TORSIONAL VIBRATIONS OF A SYSTEM WITH HOOKE'S JOINT

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We consider torsional vibrations of a system whose dynamic model is shown in Fig. 1. Here we have used the following notation: 1 and 2, driving and driven shafts of the system; 3, Hooke's joint connecting these shafts; ω_1, angular velocity of the end of the driving shaft; s_1 and s_2, rigidities of the driving and the driven shafts; I_1 and I_2, moments of inertia of the reduced driving and driven masses of the system; λ and β, the angles of rotation of the driving and driven forks of joint 3; θ, the angle of rotation of the driven mass; γ, the angle between shafts 1 and 2; M_{fr}, the moment of friction in the joint referred to the driving shaft; M_f, the moment of forces applied to the driven shaft 2.

In setting up and in the analysis of the model we have assumed that $I_1 \gg I_2$, the moments of inertia of shafts 1 and 2 are small compared with I_2, and M_f = const.

This problem has been solved in many studies, for example, [7, 8], in an analogous formulation; however, these investigations have been carried out mainly on the basis of the linear model and also without taking into consideration certain factors which are typical for such systems. Such a simplified approach does not permit one to display many interesting phenomena which can be observed in such systems. In the present article the vibrations of the system are investigated on the basis of the nonlinear model using the asymptotic methods developed by Krylov, Bogolyubov and Mitropol'skii [2].

The motion of the investigated system under the action of a torsion moment producing the rotation of the driving shaft and taking the viscous damping and the moment applied to the driven shaft into consideration can be described by the following differential equation with a known degree of accuracy:

$$I_2\theta'' + s_2(\theta - \beta) + c(\theta' - \beta') + M_f = 0, \tag{1}$$

where the prime denotes differentiation with respect to time and c is the viscous damping constant.

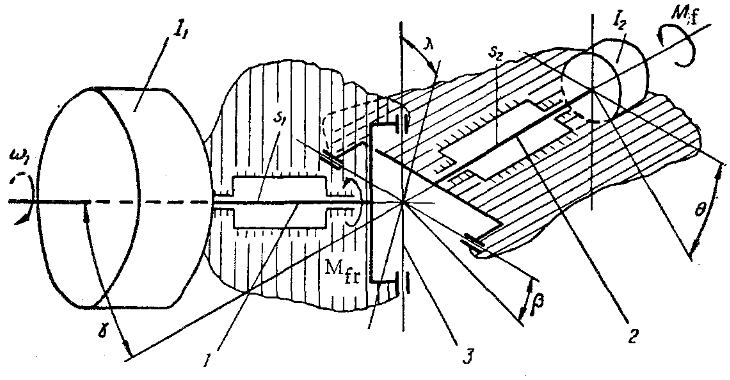

Fig. 1

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In the presence of energy losses in the joint the following equation holds:

$$s_1(\omega_1 t-\lambda)\lambda' - M_{\mathrm{fr}}\lambda' = s_2(\beta-\theta)\beta'. \tag{2}$$

Keeping in mind the well-known equation of the dynamics of Hooke's joint

$$\frac{\beta'}{\lambda'}=\frac{\cos\gamma}{1-\sin^2\gamma\cos^2\lambda} \tag{3}$$

and Eq. (2), and introducing new variables $\varphi_1 = \omega_1 t$ and $x = \lambda - \varphi_1$, Eq. (1) becomes

$$I_2\omega_1^2\ddot{x}\left(f+\frac{s_1}{s_2}F'_\lambda x+\frac{s_1}{s_2}F\right)+2I_2\omega_1^2\frac{s_1}{s_2}F'_\lambda\dot{x}(\dot{x}+1)+I_2\omega_1^2(\dot{x}+1)^2\times$$

$$\times\left(f'_\lambda+\frac{s_1}{s_2}F''_{\lambda\lambda}x\right)+c\omega_1\frac{s_1}{s_2}F'_\lambda(\dot{x}+1)+c\omega_1\frac{s_1}{s_2}F\dot{x}+s_1Fx+M_{\mathrm{f}}=0. \tag{4}$$

Here

$$f=f(\lambda,\gamma)=\cos\gamma(1-\sin^2\gamma\cos^2\lambda)^{-1};$$

$$F=F(\lambda,\gamma)=\frac{1}{f}\left\{2-\left[1-g_0\,\mathrm{tg}\,\gamma\left(\frac{\cos\gamma\cos\lambda}{\sqrt{1-\sin^2\gamma\cos^2\lambda}}-\sin\lambda\right)\right]^{-1}\right\};$$

g_0 is a constant depending on the parameters of the joint [4, 5]: the dot denotes differentiation with respect to the angle of rotation φ_1 of the driving shaft.

