DIFFRACTION OF ELASTIC WAVES

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Introduction

The diffraction of elastic waves poses one of the most complex and timely problems from the standpoint of applications in the dynamics of deformable bodies. Its timeliness owes to the almost certain inevitability of inhomogeneities (inclusions, cavities, cracks, local fluctuations of properties, etc.) being present in connection with the design of new structures operating under dynamic loads in various branches of industry, in the development of new composite heterogeneous materials, and in geophysical and seismological investigations, plus the fact that information about the dynamic stresses near such inhomogeneities is essential for a variety of objectives. On the other hand, elastic-wave diffraction problems are among the classical problems of the dynamics of deformable bodies, and their solution entails complex mathematical machinery. The latter consideration, among other things, has long impeded the investigation of broad classes of problems involving assessment of the stress–strain state near inhomogeneities. Major advances have been gained mainly in such areas as the formulation of analytical solutions of a vast number of individual problems, in the majority of cases without analysis of dynamic stresses near an obstacle; the reduction of large classes of elastic-wave diffraction problems to systems of multivariate singular and regular integral equations with subsequent proof of the existence and uniqueness of a solution; and the development of asymptotic methods of solution, which generally are inapplicable for determination of the stress state near an interface. One of the basic objectives in the study of elastic-wave diffraction by inhomogeneities is to obtain not only a formal mathematical solution, but also a solution that will permit effective determination of the diffraction field of stresses and strains near inhomogeneities. Two basic trends in the study of dynamic stress near inhomogeneities have emerged in recent years in connection with the advent and utilization of computers. The first trend involves the development of numerical methods and concomitant quantization of problems into discrete form with the application of computers in every stage of solution. The future evolution of this trend, in light of the universality of its algorithms, will clearly provide the means for investigation of exceedingly general classes of problems. Nonetheless, the fundamental results obtained in the last few years, mainly in the USSR and the United States, fall within the second main trend, which is associated with the first stage of solution of problems on the basis of analytical methods (separation of variables and its generalizations, perturbation methods, reduction to integral equations after partial separation of variables, etc.) and then the final stage with recourse to computers.

Following is a survey of research conducted to date within the context of the latter trend.

Simply Connected Domains

The broadest category of diffraction problems in an elastic medium is investigated for steady-state waves. If we limit our perspective to steady-state (stationary) wave motions, we find it possible to separate one of the independent variables—the time t—by taking out the exponential factor $e^{-i\omega t}$ (ω is the cyclic frequency), thereby simplifying the solution of the problem. On the other hand, the investigation of steady-state waves is important for many practical problems. Moreover, if it is possible to calculate the response of an elastic medium to steady-state disturbances over a wide range of frequencies, it will then be possible to analyze transient processes by means of the mathematical machinery of integral Fourier transforms.

Circular Hole. One of the first problems to be investigated in the class of steady-state wave problems was the diffraction of elastic waves by a cylindrical obstacle. The reason is that the solution of the Helmholtz equation in cylindrical (polar) coordinates had long been known, but a digital computer was needed in order to obtain numerical results. For example, in the case of the plane problem for a medium containing a circular cylindrical cavity (hole), introducing the scalar and vector potentials for the elastic displacement vector $\vec{u}$,

$$\vec{u} = \operatorname{grad}\varphi + \operatorname{rot}\vec{\psi}; \quad \vec{\psi} = \vec{k}\psi, \tag{1}$$

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where $\vec{k}$ is the unit vector normal to the plane of motion, we can formulate the diffraction problem as follows: Find a solution of the Helmholtz equation

$$\Delta\varphi + \frac{\omega^2}{c_1^2}\varphi = 0; \quad \Delta\psi + \frac{\omega^2}{c_2^2}\psi = 0 \tag{2}$$

subject to the boundary conditions at the edge of the cavity

$$\sigma_{rr}|_{r=R} = 0; \quad \sigma_{r\theta}|_{r=R} = 0 \tag{3}$$

and the radiation conditions [53] at infinity.

In expressions (2) and (3), Δ is the Laplace operator, c_1 and c_2 are the velocities of expansion and shear waves, σ_{rr} and $\sigma_{r\theta}$ are the components of the stress state, R is the radius of the cavity, and the exponential time factor $e^{-i\omega t}$ is dropped.

The solution of (2) satisfying the conditions at infinity is expressed in terms of cylindrical Hankel functions of the first kind $H_n^{(1)}$,

$$\varphi = \sum_{n=0}^{\infty} A_n H_n^{(1)}\left(\frac{\omega}{c_1}r\right)\cos n\theta; \quad \psi = \sum_{n=0}^{\infty} B_n H_n^{(1)}\left(\frac{\omega}{c_2}r\right)\sin n\theta. \tag{4}$$

Here r and θ are polar coordinates, and A_n and B_n are indeterminate constants.

In this setting the diffraction of a plane dilatational elastic wave near a circular hole in an unbounded thin plate under conditions of a generalized plane stress state has been investigated in [62]. The author determines the stress state in the vicinity of the edge of the hole and analyzes the dynamic stress concentration. It is established, in particular, that the dynamic stress coefficient exceeds its static value by about 10% in a certain frequency range. The stress distribution around the edge of the hole depends strongly on the frequency, varying quite abruptly as the latter is varied. The short- and long-wavelength limits are investigated, along with standing waves.

In [55] the analogous problem is solved for an incident plane harmonic shear wave. It is verified that the dynamic stress is approximately 20% greater than the static value.

