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The Transient Response of a Dielectric Layer

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Abstract. The transient response of an atmospheric surface duct will be studied when the distance between receiving and transmitting end is arbitrarily chosen. The duct model used is that of Kahan and Eckart, consisting of a layer of relative permittivity ε_1 overlying an infinitely conducting plane earth. At height h, this permittivity decreases discontinuously to the value ε_2. The source of the electromagnetic field is assumed to be a vertical magnetic dipole at the height ξ ($\xi < h$) above the surface of the earth with arbitrary time varying moment. The application of two integral transforms to the wave equation for the Fitzgerald vector – a Laplace transform in time and a two-dimensional Fourier transform in the horizontal coordinates in space – leads, under consideration of initial, boundary and transition conditions, to an integral representation of the solution of the wave equation in transform space. A series expansion with respect to the images of the primary source permits us to extend a method of Cagniard, de Hoop and Frankena to the case where the position of the source is in the medium of greater permittivity. Thus we get the step-function solution of the problem as an infinite sum of definite integrals over finite intervals by distinguishing between cases where the distance between receiving and transmitting end is greater or less than the total reflection distance. Thus we can give a physically intuitive description of the pulse propagation in a dielectric layer.

Index Headings: Transient response – Dielectric layer – Tropospheric propagation

For the last thirty years, the correlation between the propagation of electromagnetic waves in the atmosphere and metereological conditions has been investigated by radiometereologists. This paper treats the special problem of propagation of an electromagnetic impulse in an abnormal stratification of the lower troposphere, a so-called surface duct. Such inversions occur particularly in the boundary layer along the sea.

During this time a number of experimental and theoretical models of the duct refractive-index profiles have been developed [1, 2], all giving more or less satisfactory insight into the mechanism of waveguiding and differing mainly in the sophistication of the mathematics used. In considering our problem we will restrict ourselves to one of the simpler duct models. It was developed by Kahan and Eckart [3] and assumes a discontinuous drop in the otherwise constant refraction index at the upper duct boundary. The earth is assumed to be an ideal conductor and ideally plane, which corresponds well to the situation at sea in the microwave frequency range.

We wish to extend the steady-state duct propagation theory of Kahan and Eckart to transient excitation when no restrictions on the distance between receiving and transmitting end are made. The unit step function is considered as the time dependence of the dipole moment. Thus we get the transient response of a dielectric layer overlying a dielectric half-space and bounded by an infinitely conducting plane.

From the mathematical point of view both the geometry and the time dependence in this problem are an extension of Sommerfeld's half-space problem [4], which deals with the steady-state propagation of electromagnetic waves over a dielectric earth, emitted by a vertical dipole. Early attempts to get the transient response of a dielectric half-space were made by Gerjouy [5], Jeffreys [6], Muskat [7], Ott [8], Friedrichs *et al.* [9], and Friedlander [10]. In 1955, Poritsky [11] published a paper which, generalizing Weyl's method [12], expressed the primary field of the source in terms of a complex integral over plane waves. In 1956, van der Pol [13] found a simple analytic expression for the transient solution of the half-space problem when both transmitter and receiver are situated in the plane separating the two media; thus complications because of dispersion are avoided. In his work, van der Pol stimulated further papers dealing with the transient response of a vertical antenna in one of the two dielectric half-spaces (Pekeris *et al.* [14], de Hoop *et al.* [15, 16]). Bremmer [17] and Vlaar [18, 19] avoided the steady state solution of the problem and determined the field of a source with unit step-function time-dependence directly from an integral representation of the primary field in the time domain. Vlaar also discussed the case of the position of the source being in the medium of greater refractive index.

Nearly all these authors also apply their method to the corresponding seismic half-space problem (de Hoop [20], Pekeris *et al.* [21–23], Vlaar [24]). For this special seismic problem, Cagniard [25] developed in 1939 a certain sophisticated method of solution: he used two integral transforms – a Hankel transform concerning the coordinates in space and a Laplace transform concerning the time. Now, the key of his method is not to use the Laplace inversion formula but to change the integral representation of the solution in the transform space to an explicit Laplace integral, thus being able to read off the solution in the time domain. Pekeris [26] slightly modified Cagniard's method in solving a special integral equation occurring in propagation problems of electromagnetic and seismic pulses. This solution was applied by Pekeris *et al.* [27] to the special case of propagation of a seismic pulse in a layered liquid.

De Hoop and Frankena [15, 16] gave a further modification of Cagniard's method in finding the transient response of a dielectric half-space with arbitrary position of the source in the medium of smaller refractive index. These authors make use of a two-dimensional Fourier transform instead of the Hankel transform used by Cagniard, thus simplifying considerably the procedure of changing the solution in the Laplace transform space to an explicit Laplace integral.

We will use this method of de Hoop and Frankena for the solution of our duct problem, extending it to the case where the source is situated in the medium of greater refractive index since our dipole is within the surface duct. Thus we find expressions for the potential which are numerically easy to handle and which give a physically intuitive description of the transient response of a surface duct when no restrictions are made concerning the distance between receiving and transmitting end. For large distances these expressions become very impracticable, but this case has been treated by Pekeris in 1948 [28] for the seismic pulse and by the author [29] for the electromagnetic pulse using the approximate mode theory. This gives us the possibility of comparing the approximate solution with the exact one, coming to a clear definition of the term "great distance".