Expanding functions $f(\lambda, \gamma)$, $F(\lambda, \gamma)$, and their derivatives in power series of x and $\mu_1 = \tan(\gamma/2)$ we substitute the results into Eq. (4). Next, multiplying both sides of the obtained equation by $s_2/I_2\omega_1^2(s_1+s_2)$, we write this equation in the following form:

$$\ddot{x}+k^2(x+x_0)=\mu f_1(x,\dot{x},\ddot{x},\varphi_1)+\mu^2 f_2(x,\dot{x},\ddot{x},\varphi_1)+\ldots. \tag{5}$$

Here we have retained second-order nonlinearity and have introduced the following notation:

$$\frac{s_2}{s_1+s_2}=\rho;\qquad \sqrt{\frac{s_1s_2}{I_2(s_1+s_2)}}=\omega_{\mathrm{r}};\qquad \frac{\omega_{\mathrm{r}}}{\omega_1}=k;\qquad \frac{c}{2I_2\omega_1}=\nu\mu_1^2;$$

$$g_0=g\mu_1;\qquad x_0=\frac{M_{\mathrm{f}}}{s_1};\qquad \mu_1^2=\mu, \tag{6}$$

where ω_{r} is the eigenfrequency of the "rectified system," i.e., the system in which $\gamma = 0$ (see Fig. 1); μ is a quantity taken as the small parameter.

Following the asymptotic methods of [2], we find the solution of Eq. (5) with an accuracy up to quantities of second order of smallness in the following form:

$$x+x_0=a\cos\psi+\mu\sum_{m=1}^{2}\sum_{n=-3}^{3}\alpha_{mn}(a)\cos(m\varphi_1+n\psi)+\beta_{mn}(a)\sin m\varphi_1+n\psi), \tag{7}$$

where the amplitude a and phase ψ of the first harmonic of the vibrations are determined from the system

$$\dot{a}=-\mu a\left\{\nu k(1-\rho)-\frac{32x_0\mu\rho(3-4\rho)}{k^2-4}+6x_0\mu k^2a^2\left[\frac{3g^2(1-\rho)^2}{4k^2-1}+\frac{(2-3\rho)(3-4\rho)}{k^2-1}\right]\right\}; \tag{8}$$

$$\dot{\psi}=k+\mu^2B_2(a); \tag{9}$$

$\alpha_{mn}(a)$, $\beta_{mn}(a)$, and $B_2(a)$ are expressions which are proper functions of the amplitude a and the parameters of the system.

Equating the right-hand side of Eq. (8) to zero, we obtain an equation for determining the stationary values of the amplitudes of vibrations; this equation admits of two solutions:

$$a = 0 \tag{10}$$

and

$$a^2 = \frac{(4k^2-1)(k^2-1)[32x_0\mu\rho(3-4\rho) - \nu k(1-\rho)(k^2-4)]}{6x_0\mu k^2(k^2-4)[3g^2(1-\rho)^2(k^2-1)+(2-3\rho)(3-4\rho)(4k^2-1)]}. \tag{11}$$

An investigation of the stability of the obtained solutions shows that the inequality

$$\nu k(1-\rho) > \frac{32x_0\mu\rho(3-4\rho)}{k^2-4} \tag{12}$$

is the condition of stability of the heteroperiodic regime (10) and the condition of instability of self-oscillatory regime (11), while inequality

$$\nu k(1-\rho) < \frac{32x_0\mu\rho(3-4\rho)}{k^2-4} \tag{13}$$

characterizes, respectively, the instability and stability of the indicated regimes.

