(\sigma - \bar{\sigma});$$

$$G_y - G_x + 2iH_{xy} = \frac{1-\nu}{4\beta^2 R}\frac{\partial^2}{\partial z^2}(\sigma + \bar{\sigma});\ Q_x - iQ_y = -\frac{1}{8\beta^2 R}\frac{\partial}{\partial z}\Delta(\sigma + \bar{\sigma}). \tag{1.3}$$

The upper bar here denotes the complex conjugate, and $2(\partial/\partial z) = \partial/\partial x - i(\partial/\partial y)$.

Let us represent the solution of (1.1) by a series with constant $a_n^{\pm}$ [6, 7]:

$$\sigma = e^{i\alpha x}\sum_{-\infty}^{\infty} i^n a_n^+ H_n^{(1)}(\alpha r)e^{in\vartheta} + e^{-i\alpha x}\sum_{-\infty}^{\infty} i^{-n} a_n^- H_n^{(1)}(\alpha r)e^{in\vartheta}, \tag{1.4}$$

where $H_n^{(1)}(\alpha r)$ is the Hankel function and $z = re^{i\vartheta}$; $\alpha = (1+i)\beta$.

For the derivatives of the function $\sigma(x, y)$ we obtain

$$\frac{\partial\sigma}{\partial z} = \frac{\alpha}{2}\sigma(b_n^{\pm});\ \frac{\partial\sigma}{\partial\bar{z}} = \frac{\alpha}{2}\sigma(b_{n-1}^{\pm});\ \pm b_n^{\pm} = ia_{n+1}^{\pm} + ia_n^{\pm}, \tag{1.5}$$

where $\sigma(b_{n-1}^{\pm})$ denotes (1.4) in which the a_n^+ and a_n^- are replaced by b_{n-1}^+ and b_{n-1}^-.

It follows from (1.5) that

$$\Delta\sigma = 4\frac{\partial^2\sigma}{\partial z\partial\bar{z}} = \alpha^2\sigma(c_n^{\pm});\ c_n^{\pm} = -a_{n+1}^{\pm} - 2a_n^{\pm} - a_{n-1}^{\pm}. \tag{1.6}$$

If the function $\sigma(x, y)$ possesses symmetry of the type $\cos 2n\vartheta$ according to the nature of the problem, then the dependences $a_n^+ = a_n^- = a_{-n}^{\pm}$ are valid; where for $\cos(2n+1)\vartheta$ we have $a_n^+ = -a_n^- = a_{-n}^+$ while for $\sin(2n+1)\vartheta$ we have $a_n^+ = -a_n^- = -a_{-n}^{\pm}$ and in the case $\sin 2n\vartheta$ we have $a_n^+ = a_n^- = -a_{-n}^{\pm}$.

2. Let us give the equation of the closed curve L on the middle surface in the parametric form

$$r = r(\lambda);\ \vartheta = \vartheta(\lambda)\ (0 \leqslant \lambda \leqslant \lambda_0). \tag{2.1}$$

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The parameter λ increases as the boundary L is traversed in such a direction $\vec{\tau}$ that the external normal to the middle surface $\vec{\nu}$ is directed to the right of the boundary L seen from the external surface of the shell. Henceforth, differentiation is with respect to the arclength s of the boundary L.

From the relation (2.1) we obtain

$$ds = \sqrt{[r\vartheta'(\lambda)]^2 + [r'(\lambda)]^2}\, d\lambda. \tag{2.2}$$

In particular, we have for the angle γ between the x axis and the normal $\vec{\nu}$

$$e^{i\gamma} = -i\frac{dz(\lambda)}{ds}. \tag{2.3}$$

We find from the dependences (2.1), (2.2), (2.3)

$$e^{i\gamma} = \frac{r\vartheta'(\lambda) - ir'(\lambda)}{\sqrt{[r\vartheta'(\lambda)]^2 + [r'(\lambda)]^2}}\, e^{i\vartheta}. \tag{2.4}$$