1. Mathematical Formulation of the Problem and Integral Representation of the Solution

A dielectric layer is assumed of relative permittivity ε_1 overlying an infinitely conducting plane earth which is confined by the plane $z=0$ of a cartesian coordinate system (x, y, z). At the height h this permittivity decreases discontinuously to the value ε_2 (duct model of Kahan and Eckart). The relative permeability μ is assumed to be constant throughout the half-space $z>0$. We refer to the layer as medium 1 and to the half-space $z>h$ as medium 2. The potentials and fields which belong to the two media are marked by corresponding indices. The source of the field is assumed to be a vertical magnetic dipole in medium 1 at the point $x=y=0$, $z=\xi>0$, whose moment is given by $C \cdot F(t) \cdot \boldsymbol{e}_z$. The vector $\boldsymbol{e}_z$ denotes the unit vector in the z-direction, t is the time variable, C is some arbitrary constant to which we give the value $\mu_0 \mu$, μ_0 being the vacuum permeability. Regarding $F(t)$ we make the causality assumption $F(t)=0$ for $t<0$. This guarantees us the uniqueness of our solution.

The radiation field of this vertical magnetic dipole can be expressed in terms of the z-component of the Fitzgerald vector $\boldsymbol{\Pi}(x, y, z; t)$ denoted by $\Pi^{(i)}(x, y,$

$z;t)$, $i=1,2$. This component fulfills the following wave equation

$$\Delta\Pi^{(i)} - \frac{1}{v_i^2}\frac{\partial^2\Pi^{(i)}}{\partial t^2} = \begin{cases} 0 \text{ for } i=2 \\ -\delta(x,y,z-\xi)F(t) \text{ for } i=1, \end{cases} \quad (1)$$

where v_i denotes the phase velocity in medium i.
The application of a Laplace transform in time and a two-dimensional Fourier transform in the horizontal coordinates x and y leads, under consideration of the initial, boundary and transition conditions for $\Pi^{(i)}(x,y,z;t)$, to an integral representation for $\pi^{(i)}(x,y,z;s)$, the Laplace transform of $\Pi^{(i)}(x,y,z;t)$, s being the variable in the transform space. We get for $0<z<h$ [30]

$$\pi^{(1)}(x,y,z;s) = \frac{sf(s)}{8\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[\frac{e^{-s\gamma_1|z-\xi|}}{\gamma_1} + \frac{e^{-s\gamma_1(2h-\xi-z)}c_{12}(1-e^{-2s\gamma_1\xi}) - e^{-s\gamma_1(\xi+z)}(1+c_{12}e^{-2s\gamma_1(h-\xi)})}{\gamma_1(1+c_{12}e^{-2s\gamma_1 h})}\right]e^{js\alpha x+js\beta y}\cdot d\alpha d\beta \quad (2)$$

where

$$c_{12}(\alpha,\beta) = \frac{\gamma_1(\alpha,\beta)-\gamma_2(\alpha,\beta)}{\gamma_1(\alpha,\beta)+\gamma_2(\alpha,\beta)} \quad (3)$$

and

$$\gamma_i(\alpha,\beta) = (\alpha^2+\beta^2+v_i^{-2})^{1/2}; \quad i=1,2 \quad (4)$$

with $\operatorname{Re}\gamma_i \geqq 0$, $i=1,2$. Here α and β are the variables in the transform space of the two-dimensional Fourier transform, $f(s)$ is the Laplace transform of $F(t)$. A similar expression can be derived for $\pi^{(2)}(x,y,z;s)$, but it will not be given here since we restrict our attention to $\Pi^{(1)}(x,y,z;t)$.
The first term of the integrand in (2) is the potential due to the primary field, the second term denotes the diffracted field, and c_{12} corresponds to the Fresnel reflection coefficient.
Considering only real β, the integrand of (2) has four branch points $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ in the complex α-plane, namely

$$\alpha_{1,4} = \pm j(\beta^2+v_1^{-2})^{1/2} \quad (5)$$

$$\alpha_{2,3} = \pm j(\beta^2+v_2^{-2})^{1/2} \quad (6)$$

where $|\alpha_{1,4}(\beta)| > |\alpha_{2,3}(\beta)|$ since medium 1 is the medium of the lower phase velocity.
Our aim is to determine the potential $\Pi^{(1)}(x,y,z;t)$ at some fixed point (x,y,z) within the duct layer as a function of time, having chosen a suitable function $F(t)$. Therefore, we make use of the method of de Hoop and Frankena [15, 16], extending it to our geometry and source position in the medium of greater refractive index.