In the resonance case we seek the solution of Eq. (5) in the form

$$x + x_0 = a\cos(k\varphi_1 + \vartheta) + \mu u_1(a, k\varphi_1, \vartheta) + \dots . \tag{14}$$

Here the variables a and ϑ are determined from the system

$$\dot{a} = \mu A_1(a,\vartheta) + \mu^2 A_2(a,\vartheta) + \dots; \quad \dot{\vartheta} = \mu B_1(a,\vartheta) + \mu^2 B_2(a,\vartheta) + \dots . \tag{15}$$

Following [2], in the first approximation we can detect the appearance of resonances of the form

$$k \approx \frac{1}{3}; \quad \frac{2}{3}; \quad \frac{1}{2}; \quad 1; \quad 2.$$

We shall analyze the cases of fundamental resonance of the system: $k \approx 1$ and $k \approx 2$.

$k \approx 1$. Assuming that $k^2 = 1 + \Delta\mu$ we write the solution in the form

$$x + x_0 = a\cos(\varphi_1 + \vartheta), \tag{16}$$

where a and ϑ are determined by the system of the first approximation,

$$\begin{aligned} \dot{a} &= \mu\left[\frac{1}{4}g\rho a^2(\cos\vartheta + \sin\vartheta) + x_0\rho g(\cos\vartheta - \sin\vartheta) - a\rho\sin 2\vartheta - 3x_0 a\rho\cos 2\vartheta - a\nu(1-\rho)\right]; \\ a\dot{\vartheta} &= \mu\left[\frac{3}{4}g\rho a^2(\cos\vartheta - \sin\vartheta) + x_0\rho g(\cos\vartheta + \sin\vartheta) - a\rho\cos 2\vartheta + 3x_0 a\rho\sin 2\vartheta + \frac{\Delta}{2}a\right]. \end{aligned} \tag{17}$$

System (17) can be investigated by the qualitative methods of the phase plane [1, 2, 3]. Since the right-hand sides of Eqs. (17) are periodic functions of variable ϑ, the phase space of the system can be assumed to be the cylindrical surface (a, ϑ) with the identifying edges $\vartheta = -\pi$ and $\vartheta = \pi$, where ϑ is the angular coordinate.

In view of the unwieldy nature of the computations for the general case of systems of type (17) and also considering the smallness of parameter x_0, we carry out the analysis for the limiting case $x_0 = 0$. Thus, as the stationary solutions of (17) we get

$$a = 0; \quad \cos 2\vartheta = \Delta_1 \tag{18}$$

and

$$g_1 a(\cos\vartheta + \sin\vartheta) - \sin 2\vartheta - \nu_1 = 0; \quad 3g_1 a(\cos\vartheta - \sin\vartheta) - \cos 2\vartheta + \Delta_1 = 0. \tag{19}$$

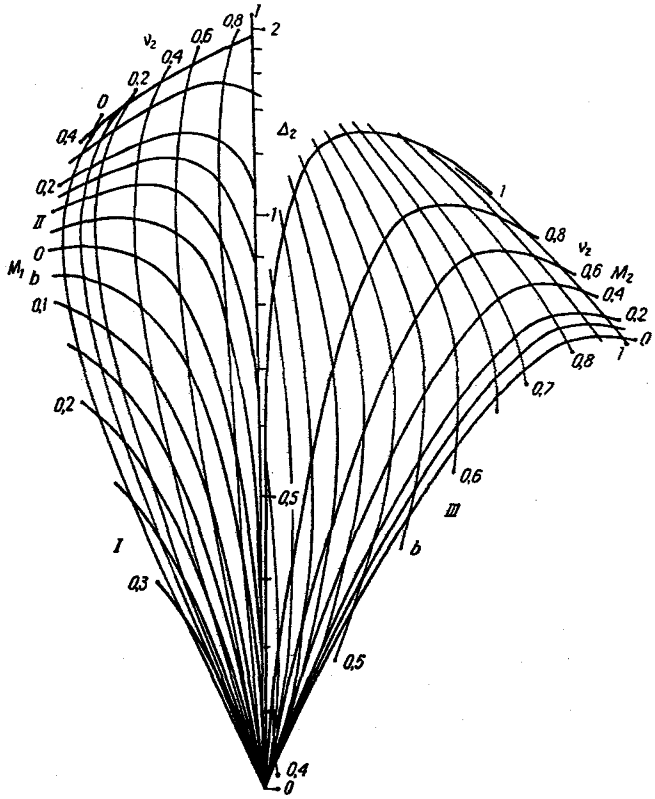

Fig. 2

Here

$$g_1=\frac{g}{4};\qquad \nu_1=\frac{\nu(1-\rho)}{\rho};\qquad \Delta_1=\frac{\Delta}{2\rho}.$$