Let us represent the formulas to transform the stress resultants, the moments, and rotations [4] upon going over to the directions $\vec{\nu}$, $\vec{\tau}$ on the boundary L as

$$\begin{gathered} T_\nu + T_\tau = T_x + T_y;\ T_\tau - T_\nu + 2iS_{\nu\tau} = e^{2i\gamma}(T_y - T_x + 2iS_{xy}); \\ G_\nu + G_\tau = G_x + G_y;\ G_\tau - G_\nu + 2iH_{\nu\tau} = e^{2i\gamma}(G_y - G_x + 2iH_{xy}); \\ Q_\nu - iQ_\tau = e^{i\gamma}(Q_x - iQ_y);\ \vartheta_\nu - i\vartheta_\tau = e^{i\gamma}(\vartheta_x - i\vartheta_y). \end{gathered} \tag{2.5}$$

3. Let us define the constants $a_n^{\pm}$ from the boundary conditions for the perturbed state on the contour L. The boundary conditions of the first fundamental problem are [4]

$$T_\nu = f_1(\lambda);\quad S_{\nu\tau} = f_2(\lambda);\qquad G_\nu = f_3(\lambda);\qquad Q_\nu + \frac{dH_{\nu\tau}}{ds} = f_4(\lambda). \tag{3.1}$$

Let us substitute the expansion (1.4) into (1.3), and the expressions obtained for the stress resultants and moments into (2.5). Now, let us write down an explicit expression for the moment $H_{\nu\tau}$ in terms of the constants $a_n^{\pm}$. After differentiating it with respect to the arclength s, the collocation method can be used to form the infinite system of algebraic equations in $a_n^{\pm}$. Conditions (3.1) are hence satisfied exactly for a finite number of values of λ.

Another means [2] is to expand (2.5) in a Fourier series in the functions $e^{2\pi i n(\nu/\pi_0)}$. Integrals of the form

$$\int_0^{\lambda_0} H_n^{(1)}(\alpha r)\exp\left[i\left(n\vartheta \pm \alpha x + m\gamma - 2\pi k\frac{\lambda}{\lambda_0}\right)\right]d\lambda \qquad (m = 0, 1, 2;\quad k = 0, \pm 1, \pm 2, \ldots) \tag{3.2}$$

must be evaluated.

For arbitrary (2.1) the integrals (3.2) can be evaluated numerically, where it is convenient to use the method of Phelan [5].

After expansion, we obtain on the boundary L

$$\begin{gathered} T_\nu + T_\tau - \frac{8i\beta^2 R}{1+\nu}(G_\nu + G_\tau) = \sum_{-\infty}^{\infty} p_k e^{2\pi i k\frac{\lambda}{\lambda_0}};\qquad Q_\nu - iQ_\tau = \sum_{-\infty}^{\infty} q_k e^{2\pi i k\frac{\lambda}{\lambda_0}}; \\ T_\tau - T_\nu + 2iS_{\nu\tau} = \sum_{-\infty}^{\infty} t_k e^{2\pi i k\frac{\lambda}{\lambda_0}};\qquad G_\tau - G_\nu + 2iH_{\nu\tau} = \sum_{-\infty}^{\infty} g_k e^{2\pi i k\frac{\lambda}{\lambda_0}}. \end{gathered} \tag{3.3}$$

Here p_k, t_k, g_k, q_k are number series with coefficients $a_n^{\pm}$.

From the expansion (3.3) we find T_ν, $S_{\nu\tau}$, G_ν, Q_ν, $H_{\nu\tau}$ on the boundary L. For example

$$H_{\nu\tau} = \frac{i}{4}\sum_{-\infty}^{\infty}(\bar{g}_{-k} - g_k)\, e^{2\pi i k\frac{\lambda}{\lambda_0}}. \tag{3.4}$$