2. Application and Extension of the Method of De Hoop and Frankena

As already mentioned, the essence of this method is to change the integral representation (2) to an explicit Laplace integral, an idea credited to Cagniard [25]. In order to apply this method we must expand the "duct denominator" $[1+c_{12}\exp(-2s\gamma_1 h)]^{-1}$ in a geometric series. It can be shown [30] that this series converges in the whole α-plane except at the points α_1 and α_4. Then we get instead of (2)

$$\pi^{(1)}(x,y,z;s) = \frac{sf(s)}{8\pi^2}I_p(x,y,z;s) + \frac{sf(s)}{8\pi^2}\sum_{n=0}^{\infty}(-1)^n[I_n^{(1)}(x,y,z;s) - I_n^{(2)}(x,y,z;s) - I_n^{(3)}(x,y,z;s) - I_n^{(4)}(x,y,z;s)] \quad (7)$$

where

$$I_p(x,y,z;s) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{\gamma_1}\cdot e^{-s\gamma_1|z-\xi|+js\alpha x+js\beta y}\,d\alpha d\beta \quad (8)$$

$$I_n^{(i)}(x,y,z;s) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{c_{12}^{n+1}}{\gamma_1}\cdot e^{-s\gamma_1\xi_n^{(i)}+js\alpha x+js\beta y}\,d\alpha d\beta \quad (9)$$

$$i=1,2,3$$

$$I_n^{(4)}(x,y,z;s) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{c_{12}^{n}}{\gamma_1}\cdot e^{-s\gamma_1\xi_n^{(4)}+js\alpha x+js\beta y}\,d\alpha d\beta \quad (10)$$

and

$$\xi_n^{(1)} = 2h(n+1) - \xi - z$$
$$\xi_n^{(2)} = 2h(n+1) + \xi - z$$
$$\xi_n^{(3)} = 2h(n+1) - \xi + z \qquad (11)$$
$$\xi_n^{(4)} = 2nh + \xi + z\,.$$

The subscript p in (7) stands for primary source. We realize that all other integrals $I_n^{(i)}$ $(i=1, \ldots, 4)$ as shown in (9) and (10) are of the same structure as I_p except the occurrence of the coefficients c_{12}^{n+1}, c_{12}^{n}, respectively. This is due to the fact that our series expansion corresponds to the infinite number of reflections of the primary source at both planes $z=0$ and $z=h$, where each reflection at the latter plane yields a multiplication by the reflection coefficient c_{12}. We denote the reflected image source by Q_{ni} when it corresponds to the integral $I_n^{(i)}$. In the following we restrict our attention to $I_n^{(1)}$, since it is representative of all the integrals $I_n^{(i)}$ $(i=1, \ldots, 4)$.

2.1. The Region below Total Reflection

We introduce polar coordinates (r, φ) regarding the space variables x, y and new variables p, q in the Fourier transform space through

$$\alpha = p\cos\varphi - q\sin\varphi$$
$$\beta = p\sin\varphi + q\cos\varphi\,. \qquad (12)$$

Further spherical polar coordinates with origin at the reflected image source Q_{n1} are introduced, viz

$$r = R_n^{(1)}\sin\vartheta_n^{(1)}$$
$$\varphi = \varphi_n^{(1)} \qquad (13)$$
$$\xi_n^{(1)} = R_n^{(1)}|\cos\vartheta_n^{(1)}|\,.$$

Then we get instead of (9)

$$I_n^{(1)}(x, y, z; s) = \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} \frac{c_{12}^{n+1}(p, q)}{\gamma_1(p, q)} \cdot e^{-s\gamma_1 R_n^{(1)}|\cos\vartheta_n^{(1)}| + jspR_n^{(1)}\sin\vartheta_n^{(1)}}\, dp\,. \qquad (14)$$

The integrand of (14) has two branch points $p_1(q)$ and $p_2(q)$ in the upper complex p-plane corresponding to the branch points $\alpha_1(\beta)$ and $\alpha_2(\beta)$. We have $|p_1(q)| > |p_2(q)|$.

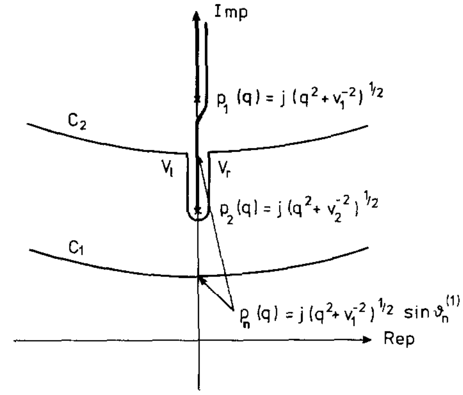

Fig. 1. Branch points, branch cuts and integration path in the upper complex p-plane in the region below total reflection denoted by C_1 and in the region beyond total reflction denoted by C_2

In the Riemann sheet of the complex p-plane given by $\mathrm{Re}\,\gamma_{1/2} \geqq 0$ we search for an integration path on which $f(p, q) = \tau$ is real and positive and

$$f(p, q) = \gamma_1(p, q)\, R_n^{(1)}|\cos\vartheta_n^{(1)}| - jpR_n^{(1)}\sin\vartheta_n^{(1)}\,. \qquad (15)$$

This path is a hyperbola yielding $R_n^{(1)}(q^2 + v_1^{-2})^{-1/2} \leqq \tau < \infty$. This is illustrated in Fig. 1, together with the branch points and branch cuts of the upper p-plane. The point of intersection of the hyperbola with the imaginary p-axis is given by

$$p_n(q) = j(q^2 + v_1^{-2})^{1/2}\sin\vartheta_n^{(1)}\,. \qquad (16)$$

We see that $|p_n(q)| < |p_1(q)|$. In addition $|p_n(q)| < |p_2(q)|$ if

$$\sin\vartheta_n^{(1)} \leqq \left(\frac{q^2 + v_2^{-2}}{q^2 + v_1^{-2}}\right)^{1/2} \qquad (17)$$

which yields the condition

$$\sin\vartheta_n^{(1)} \leqq \frac{v_1}{v_2} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\,. \qquad (18)$$