An analysis of the roots of the characteristic equation of the linear system corresponding to solution (18) shows that the singular points of the phase plane $a = 0$; $\vartheta = \frac{1}{2}\arccos\Delta_1 + \pi n$ are unstable equilibrium positions, i.e., saddle points, and the points $a = 0$, $\vartheta = -\frac{1}{2}\arccos\Delta_1 + \pi n$ are the stable equilibrium positions, i.e., the nodes, if the relation $\Delta_1^2 > 1 - \nu_1^2$ holds, and are unstable equilibrium positions, i.e., saddle points, if the relation $\Delta_1^2 < 1 - \nu_1^2$ holds. It must be noted that the existence of region (18) is ensured by the condition $|\Delta_1| \le 1$.

The characteristic equations corresponding to solutions (19) have the form

$$\lambda^2+2\nu_1\rho\mu\lambda+\rho^2\mu^2\left(\frac{8}{3}\cos^2 2\vartheta-1-\frac{1}{3}\Delta_1^2-3\nu_1^2-4\nu_1\sin 2\vartheta-\frac{4}{3}\Delta_1\cos 2\vartheta\right)=0. \tag{20}$$

In order to investigate the corresponding stationary regimes we eliminate successively the amplitude and phase of the vibrations from (19) and obtain the dependences of the quantities on the parameters of the system:

$$3(\cos\vartheta-\sin\vartheta)(\sin 2\vartheta+\nu_1)-(\sin\vartheta+\cos\vartheta)(\cos 2\vartheta-\Delta_1)=0; \tag{21}$$

$$[\sqrt{g_1^2a^2+4(1-\nu_1)}\pm 5g_1a]^2\,[8-(g_1a\mp\sqrt{g_1^2a^2+4(1-\nu_1)})^2]-16\Delta_1^2=0. \tag{22}$$

The amplitude dependence (22) is shown graphically in the form of nomograms (Fig. 2).

It is not difficult to verify that the number of solutions of system (19) is not more than three. All possible equilibrium states determined by this system can be investigated from the characteristic equation (20) using these nomograms. We can distinguish three regions on the amplitude nomogram (Fig. 2): I and II are the regions of the nomogram occurring in a single half-plane with the point M_1 referred to scale Δ_1 and arranged, respectively,