Circular Inclusion. If a perfectly rigid disk is embedded in a circular hole, two types of boundary conditions are pertinent. The simplest prescribe zero displacements on the part of the inclusion. The solution obtained by Pao and Mow [96] for this problem shows that as the frequency tends to zero, the force keeping the inclusion immobile must grow without bound. It is more natural to state the problem in such a way that the inclusion can move together with the medium, its motion being determined from the Newtonian equation of motion. The force acting on the inclusion in this case is specified by the stress state in the vicinity of the cavity.

A similar problem is investigated in [55] for an incident plane shear wave. It is found that the ratio of the plate density to the inclusion density has a considerable influence on the stress–strain state in the plate. In certain cases the maximum dynamic stresses can be 37 to 105% greater than their values in the static case.

The representation of an external wave disturbance by a plane compressional or shear wave is an idealization relative to the real wave source, which usually has definite dimensions and is situated at a finite distance from the object. Consequently, a vital question in practice is how strong is the influence of a nearby source on the validity of the results obtained for a plane wave. The singularities of the stress state in the vicinity of a circular cylindrical cavity under the action of a cylindrical dilatational wave have been investigated in [80]. It is postulated that the incident wave is generated by a source located at a certain point (plane problem). It is determined that the disparity between the cases of a plane and a cylindrical wave exist primarily at low frequencies.

The stress state in certain cross sections of an elastic plate containing a hole under the action of a plane wave is determined in [52]. It is confirmed that the normal stress level increases at a certain distance from the edge of the hole, acting as a possible cause of cleavage effects.

The action of a plane harmonic wave incident at a certain angle with respect to the axis of a circular cylindrical cavity in an elastic solid is analyzed in [109].

Noncircular Hole. Whereas in the case of bodies bounded by circular cylindrical surfaces the problem is solvable by separation of variables, the latter method is no longer feasible in the case of cylindrical boundaries of more complex transverse section. For such bodies (boundaries), accordingly, special approximative methods have been developed. One effective method for the solution of problems in this category is the boundary-perturbation method proposed by Guz' [30, 33]. This method was developed in application to domains whose exterior is obtained for conformal mapping of a plane with a hole in the form of a unit disk by means of a mapping function of the type

$$z=\zeta+\varepsilon f(\zeta);\quad z=re^{i\theta};\quad \zeta=\rho e^{i\gamma}. \tag{5}$$

By selecting various functions $f(\zeta)$ and manipulating the parameter ε it is possible to generate a large set of hole configurations. Essentially, the method entails reduction of the problem for a noncircular hole to a sequence of problems in circular cylindrical coordinates with variable right-hand sides of the boundary conditions in each approximation. This method has been used by Kubenko [2] in solving various diffraction problems. The stress state of an elastic plate containing a curvilinear hole with a time-harmonic pressure applied at its circumference has been investigated in [46]. A solution is obtained in the form of a power series with respect to ε, in which the first three terms (three approximations) are determined. Final results are obtained for an elliptical, a square, and a triangular hole (the latter two with rounded corners). The practical convergence of the methods used is determined. The action of a steady-state plane compressional or shear wave on an elliptical or square hole has been investigated in [48-50]. The numerical results indicate that the peak dynamic stresses are 10 to 15% greater than the static value in a certain frequency range.

For a parabolic crack in an elastic medium the diffraction problem admits separation of variables in the equation and boundary condition in the event of antiplane deformation. The action of a plane wave on a cavity or rigid inclusion has been investigated for this case in [65]. The limiting case of a line crack or rigid cavity of zero thickness is also derived.

Spherical Cavity (inclusion). If we consider a spherical cavity (inclusion) under conditions of an axisymmetric stress–strain state, the general solution of the Helmholtz equations (2), written in spherical coordinates, can be expressed in the following form with regard for the radiation conditions:

$$\varphi=\sum_{n=0}^{\infty}A_n h_n^{(1)}\left(\frac{\omega}{c_1}r\right)P_n(\cos\theta);\quad \psi=\sum_{n=0}^{\infty}B_n h_n^{(1)}\left(\frac{\omega}{c_2}r\right)P_n(\cos\theta), \tag{6}$$

where $h_n^{(1)}$ denotes spherical Hankel functions of the first kind, P_n denotes Legendre polynomials, and r and θ are spherical coordinates.

The solution (6) is used in [97] to investigate the action of a plane harmonic dilatational wave on a spherical cavity, rigid and elastic inclusions, and a liquid sphere in an elastic medium. The influence of the curvature of spherical waves on the dynamic stress concentration is discussed in [57]. It is shown in these papers that in the case of plane waves the dynamic stress-concentration factor is approximately 10% higher than the static value and depends on the frequency. In the case of spherical incident waves it also depends on the distance from the source. Even when the distance from the source to a spherical cavity is 50 times the cavity radius, the maximum stress concentration is roughly 30% higher than in the plane-wave case. The plane-wave approximation of spherical waves is admissible when the wavelength is less than the distance from the source to the center of the obstacle.

Axisymmetric Body. For bodies of revolution other than a sphere it is extremely difficult to investigate the diffraction of elastic waves, because separation of variables is impossible in the wave equation. An exception is the case of a spheroidal body. In spheroidal coordinates, however, the wave equation admits separation of variables only in the axisymmetric case. The axisymmetric problem of diffraction of a plane harmonic dilatational wave by a moving rigid spheroidal inclusion injected into an elastic medium has been solved in [60]. The solution is formulated as a series in spheroidal functions. For evaluation of the unknown constants an infinite system of algebraic equations is derived, from which an approximate solution can be deduced by the truncation method. It follows from the numerical results that a frequency range exists wherein the stress concentration is somewhat enhanced.

For the case of arbitrary almost-spherical bodies of revolution Guz' [34] has proposed an approximate method of solution (the "boundary-perturbation method"), which makes it possible to reduce the problem in each successive approximation to the problem for a spherical cavity with variable right-hand sides of the boundary conditions. Nemish [58, 59] has formulated expressions for any approximation of the boundary-perturbation method and analyzed the convergence of the method; the method has also been used in [39] to investigate diffraction problems for torsional waves at bodies of revolution of almost-spherical configuration.