In geometrical optics, this is the condition that a ray originating from the source Q_{n1} is not yet totally reflected from the layer $z=h$; that is to say, the sign of equality in (18) determines the critical angle of total reflection. (18) yields for the horizontal distance r between source and point of observation

$$r \leqq \xi_n^{(1)} \cdot \sqrt{\frac{\delta}{1-\delta}} \qquad (19)$$

where $\delta = \varepsilon_2/\varepsilon_1$. If r is chosen in such a manner, that (18) is fulfilled for some reflected image source

Q_{N1}, then it is fulfilled for all $n > N$. In the following we have taken $N = 0$.

Then integrating (14) along the hyperbolic integration path C_1 of Fig. 1 and changing the order of integration yields [15, 16, 30]

$$I_n^{(1)}(r,z;s) = \int\limits_{R_n^{(1)}/v_1}^{\infty} \left[\frac{4}{R_n^{(1)}} \int\limits_0^{\pi/2} \mathrm{Re}\{c_{12}^{n+1}(\tau,\psi)\} d\psi \right] e^{-s\tau} d\tau, \tag{20}$$

where we have introduced the new variable ψ through

$$q = \left(\frac{\tau^2}{R_n^{(1)2}} - \frac{1}{v_1^2} \right)^{1/2} \sin\psi . \tag{21}$$

Eq. (20) denotes the Laplace transform of the function $i_n^{(i)}(r,z;t)$ where

$$i_n^{(1)}(r,z;t) = \begin{cases} 0 \text{ for } t < R_n^{(1)}/v_1 \\ \dfrac{4}{R_n^{(1)}} \displaystyle\int\limits_0^{\pi/2} \mathrm{Re}\{c_{12}^{n+1}(\tau,\psi)\} d\psi \text{ for } t > R_n^{(1)}/v_1 . \end{cases} \tag{22}$$

The same procedure as described above can be applied to the integrals (8), (9) for $i = 2, 3$ and (10).

Now choose $f(s) = 1/s$, that is to say, take $F(t)$ as the unit step function. Then in connection with (7) and (22) we have reached our aim of representing the potential of the vertical magnetic dipole within the duct layer as a function of time for some fixed point of observation in the region below total reflection. We now turn our attention to the region beyond total reflection.

2.2. *The Region beyond Total Reflection*

We again consider the integral $I_n^{(1)}(x,y,z;s)$ given by (9) and choose the distance r in such a manner that (19) is not fulfilled for the reflected image source Q_{n1}, that is to say, $|p_2(q)| < |p_n(q)| < |p_1(q)|$. The resulting integration path C_2 in the complex p-plane is shown in Fig. 1. It is clear that the part of the integral due to the hyperbolic part of the integration path is equal to (20). But now we have in addition to (20) the parts due to the borders V_r and V_l of the branch cut which can be combined to the following integral [30], since c_{12} on V_l is the complex conjugate of c_{12} on V_r:

$$V_n^{(1)}(r,z;s) = 4j \int\limits_0^{\infty} dq \int\limits_{p_2(q)}^{p_n(q)} \frac{\mathrm{Im}\{c_{12}^{n+1}(p,q)\}}{\gamma_1(p,q)} \cdot e^{-s\gamma_1 R_n^{(1)} |\cos\vartheta_n^{(1)}| + jspR_n^{(1)}\sin\vartheta_n^{(1)}} dp . \tag{23}$$

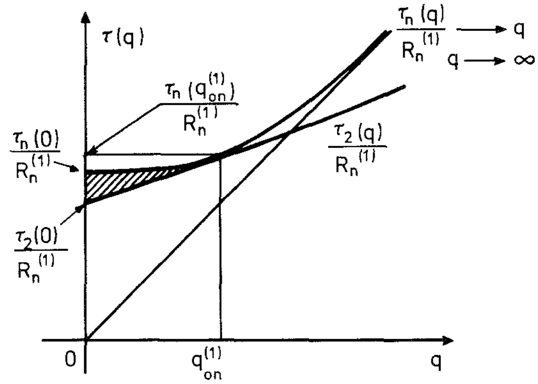

Fig. 2. Region of integration (shaded) in the τ, q-plane

In (23) p takes on only purely imaginary values, so we substitute $p = ju$. The new end points of the integration interval regarding p are denoted by $u_2(q)$ and $u_n(q)$. These curves $u_2(q)$ and $u_n(q)$ have one point of intersection at $q_{0n}^{(1)}$, where

$$q_{0n}^{(1)} = \frac{1}{|\cos\vartheta_n^{(1)}|} \left(\frac{1}{v_1^2} \sin\vartheta_n^{(1)} - \frac{1}{v_2^2} \right)^{1/2} . \tag{24}$$

That means that for $q \geqq q_{0n}^{(1)}$ we have no longer a branch cut integration, since $u_2(q) \geqq u_n(q)$ for $q \geqq q_{0n}^{(1)}$; or, restated, for $q \geqq q_{0n}^{(1)}$ γ_1 and γ_2 are real and hence $\mathrm{Im}\, c_{12} = 0$.