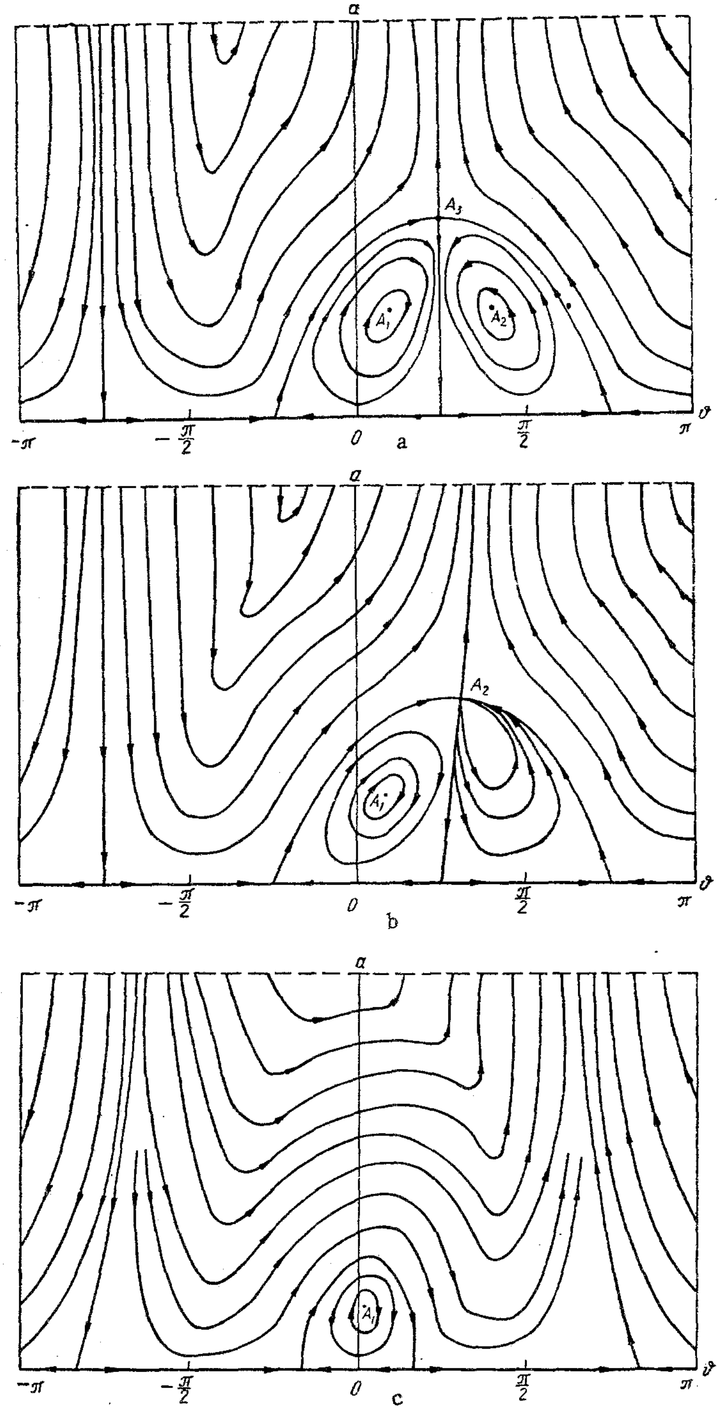

Fig. 3

below and above the line $b = g_1 a = 0$ of the family b; III is the region located in a single half-plane with point M_2 related to scale Δ_1.

An analysis shows that the solution belonging to region I of the nomogram corresponds to the stable equilibrium position of the system, i.e., to the focus or nodes; region II gives unstable equilibrium points, i.e., saddle points. With regard to region III, we can note the following: if the resolving straight line of the nomogram intersects the fixed line ν_1 (for known Δ_1 and ν_1) at two points, then the intersection point closer to scale Δ_1 determines the stable equilibrium position, while the point closer to M_2 determines the unstable equilibrium position.

Let us investigate the behavior of the system on varying one parameter (Δ_1) with the other parameter fixed ($\nu_1 = 0$) (Fig. 3).