Cracks. An interesting and important practical problem area comprises dynamic problems in the theory of cracks and the problems of diffraction by cracks. Few such problems have been solved to date, primarily because of the difficulty of obtaining an effective mathematical solution insofar as the classical method of separation of variables is inapplicable in the dynamic case for an elastic body with a crack. Dynamic problems for bodies with cracks fall into two categories: 1) wave diffraction by stationary cracks; 2) crack propagation. Studies have recently been published, treating the interaction of elastic waves with mobile cracks.

We briefly summarize the fundamental results in the first category of problems. The diffraction problem for a plane wave at a semiinfinite line crack under conditions of antiplane deformation has been solved in parabolic coordinates in [65]. It has been confirmed that near the tip of the crack the stress has a singularity of order $1/\sqrt{\omega r}$, where r is the distance from the tip. Also investigated in the same paper is the diffraction problem for a plane wave at a rigid stationary ribbon of zero thickness. Unlike the crack problem, wave diffraction at a ribbon takes place even at zero angle of incidence. The interaction of a plane harmonic compressional wave with a semiinfinite line crack in an elastic half-space under conditions of plane deformation is discussed in [74, 92]. The wave potentials are constructed in the form of Fourier integrals. The solution of the problem is sought by the Wiener–Hopf method. The interaction of waves with a crack of finite dimensions can be analyzed in elliptical coordinates, as in [3, 64]. In [104] the diffraction of an antiplane wave by a finite crack is reduced to a system of dual integral equations, which is then reduced to an integral equation of standard type for a complex-valued unknown function. It is established that the dynamic stress-concentration factor is 27.5% higher than in the static case. In [106] the diffraction of a plane harmonic wave by a crack of finite length is investigated for conditions of plane deformation. The problem is reduced to a system of two consistent Fredholm integral equations, which admits numerical solution. In the vicinity of the tip of the crack it is possible to separate out principal terms and then use those terms to determine the stresses. The dynamic stress-concentration factor is 20 to 30% higher than the static value. The interaction of a standing torsional wave with an elastic space weakened by a penny-shaped crack has been investigated in [105]. The diffraction of a dilatational wave by a penny-shaped crack or a rigid inclusion has been studied in [85, 91, 98], and the same for a shear wave in [88].

Parton and Kudryavtsev [63] have investigated the important problem of diffraction of a plane shear wave by a rigid cylindrical inclusion that does not have continuous contact with the elastic medium around its entire contour, i.e., a finite curvilinear fracture exists at the interface between the elastic medium and the rigid inclusion. The problem is solved under conditions of antiplane deformation for an incident plane wave.

Circular Hole. Flexural Waves. Of considerable importance is the solution of problems in the diffraction of flexural waves in plates containing an obstacle in the form of a hole or an inclusion. In the majority of situations the solution is analyzed within the context of the classical theory of bending of plates, but extensive use has been made lately of various refinements of that theory.

Using the classical theory, Konenkov [42] has studied the solution of the problem of diffraction of flexural waves in a plate with a hole, whose edge can be free, clamped, or supported, as well as in a plate with an inclusion. Shvets [77] has obtained quantitative results for the case of a plane flexural wave incident on a stress-free hole, making use of the classical equations for bending of plates. Pao and Chao [95] have obtained quantitative results for a similar problem, invoking theoretical equations of the Timoshenko type.

More complex diffraction problems for flexural waves in a plate with a single circular hole have been solved in [84] for the case in which waves are excited by a point source near the hole. Saito and Nagaya [102] have investigated the diffraction of flexural waves in a plate with a hole subjected to dynamic loading in the edge regions.

Circular Inclusion. Flexural Waves. Three types of inclusions are considered in relation to the diffraction of flexural waves in a plate. A perfectly rigid stationary inclusion is discussed in [42], and the problems for a rigid mobile inclusion are solved in [76, 99]. The diffraction of a plane flexural wave by an elastic inclusion is solved in [99]. These problems are solved within the context of the classical theory.

Transient Waves. Inadequate attention has been given to the study of transient elastic-wave diffraction. The main reason for this deficit is the obvious difficulty of solving transient problems. As a rule, they are solved with recourse to integral transforms permitting separation of the time variable, whereupon the method of separation of variables is applied.

The propagation of transient axisymmetric and centrosymmetric waves generated by the application of dynamic loading to the surface of a cylindrical or a spherical cavity in a homogeneous isotropic elastic medium has been investigated in [89, 103, 108]. The case of a cylindrically (spherically) anisotropic medium is discussed in [86]. A cylindrical shear wave in an inhomogeneous medium is studied in [107], and the propagation of transient dilatational waves from a cylindrical (spherical) cavity in a cylindrically (spherically) anisotropic inhomogeneous elastic medium is covered in [44, 45, 47]. The authors of these publications determine the stress–strain state near the periphery of the cavity, at the wave front, and behind it. The propagation of a transient cylindrical wave is investigated in [93] in connection with application to the process of hole-punching in a thin infinite plate. The observed increase in the tangential stresses at the edge of the hole is interpreted by the author as responsible for the formation and growth of radial cracks in the material.