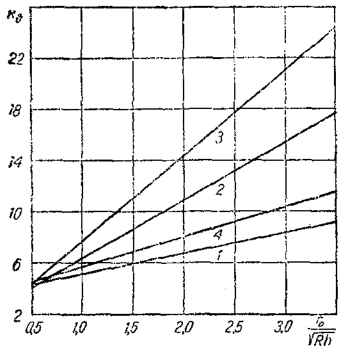

Fig. 1

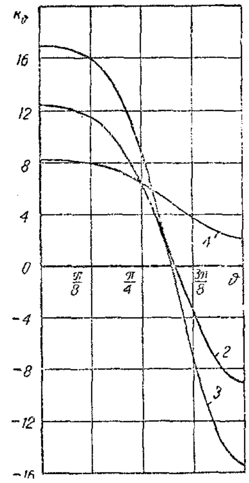

Fig. 2

To evaluate the derivative with respect to the arclength s, let us expand the expression

$$\{[r\vartheta'(\lambda)]^2 + [r'(\lambda)]^2\}^{-\frac{1}{2}} = \sum_{-\infty}^{\infty} \alpha_n e^{2\pi i n\frac{\lambda}{\lambda_0}} \tag{3.5}$$

and taking account of (2.2) and (3.5) we obtain

$$\frac{dH_{\nu\tau}}{ds} = \{[r\vartheta'(\lambda)]^2+[r'(\lambda)]^2\}^{-\frac{1}{2}}\frac{dH_{\nu\tau}}{d\lambda} = \frac{\pi}{2\lambda_0}\sum_{m=-\infty}^{\infty} e^{2\pi i m\frac{\lambda}{\lambda_0}} \sum_{k=-\infty}^{\infty} k\alpha_{m-k}(g_k - \bar{g}_{-k}). \tag{3.6}$$

Let us expand the right sides of conditions (3.1) into the Fourier series

$$f_k(\lambda) = \sum_{-\infty}^{\infty} \beta_{kn} e^{2\pi i n\frac{\lambda}{\lambda_0}}. \tag{3.7}$$

Substituting (3.3), (3.6), (3.7) into (3.1) and equating the expressions with identical exponentials, we obtain an infinite system of linear algebraic equations in $a_n^{\pm}$. Let us add [6, 7] the condition of self-equilibration of the stresses and uniqueness of the tangential displacements

$$\sum_{-\infty}^{\infty}(a_n^{+} - a_n^{-}) = 0 \tag{3.8}$$

to the infinite system.

4. The stresses in a shell with a free circular hole stretched uniformly by stress resultants distributed over the endfaces p, and in a shell with circular hole of radius r_0 loaded by uniform internal pressure q_0 and covered by a cap whose effect on the shell is replaced by just transverse forces, have been determined by the method elucidated above. The following transverse force distribution laws have been considered:

$$f_4(\lambda) = -\frac{q_0 r_0}{2} \tag{4.1}$$

$$f_4(\lambda) = -q_0 r_0 \sin^2\lambda; \tag{4.2}$$

$$f_4(\lambda) = -q_0 r_0 \cos^2\lambda. \tag{4.3}$$

The integrals (3.2) were evaluated by the Phelan method [5]. We put $r = r_0$; $\lambda = \vartheta$ in (2.1) for the circular hole. We find $\gamma = \vartheta$ from (2.4). The infinite system was solved by the method of reduction. The accuracy of the solution was estimated by the degree to which the expansion (1.4) satisfied the boundary conditions (3.1) after the values of the constants $a_n^{\pm}$ had been substituted. The calculations were performed on the "Minsk-22" electronic computer for values of the parameters $\nu = 0.3$; $r_0/\sqrt{Rh} = 0.5$-3.5.

The greatest stress concentration coefficients k_ϑ are shown in Fig. 1 as a function of the parameter $r_0/\sqrt{Rh}$. Curve 1 corresponds to axial tension on the shell. Hence

$$pk_\vartheta = T_\vartheta\left(r_0, \frac{\pi}{2}\right) - \frac{6}{h}G_\vartheta\left(r_0, \frac{\pi}{2}\right).$$

Curves 2, 3, 4 characterize loading by uniform pressure for transverse force distributions corresponding to (4.1), (4.2), (4.3), where

$$q_0 R k_\vartheta = T_\vartheta(r_0, 0) - \frac{6}{h} G_\vartheta(r_0, 0).$$

Presented in Fig. 2 are graphs of the change in the coefficients k_ϑ along the hole contour ($r_0/\sqrt{Rh} = 2.5$). As is seen, the case (4.3) turns out to be the best of the transverse force distribution versions considered.

The numerical values of the stresses agree well with those presented in [3, 7].

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