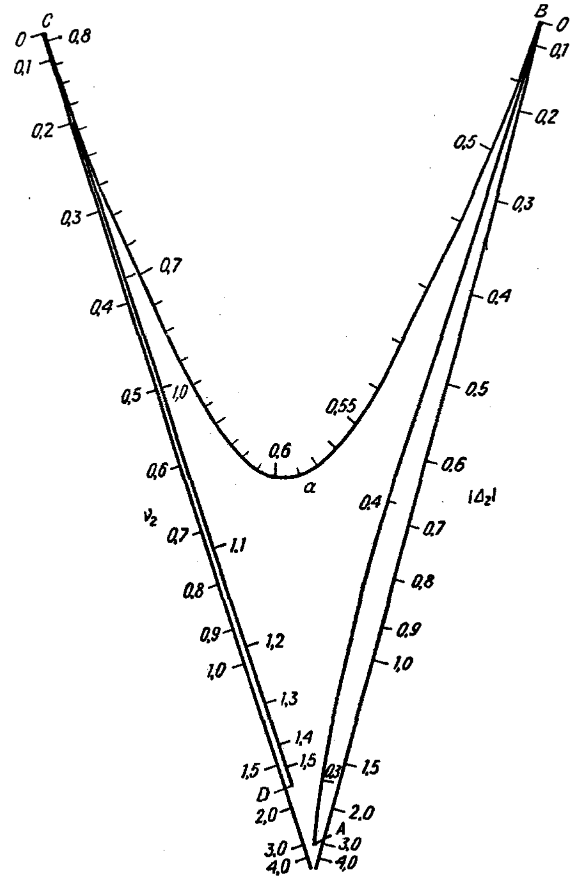

Fig. 4

Thus, for small values of $\Delta_1 (\Delta_1 < \Delta_{10} \approx 0.22)$ system (19) has three solutions which can be found from the nomogram (Fig. 2). One solution belonging to region I gives a singular point, i.e., the center A_1 (Fig. 3a); the other two solutions are obtained in region III of the nomogram. To these solutions correspond the singular points A_2, i.e., the center, and A_3, the saddle point.

Furthermore, the four singular points $a = 0$; $\vartheta = \pm \pi n$ $(n = 0; 1)$ are unstable points of the type of saddle points. For a slow increase of parameter Δ_1, center A_1 on the phase cylinder slides downward towards the ϑ axis, and the second center A_2 and saddle point A_3 approach each other, merging for $\Delta_1 = \Delta_{10}$ into the singular point which is a result of merging of the center and the saddle point (Fig. 3b). This case corresponds to the position where the resolving straight line in the nomogram is tangent to the line $\nu_1 = 0$. For further increase of the parameter $\Delta_1 (\Delta_{10} < \Delta_1 < 1)$ the phase pattern of the system becomes "poorer" (Fig. 3c); besides the saddle point $a = 0$, on this pattern there remains only one equilibrium position, i.e., center A_1 (in this case the nomogram gives only one solution belonging to region I). It is obvious that for small damping $(\nu_1 \neq 0)$ the centers on the phase plane of the system go into stable nodes or foci. It should be noted that the separatrices of the corresponding saddle points will divide the phase plane of the system into regions of stable and unstable motions.

The dependence characterizing the resonance $k \approx 2$ can be represented in a similar way. In particular, its stationary amplitude a is related to the parameters of the system by the equation

$$\begin{aligned}[16x_0^2 + (1.5a^2 + 6x_0^2 - 1)(4.5a^2 + 6x_0^2 - 1)]^2 = a^2\{[4x_0\nu_2 + \\ + \Delta_2(1.5a^2 + 6x_0^2 - 1)]^2 + [4x_0\Delta_2 - \nu_2(4.5a^2 + 6x_0^2 - 1)]^2\}.\end{aligned} \tag{23}$$