The elementary motions of a rigid cylinder or sphere included in an elastic medium have been investigated in [1, 2, 41]. Several publications have been devoted to the transient diffraction of elastic waves by obstacles. In [81-83] the integral Fourier transform is used to solve the problem of the action of a plane step wave on a cylindrical cavity in an elastic medium. The solution obtained in the papers is valid for later times, since only the two terms of the Fourier series left in the solution for large times are calculated. The displacements, velocity, and acceleration of a rigid cylinder under the action of a plane wave in an elastic medium have been determined in [66]. The case of a cylindrical cavity reinforced by a shell and subjected to the action of a plane step wave is treated in [82]. The transient stress concentration around a spherical cavity in an elastic medium under the action of a plane wave is analyzed in [67] by the methods of residue theory and in [61] by the method of Volterra integral equations, proposed by Kubenko [51]. The effects of a plane step dilatational wave on a spherical inclusion are discussed in [54].

Multiply Connected Domains

Quantitative results pertaining to the diffraction of elastic waves in multiply connected bodies have been obtained on the basis of addition theorems for special functions entering into solutions of the type (4) and (6). The solutions of multiply connected problems come in two versions. The first approach was developed by Guz' [31, 32, 36] and reduces the problem to the solution of infinite systems of algebraic systems. The second approach, which is adopted in [68] and other papers by non-Soviet authors, is the multiple-scattering technique. It is a special case of the first approach and essentially entails the solution of an infinite system by successive approximations [72].

Circular Obstacles. Among the first problems to be solved were those involving diffraction of elastic waves in a plate containing a finite set of circular holes subjected to harmonic pressure [5, 6]. Golovchan [7] has solved the problem of diffraction of a plane longitudinal wave by circular holes in an infinite plate. The same problem is solved in [75] by the multiple-scattering technique. A solution of the diffraction problem for a plane dilatational wave at several perfectly rigid circular cylinders is given in [68]. The multiple-scattering technique is used in [90] to determine the stress state of an elastic body with several elastic inclusions of circular cylindrical configuration, upon which is incident a cylindrical wave generated by a harmonic source situated at a certain distance from the set of inclusions.

Cherevko [69], using the classical theory of bending of plates, has solved the problem of diffraction of flexural waves in an infinite plate containing several circular holes, at the edges of which the bending moment is specified. The same problem is solved in [38] within the framework of Timoshenko theory. The problem of scattering of a plane flexural wave by several penny-shaped cracks in a plate is analyzed in [70], both in the classical setting and with the application of Timoshenko theory. On the basis of the multiple-scattering technique the problem of scattering of a plane flexural wave by holes in a plate is solved in [100] (classical theory) and [101] (Timoshenko theory). Gritsai [29] has derived fundamental relations for the problem of diffraction of flexural waves in a transverse isotropic plate weakened by two identical circular holes, taking into account the inertia of rotation and transverse shear. In the case of a plate with several circular inclusions Nagaya and Saito [94] have obtained a solution of the diffraction problems for flexural and torsional waves, using equations of Timoshenko theory for the plate.

The solution of concrete problems for bodies containing several circular obstacles has disclosed the "local resonance" effect in the domain between reflecting surfaces, where considerable stress enhancement, several times the stress in a body with one obstacle, is observed.

The method developed by Guz' for the solution of elastic-wave diffraction problems has resulted in the effective solution of the diffraction problem for bodies containing an array of circular obstacles. Applications of this method for the solution of periodic and doubly periodic elastic-wave diffraction problems are described in [24, 26, 35].

For an elastic body containing a line array of identical circular holes Golovchan [14] has obtained numerical results for the stress field when the surfaces of the cavities are acted upon by tangential forces varying harmonically with time and when a plane SH wave is incident on an array of holes. The field in the vicinity of holes in a body containing several identical cavities is determined in [15, 16, 25] for the cases of a harmonic pressure applied to the surfaces of the holes and incidence of a plane longitudinal wave on the holes. The problem of diffraction of a plane SH wave by an array of circular cylindrical elastic inclusions in a deformable body is solved in [40]. This study reveals a considerable increase in the radial tangential stresses along the junction lines as the rigidity of the inclusion is increased, resulting in peeling of the inclusion. The scattering of a plane expansion wave by an array of elastic inclusions is discussed in [73], in which the stress fields are compared for various combinations of elastic properties of the inclusions.

The problem of diffraction of flexural waves in a plate containing an array of identical penny-shaped cracks with a given bending moment at the edges has been solved by Guz', Golovchan, and Cherevko [37]. The case of scattering of a plane flexural wave by an array of circular holes in a plate is considered in [71], in which quantitative results are obtained for the stress field. Shvets and Gritsai [78] have derived fundamental relations for the problem of diffraction of flexural waves in a transverse isotropic plate with an annular array of identical circular holes.

The solution of periodic problem has disclosed anomalous variations of the stress fields, namely a large increase or abrupt variation around "glide" points, which are determined by the expression

$$2\pi m = \alpha\delta(1 \pm \cos\gamma) \quad (m = 0, 1, \ldots).$$

Here α is the incident wave number, γ is the angle of wave incidence, and δ is the distance between inclusion centers.

For SH-wave diffraction problems and for flexural waves in a plate, in the classical setting and within the framework of Timoshenko theory, we have a single family of glide points below the critical frequency. In longitudinal-wave diffraction problems there are two families of glide points. In the solution of flexural-wave diffraction problems for plates in connection with the application of Timoshenko theory at driving frequencies above the critical point there are three families of glide points.

Studies by Golovchan [8, 10] are devoted to the solution of problems of elastic-wave propagation in a cylinder with longitudinal cavities. Kosmodamianskii and Moiseenko [43, 56] have investigated the problems of determining the dynamic stress state of a circular slab with eccentrically distributed holes or inclusions.