So we get

$$V_n^{(1)}(r,z;s) = -4 \int\limits_0^{q_{0n}^{(1)}} dq \int\limits_{u_2(q)}^{u_n(q)} \frac{\mathrm{Im}\, c_{12}^{n+1}}{\gamma_1} \cdot e^{-s(\gamma_1 R_n^{(1)} |\cos\vartheta_n^{(1)}| + uR_n^{(1)}\sin\vartheta_n^{(1)})} du . \tag{25}$$

Since the function

$$f(u,q) = \gamma_1 R_n^{(1)} |\cos\vartheta_n^{(1)}| + uR_n^{(1)} \sin\vartheta_n^{(1)} \tag{26}$$

in the argument of the exponential in (25) is a real function of u and q in the region of integration, we replace it by the real variable $\tau = f(u,q)$. It follows then

$$V_n^{(1)}(r,z;s) = \frac{4}{R_n^{(1)}} \int\limits_0^{q_{0n}^{(1)}} dq \int\limits_{\tau_2(q)}^{\tau_n(q)} \frac{\mathrm{Im}\{c_{12}^{n+1}(\tau,q)\} e^{-s\tau}}{(q^2 + v_1^{-2} - \tau^2/R_n^{(1)2})^{1/2}} d\tau \tag{27}$$

where

$$\tau_2(q) = \left(\frac{1}{v_1^2} - \frac{1}{v_2^2} \right)^{1/2} R_n^{(1)} |\cos\vartheta_n^{(1)}| + (q^2 + v_2^{-2})^{1/2} R_n^{(1)} \sin\vartheta_n^{(1)} \tag{28}$$

and

$$\tau_n(q) = R_n^{(1)}(q^2 + v_1^{-2})^{1/2}. \tag{29}$$

We have $\tau_n(q) \geqq \tau_2(q)$, where equality holds for $q = q_{0n}^{(1)}$. The region of integration of (30) is illustrated in Fig. 2. By changing the order of integration we finally get [30]

$$\begin{aligned} V_n^{(1)}(r,z;s) &= \int\limits_{\tau_2(0)}^{\tau_n(0)} e^{-s\tau}\, d\tau \\ &\cdot \left[\frac{4q_2(\tau)}{R_n^{(1)}} \int\limits_0^{\pi/2} \frac{\operatorname{Im}\{c_{12}^{n+1}(\tau,\psi)\} \cos\psi}{[q_2^2(\tau)\sin^2\psi - q_n^2(\tau)]^{1/2}}\, d\psi \right] \\ &+ \int\limits_{\tau_n(0)}^{\tau_n(q_{0n}^{(1)})} e^{-s\tau}\, d\tau \left[\frac{4[q_2^2(\tau) - q_n^2(\tau)]^{1/2}}{R_n^{(1)}} \right. \\ &\cdot \left. \int\limits_0^{\pi/2} \frac{\operatorname{Im}\{c_{12}^{n+1}(\tau,\psi)\} \cos\psi}{\{q_n^2(\tau) + [q_2^2(\tau) - q_n^2(\tau)]\sin^2\psi\}^{1/2}}\, d\psi \right] \end{aligned} \tag{30}$$

where

$$q_2(\tau) = \frac{1}{\sin^2 \vartheta_n^{(1)}} \left[\frac{\tau}{R_n^{(1)}} - \left(\frac{1}{v_1^2} - \frac{1}{v_2^2} \right)^{1/2} |\cos \vartheta_n^{(1)}| \right] \tag{31}$$

and

$$q_n(\tau) = \left(\frac{\tau^2}{R_n^{(1)2}} - \frac{1}{v_1^2} \right)^{1/2}. \tag{32}$$

In the first term of (30) we have substituted

$$q = q_2(\tau)\sin\psi \tag{33}$$

and in the second term

$$q = \{q_n^2(\tau) + [q_2^2(\tau) - q_n^2(\tau)]\sin^2\psi\}^{1/2}. \tag{34}$$

Now, if $V_n^{(1)}(r,z;s)$ is the Laplace transform of $v_n^{(1)}(r,z;t)$, our solution in the time domain has the following form

$$v_n^{(1)}(r,z;t) = \begin{cases} 0 \quad \text{for} \quad t < \tau_2(0) \\ \dfrac{4q_2(t)}{R_n^{(1)}} \displaystyle\int\limits_0^{\pi/2} \frac{\operatorname{Im}\{c_{12}^{n+1}(t,\psi)\}\cos\psi}{[q_2^2(t)\sin^2\psi - q_n^2(t)]^{1/2}}\, d\psi \quad \text{for} \quad \tau_2(0) < t < \tau_n(0) \\ \dfrac{4[q_2^2(t) - q_n^2(t)]^{1/2}}{R_n^{(1)}} \displaystyle\int\limits_0^{\pi/2} \frac{\operatorname{Im}\{c_{12}^{n+1}(t,\psi)\}\cos\psi}{\{q_n^2(t) + [q_2^2(t) - q_n^2(t)\sin^2\psi]\}^{1/2}}\, d\psi \quad \text{for} \quad \tau_n(0) < t < \tau_n(q_{0n}^{(1)}). \\ 0 \quad \text{for} \quad t > \tau_n(q_{0n}^{(1)}). \end{cases} \tag{35}$$

The term $v_n^{(1)}(r,z;t)$ described by (35) must be added to (22) when the horizontal distance r comes into the region beyond total reflection for some reflected image source Q_{n1}. Expressions similar to (35) can be derived for all other sources Q_{ni}; it is only remarkable that $v_0^{(4)}(r,z;t)=0$, since the integral representation (10) has only one branch point $\alpha_1(\beta)$ if $n=0$.