Here

$$\nu_2 = \frac{2\nu(1-\rho)}{\rho}; \qquad \Delta_2 = \frac{\Lambda}{4\rho}.$$

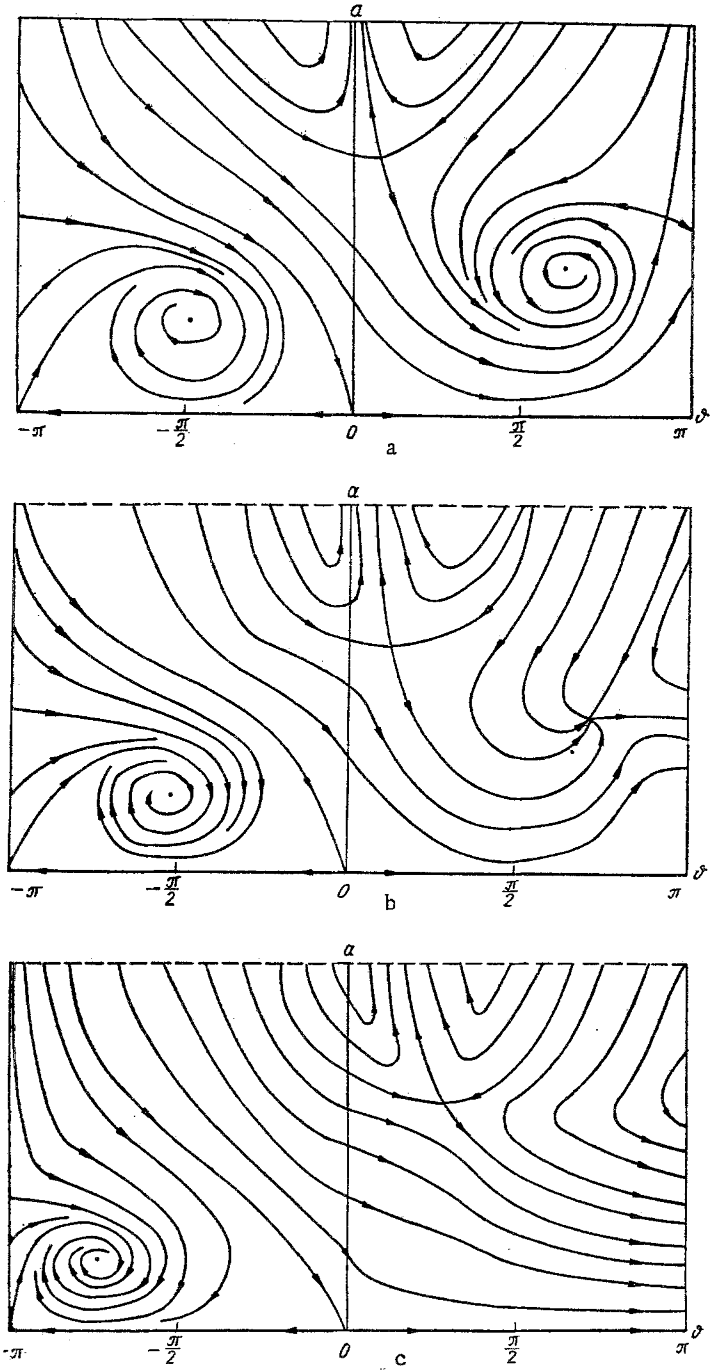

Fig. 5

For dependence (23) a nomogram consisting of the rectilinear scales of variables Δ_2 and ν_2 and the curvilinear response scale a has been computed and constructed for $x_0 = 0$ (Fig. 4). Scale a is arbitrarily divided into three segments: AB, BC, and CD.

The investigation shows that the solutions belonging to the segments AB and CD determine the stable and unstable equilibrium positions in the phase plane of the system. If the resolving straight line intersects BC at two points, the solution closer to point B corresponds to the stable singular point, while the solution closer to point C corresponds to the unstable singular point. Examples of phase patterns of the system are shown in Fig. 5a, b, c for the indicated resonance in the same way as for the case $k \approx 1$.

Experimental investigations conducted earlier on special simulating equipment reflecting the characteristics of the dynamic model used here confirm the reliability of the results of theoretical investigations presented in this paper [6].

In conclusion, we note that the use of asymptotic methods in the investigation of vibrations of these systems permitted not only the significant refinement of the results of the solutions of such problems in linear formulations [7, 8], but also the additional detection of a number of phenomena which can occur in such systems (self-oscillations, parametric resonance). This analysis gives an idea of the dynamic behavior of the system in its parameter space. With the use of analytical dependences, nomograms, and the phase patterns it is possible to estimate the effect of the parameters on the dynamics of the system. In particular, by appropriate variation of these parameters, it is possible to achieve the required reduction of the amplitudes of the vibrations and elimination of undesirable resonance phenomena.

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MOTION OF A BODY WITH A CAVITY PARTIALLY FILLED BY A LIQUID IN THE PRESENCE OF A GAS–LIQUID SEPARATOR

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1. Formulation of the Hydrodynamic Problem

The motion of bodies with cavities partially filled with liquid has been investigated in [1] in the case where a gas–liquid separator of the type of an elastic plate is present in the cavity. In the present work similar problems are investigated, but separators in the form of an ellipsoid of revolution or a right circular cylinder are considered. The cavity is assumed to be spherical (see Fig. 1, where 1 is the cavity, 2 is the liquid, and 3 is the separator).

For simplifying the writing of equations of the perturbed motion of the solid body, we assume that the center of the cavity lies on its longitudinal axis. We introduce the following rectangular right-handed coordinate system: coordinate system OXYZ executing a certain given unperturbed motion and a coordinate system oxyz attached to the center of the cavity. The displacement of the coordinate system oxyz relative to OXYZ determines the perturbed motion; in the unperturbed motion the OX axis coincides with the longitudinal axis of

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