Simultaneous application of the method of specular reflection with the solution of multiply connected diffraction problems has made it possible to solve a number of problems in the diffraction of SH waves in bodies containing linear and circular boundaries [12, 79]. The problems of diffraction of shear waves in a half-space containing a circular cavity have been solved in [9, 18]. The diffraction of SH waves by elliptical cylinders in a half-space is also discussed in [18]. A solution is obtained in [36] for the problem of SH-wave diffraction in a quarter-space containing a cavity, as well as in a layer with a hole. Golovchan and Guz' [28] have investigated the problem of elastic-wave diffraction in a layer with a line array of circular holes.

Spherical Cavities. Addition theorems for spherical functions can be used to solve elastic-wave diffraction problems in bodies containing spherical cavities. Various boundary-value problems associated with the diffraction of elastic waves by several spherical cavities have been investigated in [11, 17, 20, 27]. A numerical study of the stress state of a body containing two identical spherical cavities is described in [22, 27], where, as in the case of bodies with cylindrical boundaries, the local resonance effect is disclosed. Golovchan [19] has investigated the problem of elastic-wave diffraction by an array of spherical cavities. The same author [13] poses the objective of solving fundamental boundary-value problems in the diffraction of elastic waves in a half-space containing spherical cavities. Quantitative results for the problem of oscillations of a spherical shell of variable thickness are presented in [21]. Another paper [23] is concerned with solving the problem of torsion of an elastic cylinder containing spherical cavities.

LITERATURE CITED

1. A. I. Babichev, "Translational motion of a sphere in an elastic medium," Izv. Akad. Nauk UzbekSSR, Ser. Tekh. Nauk, No. 4, 35-40 (1966).
2. A. I. Babichev, "Analysis of the interaction of elastic waves with cylindrical and spherical obstacles by the method of characteristics," in: Proceedings of the All-Union Symposium on Propagation of Elastoplastic Waves in Continua, 1964 [in Russian], Akad. Nauk AzerbSSR, Baku (1966), pp. 443-456.

3. N. M. Borodachev, "Dynamic friction problem in the case of longitudinal shear strains," Probl. Prochn., No. 4, 23–25 (1973).
4. G. N. Watson, Theory of Bessel Functions, 2nd ed., Macmillan, New York (1945).
5. V. T. Golovchan, "Dynamic stress concentration in a plate containing two circular holes," Prikl. Mekh., 3, No. 11, 23–28 (1967).
6. V. T. Golovchan, "Dynamic stress distribution between holes in an infinite plate," Prikl. Mekh., 4, No. 4, 131–135 (1968).
7. V. T. Golovchan, "Diffraction of a longitudinal elastic wave at the edges of two circular holes in an infininte plate," Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, No. 4, 60–64 (1969).
8. V. T. Golovchan, "Plane vibrations of an eccentric cylinder," Prikl. Mekh., 5, No. 5, 120–124 (1969).
9. V. T. Golovchan, "Vibrations of a half-plane with circular holes," Prikl. Mekh., 6, No. 1, 113–115 (1970).
10. V. T. Golovchan, "Propagation of elastic waves in a cylinder containing longitudinal cavities," Dop. Akad. Nauk UkrRSR, Ser. A, No. 1, 45–47 (1970).
11. V. T. Golovchan, "Solutions of dynamic problems for an elastic body bounded by spherical surfaces," Prikl. Mekh., 6, No. 6, 30–36 (1970).
12. V. T. Golovchan, "Fundamental boundary-value problems in the dynamic theory of elasticity for domains bounded by circles and straight lines," Dop. Akad. Nauk UkrRSR, Ser. A, No. 6, 531–534 (1970).
13. V. T. Golovchan, "Solution of fundamental boundary-value problems for the wave equation in a half-space with spherical cavities," Akust. Zh., 17, No. 2, 235–239 (1971).
14. V. T. Golovchan, "Diffraction of a shear wave by an infinite line array of cylindrical cavities," Prikl. Mekh., 7, No. 3, 41–46 (1971).
15. V. T. Golovchan, "Solution of a plane problem of elastic-wave diffraction for a plate with an infinite line array of circular holes," Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, No. 5, 182–185 (1971).
16. V. T. Golovchan, "Diffraction of a longitudinal wave by an infinite line array of circular holes in an elastic plate," Prikl. Mekh., 7, No. 4, 74–81 (1971).
17. V. T. Golovchan, "Infinite algebraic systems of equations for problems in the diffraction of elastic waves at spherical surfaces," Dop. Akad. Nauk UkrRSR, Ser. A, No. 11, 1009–1013 (1971).
18. V. T. Golovchan, "Diffraction of a transverse elastic wave by elliptical cylinders in a half-space," Dop. Akad. Nauk UkRSR, Ser. A, No. 3, 246–249 (1971).
19. V. T. Golovchan, "Diffraction of elastic waves by an infinite line array of spherical cavities," Dop. Akad. Nauk UkrRSR, Ser. A, No. 2, 137–139 (1971).
20. V. T. Golovchan, "Addition theorems for the solenoidal part of the displacement vector of points of an elastic body in spherical coordinates," Dop. Akad. Nauk UkrRSR, Ser. A, No. 11, 1005–1008 (1973).
21. V. T. Golovchan, "Vibrations of a spherical shell of variable thickness," Prikl. Mekh., 10, No. 9, 90–103 (1974).
22. V. T. Golovchan, "Wave diffraction by two spherical cavities in an elastic space," Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, No. 1, 171–176 (1976).
23. V. T. Golovchan, "Torsional waves in an elastic cylinder containing spherical cavities," Prikl. Mekh., 12, No. 11, 25–32 (1976).
24. V. T. Golovchan and A. N. Guz', "Diffraction of elastic waves by an infinite line array of circular cylinders," Dokl. Akad. Nauk SSSR, 186, No. 2, 286–288 (1969).
25. V. T. Golovchan and A. N. Guz', "Diffraction of elastic waves by an infinite line array of circular holes," Dop. Akad. Nauk UkrRSR, Ser. A, No. 2, 159–161 (1970).
26. V. T. Golovchan and A. N. Guz', "Solution of two-dimensional periodic and doubly periodic problems in the theory of steady-state vibrations of elastic and viscoelastic bodies," in: Waves in Inelastic Media [in Russian], Akad. Nauk MoldSSR, Kishinev (1970), pp. 57-63.
27. V. T. Golovchan and A. N. Guz', "Solution of problems in the diffraction of elastic waves by spherical cavities," Prikl. Mekh., 8, No. 9, 118–122 (1972).
28. V. T. Golovchan and A. N. Guz', "Propagation of shear waves in an elastic layer perforated with a row of cylindrical holes," Prikl. Mekh., 12, No. 9, 18–23 (1976).
29. S. V. Gritsai, "Dynamic bending stresses in wood panels weakened by two circular holes," in: Forestry and the Timber, Paper, and Woodworking Industry [All-Republic Interinstitute Scientific-Technical Collection] [in Russian], No. 1 (1973), pp. 39–43.
30. A. N. Guz', "Approximate method for calculation of the stress concentrations around curvilinear holes in shells," Prikl. Mekh. [in Ukrainian], 2, No. 6, 605-612 (1962).
31. A. N. Guz', "Solution of dynamic problems for several parallel cylindrical cavities," in: Problems in the Mechanics of Rocks [in Russian], Akad. Nauk KazSSR, Alma-Ata (1966), pp. 137-144.