(22) and (35), together with the corresponding terms $i_p(r,z;t)$, $i_n^{(i)}(r,z;t)$ $(i=2,3,4)$ and $v_n^{(i)}(r,z;t)$ $(i=2,3,4)$ represent the exact solution of the transient response of a dielectric layer, i.e. the response of the layer when $F(t)$ is given as a unit step function. We denote this response by $\Pi_{\text{usf}}^{(1)}(r,z;t)$.

It is to be noted that $\Pi_{\text{usf}}^{(1)}(r,z;t)$ becomes logarithmically singular in the region beyond total reflection for times $t = R_n^{(i)}/v_1$, which is due to the additional integrals (35); that is to say, the wave front amplitudes of the spherical waves originating from the reflected image sources Q_{ni} (except Q_{04}) are singular in a logarithmical manner if the distance r is great enough so that total reflection has occurred. This fact has already been cited by several authors treating the transient response of a totally reflecting plane between two dielectric half-spaces [21, 27, 10]. One can show that this is mathematically due to the discontinuity of the unit step function at $t=0$ [30].

3. Numerical Results and Discussion

Numerical computation of the transient response $\Pi_{\text{usf}}^{(1)}(r,z;t)$ within the duct layer can now easily be performed, based on (22) and (35). The following parameters have been chosen to calculate the curves shown by Figs. 3–6: duct height $h = 20$ m and difference of relative permittivities at the upper duct boundary $\Delta\varepsilon = \varepsilon_1 - \varepsilon_2 = 4 \cdot 10^{-4}$. We take these numbers from measurements of Brocks *et al.* [2] who investigated an atmospheric surface duct in the

German Bight. Since our solution depends only on $\delta=\varepsilon_2/\varepsilon_1$, we take $\varepsilon_2=1$ and therefore $\varepsilon_1=1.0004$. The height of the primary source and the point of observation has been taken to be $\xi=z=15$ m. We normalize $\Pi^{(1)}_{usf}(r,z;t)$ in such a manner that the corresponding potential of the primary source is independent of the horizontal distance r and takes on the value one for $t>\tau_0=R_0/v_1$, where R_0 denotes the spherical distance between the source and the point of observation; that is to say, we multiply $\Pi^{(1)}_{usf}(r,z;t)$ by $4\pi R_0$. We define a normalized time τ by $\tau=t/\tau_0-1$, meaning that the beginning of the τ-axis coincides with the arrival time of the spherical wave originating directly from the primary source; clearly this normalization depends upon the point of observation.

At first we choose such a point of observation whose horizontal distance r from the dipole is in the region below total reflection. Then $r<r_{min}$ where r_{min} denotes the minimum of $\xi^{(i)}_n[\delta/(1-\delta)]^{1/2}$ with respect to n and i. For our numerical example this minimum will be taken on for $n=0$ and $i=1$; therefore $r_{min}=500$ m. Figure 3a shows the normalized potential as a function of the normalized time if r equals 400 m; we see the arrival of the spherical wave front originating from the primary source at the time $\tau=0$ of the normalized time axis. Time $\tau=R^{(1)}_0/R_0-1$ is the arrival time of the spherical wave front originating from the reflected image source Q_{01} whose amplitude is superimposed into the primary potential. This part of Q_{01} is formed by reflection of the primary wave front at the upper duct layer boundary; the reflection coefficient there is frequency dependent and therefore the contribution of Q_{01} is, in contrast to the primary part, no longer a unit step function. Some time later, at $\tau=R^{(4)}_0/R_0-1$, the spherical wave front of the primary wave arrives which has been reflected at the infinitely conducting earth, changing its sign. The normalized amplitude of this wave front is $R_0/R^{(4)}_0=0.995$. The contribution of all other reflected image sources can be neglected because of their greater geometrical attenuation and because the multiple reflections at the upper duct layer boundary lead to an order of 10^{-2}; that is to say, in the case $r=400$ m we can stop the summation in (7) after $n=1$.

Figure 3b shows the transient response of the duct layer for the distance $r=500$ m. The delay of the wave fronts originating from Q_{01} and Q_{04} against the primary wave front becomes smaller and their amplitudes increase because of the increase in the angle of incidence.

Figure 3c shows the case where, for the first time, total reflection has occurred; the logarithmic singularity of the wave front originating from Q_{01} is indicated. With increasing horizontal distance this singularity becomes more and more pronounced.