32. A. N. Guz', "Solution of the second plane dynamic elasticity problem for multiply connected domains," Prikl. Mekh., 2, No. 8, 126–132 (1966).
33. A. N. Guz', "A method for the solution of three-dimensional linear problems of continuum mechanics for noncanonical domains," Dop. Akad. Nauk UkrRSR, Ser. A, 352–355 (1970).
34. A. N. Guz', "Wave diffraction by finite bodies of revolution," Prikl. Mekh., 9, No. 7, 10–18 (1973).
35. A. N. Guz' and V. T. Golovchan, "Solution of two-dimensional doubly periodic problems in the theory of steady-state vibrations of viscoelastic bodies," Prikl. Mat. Mekh., 33, No. 4, 756–759 (1969).
36. A. N. Guz' and V. T. Golovchan, Diffraction of Elastic Waves in Multiply Connected Bodies [in Russian], Naukova Dumka, Kiev (1972).
37. A. N. Guz', V. T. Golovchan, and M. A. Cherevko, "Diffraction of antisymmetric waves by a line array of penny-shaped cracks in an infinite plate," Prikl. Mekh., 10, No. 8, 56–61 (1974).
38. A. N. Guz', V. T. Golovchan, and M. A. Cherevko, "Diffraction of flexural waves in an infinite plate with several penny-shaped cracks," in: Theoretical Applied Mechanics [All-Republic Interinstitute Scientific-Technical Collection] [in Russian], No. 7 (1976), pp. 8–15.
39. A. N. Guz', R. Yu. Kerimov, and S. Yu. Kerimov, "Diffraction of torsional waves by bodies of revolution," Izv. Akad. Nauk AzerbSSR, Ser. Fiz.-Tekh. Mat. Nauk, No. 2, 144–149 (1973).
40. A. N. Guz' and M. A. Cherevko, "Diffraction of shear waves by an array of circular elastic filaments," Mekh. Polim., No. 2, 337–341 (1977).
41. A. N. Kovshov and I. V. Simonov, "Certain motions of a rigid sphere embedded in an infinite elastic medium," Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, No. 5, 155–162 (1967).
42. Yu. K. Konenkov, "Diffraction of a flexural wave by a circular obstacle in a plate," Akust. Zh., 10, No. 2, 186–190 (1964).
43. A. S. Kosmodamianskii and A. A. Moiseenko, "The dynamic elasticity problem for an inhomogeneous plate," Dop. Akad. Nauk UkrRSR, Ser. A, No. 7, 624–627 (1971).
44. V. D. Kubenko, "Propagation of an elastic expansion wave from a circular hole in a cylindrically anisotropic inhomogeneous plate," in: Stress Concentration [in Russian], No. 1, Naukova Dumka, Kiev (1965), pp. 164–173.
45. V. D. Kubenko, "Propagation of elastic waves from a circular hole in an anisotropic inhomogeneous plate," Prikl. Mekh., 1, No. 2, 25–33 (1965).
46. V. D. Kubenko, "Stresses around an elliptical hole subjected to oscillating pressure," Prikl. Mekh., 1, No. 5, 133-137 (1965).
47. V. D. Kubenko, "Propagation of elastic waves from a spherical cavity in an inhomogeneous anisotropic medium," in: Proceedings of the First All-Republic Conference of Young Mathematicians of the Ukraine [in Russian], Inst. Mat. Akad. Nauk UkrSSR, Kiev (1965), pp. 378-389.
48. V. D. Kubenko, "Dynamic stress concentration around a square hole under steady-state wave motions," Prikl. Mekh., 2, No. 12, 67–75 (1966).
49. V. D. Kubenko, "Dynamic stress concentration around an elliptical hole," Dop. Akad. Nauk UkrRSR, Ser. A, No. 3, 60–64 (1967).
50. V. D. Kubenko, "Propagation of a plane harmonic shear wave in a plate with a square hole," Prikl. Mekh., 4, No. 2, 129–133 (1968).
51. V. D. Kubenko, "Solution of diffraction problems for transient elastic waves at obstacles of cylindrical and spherical configuration," Dokl. Akad. Nauk UkrSSR, Ser. A, No. 10, 901–906 (1975).
52. V. D. Kubenko and Z. M. Zul'fugarov, "Stress state around a circular cavity in the dynamic plane problem of couple elasticity theory," Izv. Akad. Nauk AzerbSSR, No. 4, 99–102 (1969).
53. V. D. Kupradze, Potential Methods in the Theory of Elasticity [in Russian], Fizmatgiz, Moscow (1963).
54. C. C. Mow, "Transient response of a rigid spherical inclusion in an elastic medium," Trans. ASME, Ser. E: J. Appl. Mech., 87, No. 3, 637 (1965).
55. C. C. Mow and L. J. Mente, "Dynamic stresses and displacements around cylindrical discontinuities due to plane harmonic shear waves," Trans. ASME, Ser. E: J. Appl. Mech., 86, No. 4, 598 (1963).
56. A. A. Moiseenko, "Influence of an eccentric dynamic load on the stress–strain state of an inhomogeneous isotropic plate," in: Theoretical and Applied Mechanics [All-Republic Interinstitute Scientific-Technical Collection] [in Russian], No. 4 (1973), pp. 62–69.
57. F. C. Moon and Y.-H. Pao, "The influence of the curvature of spherical waves on dynamic stress concentration," Trans. ASME, Ser. E: J. Appl. Mech., 89, No. 2, 373 (1967).
58. Yu. N. Nemish, "The 'boundary-perturbation' method in three-dimensional problems in the mechanics of deformable media," Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, No. 1, 17–26 (1975).

59. Yu. N. Nemish, "Foundation of the perturbation method in three-dimensional problems in the mechanics of deformable media," Prikl. Mekh., 13, No. 12, 23–33 (1977).
60. M. A. Oien and Y.-H. Pao, "Scattering of compressional waves by a rigid spheroidal inclusion," Trans. ASME, Ser. E: J. Appl. Mech., 95, No. 4 (1973).
61. N. N. Panasyuk, "Action of a plane step elastic wave on a spherical cavity," in: Waves in Continuous Media [in Russian], Naukova Dumka, Kiev (1978), pp. 79–85.
62. Y.-H. Pao, "Dynamical stress concentration in an elastic plate," Trans. ASME, Ser. E: J. Appl. Mech., 84, No. 2, 299 (1962).
63. V. Z. Parton and B. A. Kudryavtsev, "A dynamic problem of fracture mechanics for a plane with an inclusion," in: Mechanics of Deformable Bodies and Structures [in Russian], Mashinostroenie, Moscow (1975), pp. 379–384.
64. V. Z. Parton and E. M. Morozov, Mechanics of Elastoplastic Fracture [in Russian], Nauka, Moscow (1974).
65. S. A. Thau and Y.-H. Pao, "Diffractions of horizontal shear waves by a parabolic cylinder and dynamic stress concentrations," Trans. ASME, Ser. E: J. Appl. Mech., 88, No. 4, 785 (1966).
66. M. J. Forrestal and W. E. Alzheimer, "Transient motion of a rigid cylinder produced by elastic and acoustic waves," Trans. ASME, Ser. E: J. Appl. Mech., 90, No. 3, 134 (1968).
67. H. Huang and Y. F. Wang, "Transient stress concentration by a spherical cavity in an elastic medium," Trans. ASME, Ser. E: J. Appl. Mech., 94, No. 4, 1002 (1972).
68. S. L. Cheng, "Multiple scattering of elastic waves by parallel cylinders," Trans. ASME, Ser. E: J. Appl. Mech., 91, No. 3, 523 (1969).
69. M. A. Cherevko, "Diffraction of flexural waves in a thin infinite plate with cracks," Dop. Akad. Nauk UkrRSR, Ser. A, No. 4, 337–339 (1974).
70. M. A. Cherevko, "Diffraction of plane flexural waves in an infinite multiply connected plate," Prikl. Mekh., 10, No. 6, 66–72 (1974).
71. M. A. Cherevko, "Scattering of a plane flexural wave by an infinite array of penny-shaped cracks in a plate," Prikl. Mekh., 10, No. 9, 118–121 (1974).
72. M. A. Cherevko, "On the multiple-scattering technique in diffraction theory," Dokl. Akad. Nauk UkrSSR, Ser. A, No. 9, 814–817 (1975).
73. M. A. Cherevko, "Diffraction of longitudinal waves by a line array of circular elastic inclusions," Prikl. Mekh., 14, No. 2, 67–72 (1978).
74. G. P. Cherepanov, Mechanics of Brittle Fracture [in Russian], Nauka, Moscow (1974).
75. S. L. Cheng, "Dynamic stresses in a plate with circular holes," Trans. ASME, Ser. E: J. Appl. Mech., 94, No. 1, 129 (1972).
76. R. N. Shvets, "Dynamic bending stresses in a thin plate containing a foreign inclusion," Fiz.-Khim. Mekh. Mater., 7, No. 1, 82–85 (1971).
77. R. N. Shvets, "Dynamic bending-stress concentration in a thin plate with a circular inhomogeneity," in: Stress Concentration [in Russian], No. 3, Naukova Dumka, Kiev (1971), pp. 203–208.
78. R. N. Shvets and S. V. Gritsai, "Dynamic stresses in a transverse isotropic plate with an annular array of identical circular holes," in: Mathematical Methods and Physicochemical Fields [All-Republic Interinstitute Scientific-Technical Collection] [in Russian], No. 3 (1976), pp. 57–61.
79. N. A. Shul'ga, "Diffraction of waves by circular obstacles in a half-plane," Prikl. Mekh., 5, No. 5, 115–119 (1969).
80. M. T. Jakub and C. C. Mow, "On the effects of source proximity on the dynamic stresses around a cylindrical cavity," Trans. ASME, Ser. E: J. Appl. Mech., 89, No. 2 (1967).
81. M. L. Baron and A. T. Matthews, "Diffraction of pressure wave by a cylindrical cavity in an elastic medium," Trans. ASME, Ser. E: J. Appl. Mech., 83, No. 3, 347–354 (1961).
82. M. L. Baron and R. Parness, "Diffraction of a pressure wave by a cylindrical shell and an elastic medium," in: Proceedings of the Fourth U. S. National Congress on Applied Mechanics, Pergamon, New York (1962), pp. 63–75.
83. M. L. Baron and R. Parness, "Displacement and velocities produced by the diffraction of a pressure wave by a cylindrical cavity in an elastic medium," Trans. ASME, Ser. E: J. Appl. Mech., 84, No. 2, 385–395 (1962).
84. P. M. Culkowski and H. Reismann, "Diffraction of flexural wave by an inner circular boundary in an unbounded flat plate," Z. Angew. Math. Mech., 53, No, 9, 519–525 (1973).
85. S. K. Datta, "The diffraction of a plane compressional elastic wave by a rigid circular disk," Quart. Appl. Math., 28, No. 1, 1–14 (1970).
86. G. Eason, "Propagation of waves from spherical and cylindrical cavities," Z. Angew. Math. Phys., No. 1, 12–23 (1963).