Figure 4 shows the case $r=2000$ m. We see that now total reflection occurs for the spherical wave fronts originating from the reflected image sources Q_{02} and Q_{03} (since in our numerical example, $\xi^{(2)}_n=\xi^{(3)}_n$). In addition, we see another wave front arriving before the primary wave front; this is a conical front due to the reflected image source Q_{01}. It exists only within the duct layer, travels with the phase velocity v_2 at the upper boundary of the layer, and is described by the first term of (30). This conical wave front has already been mentioned by several authors [5, 9, 10, 17] who investigated the transient response of a

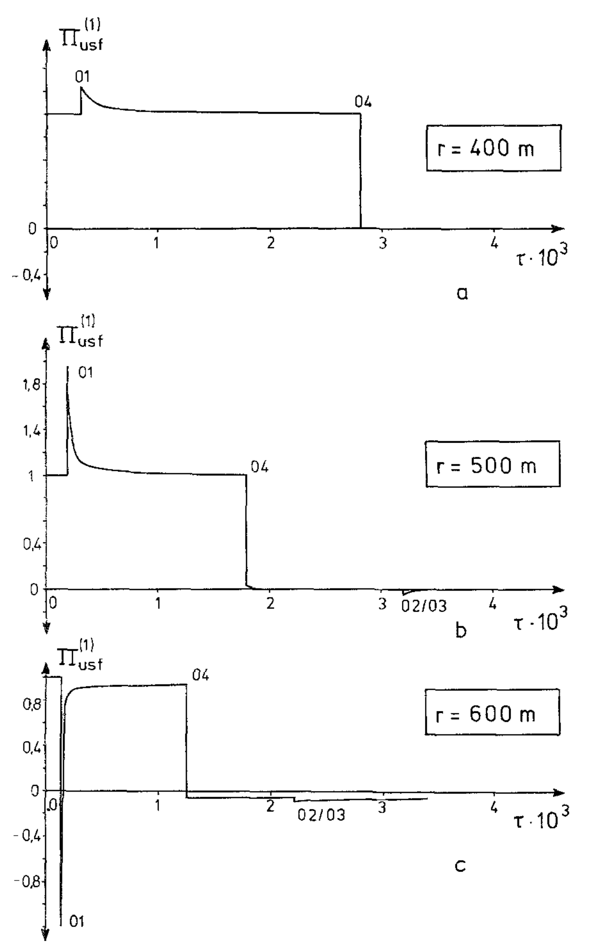

Fig. 3. Normalized potential within the duct layer as function of normalized time: a) $r=400$ m, b) $r=500$ m, c) $r=600$ m

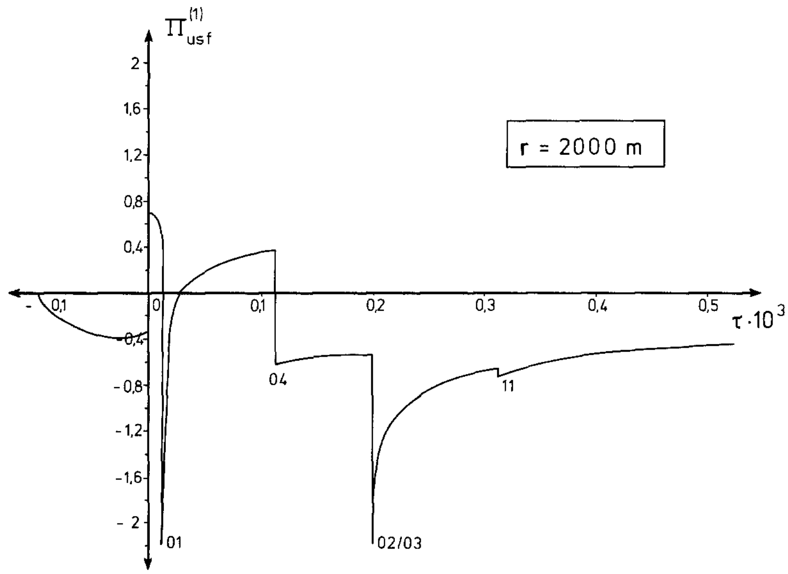

Fig. 4. Normalized potential within the duct layer as function of normalized time ($r = 2000$ m)

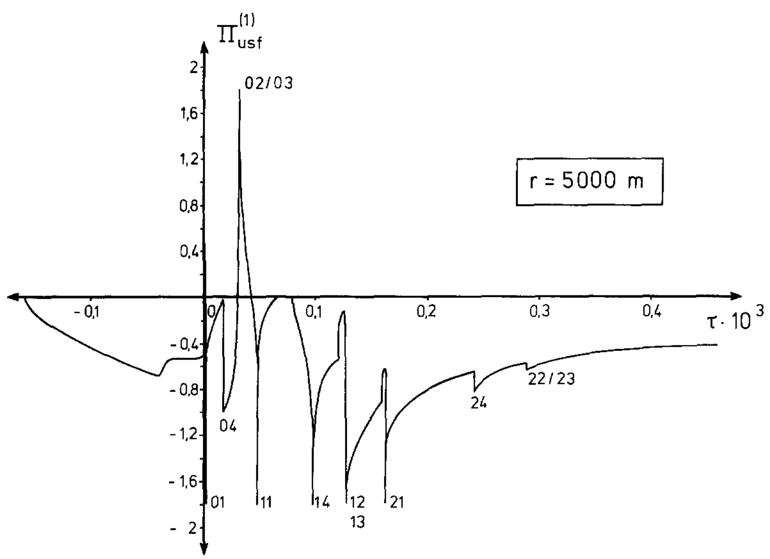

Fig. 5. Normalized potential within the duct layer as function of normalized time ($r = 5000$ m)

totally reflecting plane boundary between two dielectric half-spaces.