87. J. N. Goodier, "Propagation of a sudden rotary disturbance in an elastic plate in plane stress," Trans. ASME, Ser. E: J. Appl. Mech., 78, No. 2, 284–286 (1956).
88. D. L. Jain and R. P. Kanwal, "Diffraction of a plane shear elastic wave by a circular rigid disk and a penny-shaped crack," Quart. Appl. Math., 4, No. 3, 283–298 (1972).
89. A. Kromm, "Zur Ausbreitung von Stobwellen in Kreislochscheiben," Z. Angew. Math. Mech., 28, No. 4, 104–114 (1948).
90. I. Maekawa and T. Fukagawa, "Interference effect in dynamic stress concentration (effect of cylindrical wave on circular discontinuities)," Bull. Japan. Soc. Mech. Eng., 20, No. 140, 153–159 (1977).
91. A. K. Mal, "Dynamic stress-intensity factor for a nonaxisymmetric loading of the penny-shaped crack," Int. J. Eng. Sci., 6, No. 12, 725–733 (1968).
92. A. W. Maue, "Die Entspannungswelle bei plötzlichen Einschnitt eines gespannten elastischen Körpers," Z. Angew. Math. Mech., 34, Nos. 1/2, 1-12 (1954).
93. J. Miklowitz, "Plane-stress unloading wave emanating from a suddenly punched hole in a stretched elastic plate," Trans. ASME, Ser. E: J. Appl. Mech., 82, No. 1, 681–689 (1960).
94. K. Nagaya and H. Saito, "Transverse vibration and wave propagation in an infinite thin elastic plate with circular inclusions," Bull. Japan. Soc. Mech. Eng., 17, No. 111, 1121-1128 (1974).
95. Y.-H. Pao and C. C. Chao, "Diffraction of flexural waves by a cavity in an elastic plate," AIAA J., 2, No. 11, 145-152 (1964).
96. Y.-H. Pao and C. C. Mow, "Dynamic stress concentration in an elastic plate with a rigid circular inclusion," in: Proceedings of the Fourth U. S. National Congress on Applied Mechanics, Pergamon, New York (1962), pp. 335–345.
97. Y.-H. Pao and C. C. Mow, "Scattering of plane compressional waves by a spherical obstacle," J. Appl. Phys., 34, No. 3, 493–499 (1963).
98. J. A. Robertson, "Diffraction of a plane longitudinal wave by a penny-shaped crack," Proc. Cambridge Philos. Soc., 63, No. 2, 229–238 (1967).
99. H. Saito and K. Nagaya, "Flexural wave propagation in an infinite thin plate with a circular inclusion," Technol. Rep. Tohoku Univ., 36, No. 2, 399–412 (1971).
100. H. Saito and K. Nagaya, "Flexural wave propagation in a thin plate with circular holes," Bull. Japan. Soc. Mech. Eng., 16, No. 97, 1045–1052 (1973).
101. H. Saito and K. Nagaya, "Flexural wave propagation in a plate with circular holes," Bull. Japan. Soc. Mech. Eng., 16, No. 100, 1506–1512 (1973).
102. H. Saito and K. Nagaya, "Flexural vibrations in an infinite thick plate with a circular hole to dynamical loads at the hole," Bull. Japan. Soc. Mech. Eng., 17, No. 109, 896–903 (1974).
103. H. L. Selberg, "Transient compression waves from spherical and cylindrical cavities," Ark. Fys., No. 1, 26–30 (1952).
104. G. C. Sih, "Some elastodynamic problems of cracks," Intern. J. Fract. Mech., 4, No. 1, 51–68 (1968).
105. J. F. Loeber and G. C. Sih, "Torsional vibration of an elastic solid containing a penny-shaped crack," J. Acoust. Soc. Amer., 44, No. 5, 1237-1245 (1968).
106. J. F. Loeber and G. C. Sih, "Wave propagation in an elastic solid with a line of discontinuity of finite crack," Q. Appl. Math., 27, No. 2, 193-213 (1969).
107. E. Sternberg and L. C. Chakravorty, "On the propagation of shock waves in a nonhomogeneous elastic medium," ASME, J. Appl. Mech., 81, No. 4, 528-536 (1959).
108. V. Vodička, "V. Radial vibrations of an infinite medium with a cylindrical cavity," ZAMP No. 2, 166-172 (1963).
109. R. M. White, "Elastic wave scattering at a cylindrical discontinuity in a solid," J. Acoust. Soc. Amer., 30, No. 5, 934-939 (1958).