We now come to Fig. 5 and the horizontal distance $r = 5000$ m. The following aspects are worth being described: the conical wave front due to the reflected image sources Q_{02} and Q_{03} now arrives before the primary wave front, it is superimposed to the conical front due to Q_{01} and the time distance between the latter and the primary front becomes greater and greater. Also more and more reflected image sources must be considered, since with increasing distance more spherical wave fronts due to the reflected image sources undergo total reflection and contribute to the received potential with their logarithmic singularities: that is to say, we must sum in (7) at least those contributions from the reflected image sources which have undergone total reflection, all others being negligible. This means that with increasing distance r between receiving and transmitting end our series expansion becomes more and more impractical: it even diverges if r tends to infinity. Physically this is quite clear, mathematically it follows from the fact that if $r \to \infty$ we have $p_n(q) \to p_1(q)$ and our series can not be expanded at the point $p_1(q)$. These facts are

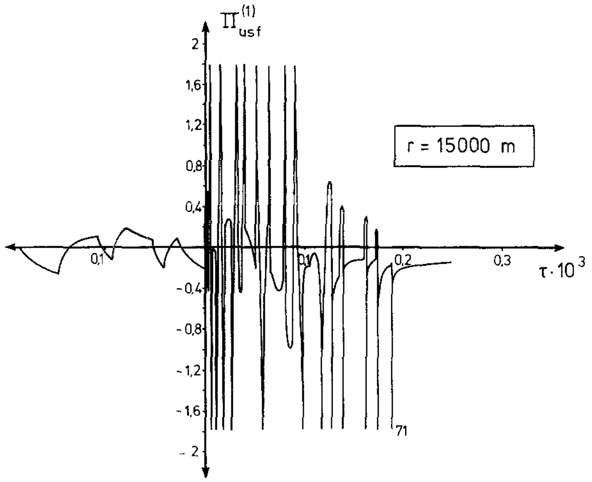

Fig. 6. Normalized potential within the duct layer as function of normalized time ($r = 15000$ m)

again illustrated in Fig. 6: we have to sum until $n = 7$. In addition, we can now characterize the principal behaviour of the transient response of a duct layer for great distances where "great" means $r \gg r_{\min}$: the received potential as a function of time consists of a relatively low frequency wave formed by the superposition of the conical wavefronts, a high frequency wave formed by the spherical wave fronts with their logarithmic singularities (which are of no physical reality) and an exponential decay for large t.

These different parts of the signal can also be found by a method very different to the one described above: the mode theory which was used by Pekeris [28] to study the seismic pulse propagation in shallow water and by the author [29] to get the solution of the corresponding electromagnetic problem. This method can only be used for great distances r which cannot be clearly defined by the method itself. But comparing the results of the mode theory (Fig. 7) with the exact theory described in this paper we can say: "great" always means $r \gg r_{\min}$. A detailed comparison of the two methods [30] reveals some other insufficiencies of the mode theory, but nevertheless it is a useful tool to get the transient response of layered structures for great distances.

4. Concluding Remarks

We gave the exact solution of the transient response of a dielectric layer as, for example, an atmospheric duct layer over sea valid for arbitrary distances between receiving and transmitting end. To this end, we used a method originally given by Cagniard and modified by de Hoop and Frankena extending it to the present case of more complex geometry and to source position in the medium of greater refractive

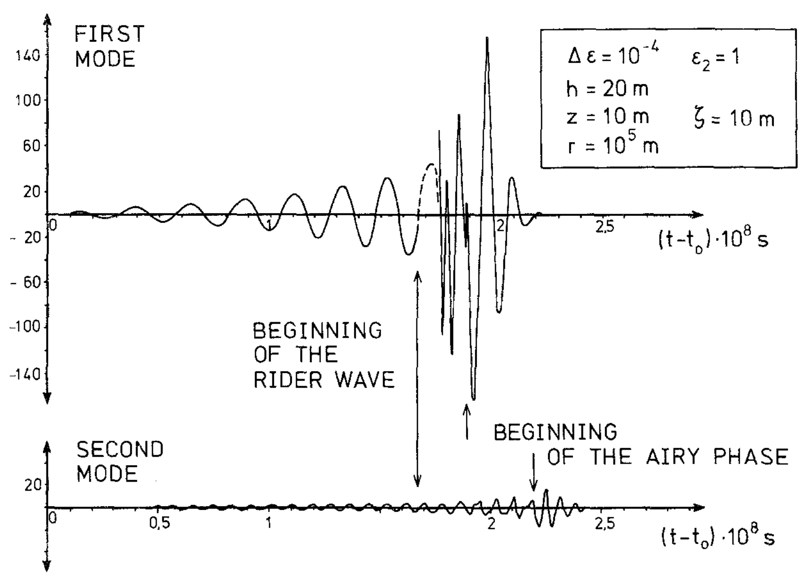

Fig. 7. Space wave, rider wave and Airy phase of the transient response of a dielectric layer at great distances

index. Its advantage is the physical concreteness and the possibility of comparing it with the asymptotic mode theory, giving a certain insight into the validity of the latter. A disadvantage of the method is that it cannot be used to calculate the potential in the dielectric half-space outside the layer in a similar manner.

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