MULTICRITERIAL OPTIMIZATION

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Traditional methods of scalar optimization, very effective for the solution of sufficiently simple problems of optimal design, are largely ineffective for the solution of complex problems in real design conditions. This is because, in essence, problems of optimal design are multicriterial, hierarchical, and decompositional [9]. In fact, the difficulty of solving large and complex problems is associated primarily with indeterminacies in estimating efficiency, conditions, and aims. The latter indeterminacy arises in that in complex problems there are always several aims, each of which corresponds to its own local criterion. In consequence, the estimation of the efficiency of solutions and the optimum choice of solutions must be made with respect to several criteria, and the problem is formalized as a vector model of the choice of solutions [2, 4, 5].

1. Vector optimization is associated with a number of problems: criteria may be incommensurate, of different importance, and contradictory, which means that the choice of optimal solutions must be made on the basis of a certain set of compromises. The difficulty in choosing an expedient set of compromises is that, in general, in the vector estimation of efficiency, there is an indefinite number of viewpoints as to what constitutes an optimal solution; each of these viewpoints corresponds to a definite principle of optimality. In choosing the principle, the following logical sequence must be followed: situation—axiomatics—optimality principle [2], i.e., the given situation determines the choice of the corresponding compromise axiomatics (set of axioms), and then the search for optimal solutions begins. Thus, the main point is that all the basic problems (choice of optimality principle, method of normalization, and principle adopted in taking account of priorities) are solved in strict accordance with the nature of the real situation of choice between solutions.

The present work considered the optimal design of rod and thin-walled systems in conditions where there are several aims: minimum weight of the structure, maximum ease of construction, and maximum rigidity of the system (minimum tendency to deformation). The criteria of the problem may be categorized into three types [9]: a) inequality type

$$H(S, X_s) \geqslant 0 \quad \text{or} \quad h_i(S, X_s) \geqslant 0 \quad (i = 1, 2, \dots, m); \tag{1.1}$$

b) equality type

$$T(S, X_s) = 0 \quad \text{or} \quad t_i(S, X_s) = 0 \quad (i = 1, 2, \dots, n); \tag{1.2}$$

c) extremal type

$$U(S, X_s) \to \text{extr} \quad \text{or} \quad u_i(S, X_s) \to \text{extr} \quad (i = 1, 2, \dots, p), \tag{1.3}$$

where S is the structure of the system; X_S are the parameters of the system for a given structure from the set of parameters Ψ_X and the set of structures Φ_S.

If the requirement of minimum tendency to deformation may be written in the form in Eq. (1.1), the search problem for a system of minimum weight with constraints on the strength, rigidity, and stability of the form in Eq. (1.1), taking into account coupling conditions of the form in Eq. (1.2), reduces to the well-known problem of mathematical programming with one criterion of the extremal type in Eq. (1.3). However, rigidity conditions — e.g., constraints on the displacement of points — cannot always be written in the form of Eq. (1.1). In practice these conditions are often indeterminate; i.e., the permissible displacement of specific points cannot be clearly specified. In some cases, also, constraints on the displacement of several points cannot simultaneously be satisfied, because they are contradictory or possibly statically unachievable. There then arises the search problem for a structure that has little tendency to deformation, and at the same time is of reasonable weight (two criteria) and relatively simple to construct (three criteria). This optimal-design problem may be formulated as follows

$$Y_p(S, X_s) \to \underset{S, X_s \in \Phi}{\text{extr}} \quad (p = 1, 2, \dots,); \tag{1.4}$$

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$$\psi:\begin{cases} h_i(S, X_s) \leqslant 0 & (i = 1, 2, \ldots, m); & (1.5)\\ g_j(S, X_s) = 0 & (j = 1, 2, \ldots, n); & (1.6)\\ X_s \in \Phi_x(S) & (S \in \Phi_s), & (1.7)\end{cases}$$

2. Suppose that, given the outline and structure of a sphere—rod system, its load and material, and the shape of the rod cross section, it is required to find the optimal solution (cross-sectional area) $F^0 \in \varphi_x^*$ corresponding to the efficiency vector $Y(y_1, y_2, y_3)$, where y_1 is the weight of the system; $y_2 = T_{co}$ is the difficulty of construction; $y_3 = \delta$ is the displacement of a point of the system, characterizing the tendency to deformation of the system; φ_x^* is the permissible discrete set of areas chosen from among a series of sets. It is also necessary to satisfy the conditions of strength, stability, and discreteness of the areas, and also the condition that the elements of the system belong to a definite group q with the same areas:

$$\left.\begin{aligned} Y_1 = G = \sum_{i=1}^{m} \gamma F_i^* l_i \\ T_{co} = K \frac{1}{a} q^b \sqrt{Gm} \\ \delta_{rs} = \sum_{i=1}^{m} \frac{N_{is}\overline{N}_i r}{EF_i^*} l_i \end{aligned}\right\} \to \min_{F^* \in \psi}; \tag{2.1}$$

$$\psi\begin{cases} \varphi_i R_i F_i \leqslant \left(N_{is}^{(0)} + \sum_{j=1}^{n} \overline{N}_{ij} X_{js}\right) \leqslant R_i F_i; & (2.2)\\ \sum_{i=1}^{m} \overline{N}_{ij} \sigma_{is} l_i = 0; & (2.3)\\ \left(N_{is}^{(0)} + \sum_{j=1}^{n} \overline{N}_{ij} X_{js}\right) \frac{1}{\sigma_{is}} \geqslant [F_{\min}] > 0; & (2.4)\\ F_i = \{F_1^*, F_2^*, \ldots, F_m^*\} \in \Phi_x^*; & (2.5)\\ F_i \in Q;\ Q = \{F_1^*, F_2^*, \ldots, F_{[q]}^*\} \in \Phi_x^*;\ [q] < m & (2.6)\end{cases}$$

$(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n;\ s = 1, 2, \ldots, k)$,

where m is the number of rods in the system; $K = \psi_T c k_T$ is a coefficient taking into account the type of construction and the use of high-strength steels [7]; a and b are parameters, constant for each type of system, determined by the statistical dependence $T_{co} = f(q)$; q is the number of type-dimensions (groups of identical cross-sectional area etc.); $\varphi_i = \varphi_i(\lambda_i)$ are nonlinear functions characterizing the dependence between the coefficient of calculated-drag reduction for compressible rods and the ductility λ_i; $\varphi_i = m_1/\lambda_i^2$ if $\lambda_i \geq \lambda_b$; $\varphi_i = 1 - m_2\lambda^2$ if $\lambda_i \leq \lambda_b$; m_1 and m_2 are coefficients depending on the material; λ_b is the boundary value of the ductility corresponding to the point of inflection of the curve of $\varphi_i = \varphi_i(\lambda_i)$. The conventional notation is used for other quantities (lengths, rod cross-sectional areas, forces, etc.).

This formulation corresponds to a vector optimization model of general form [2, 4]

$$F^{(0)} = \Phi^{-1}\{\underset{F^* \in \psi}{\mathrm{opt}}\,(Y, \Omega)\}, \tag{2.7}$$

where opt is an optimization operator, defining the optimality principle and having the meaning of an order relation; Φ^{-1} is the inverse mapping $Y \to F^* = \Phi^{-1}(Y)$, if the efficiency vector Y is related to the solution by the mapping $F^* \to Y = \Phi(F^*)$, given analytically, statistically, or heuristically; $\Omega = (\omega_1, \omega_2, \ldots, \omega_p)$ is the priority vector, indicating the relative importance of the local criteria $y_1, y_2, y_3, \ldots, y_p$; p is the number of criteria.

In order to establish in what sense the optimal solution is superior to all the other permissible solutions, it is necessary to expand the operator opt (Y, Ω) appearing in the model in Eq. (2.7).

In the case when the criteria are normalized and of the same importance, the natural tendency is to increase the quality of all the local criteria uniformly and harmoniously. This idea may be realized, e.g., using the quasiequality principle [2]

$$\mathrm{opt}\, y_1 \equiv \{y \mid \| y_j - y_\nu \| \leqslant \varepsilon;\ j, \nu \in I\} \cap Y^c. \tag{2.8}$$

Here all the local criteria are minimized under the condition that the difference in the levels of the various criteria does not exceed ε ($\varepsilon \to \min$).

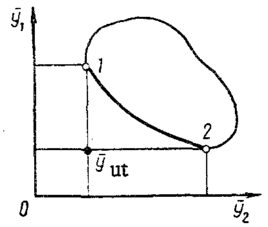

Fig. 1

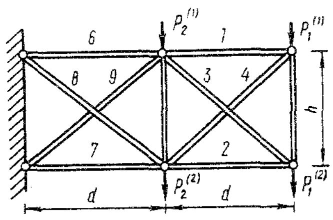

Fig. 2

The idea of a valid compromise ("validity" in this context means that the levels of all the local criteria tend to equalize) is realized in the optimality principle [2]

$$\operatorname{opt} y_2 \equiv \min_{y \in Y^c} \prod_{p \in I} Y_p. \tag{2.9}$$

If it cannot be assumed that the local criteria are of the same importance, a correction is introduced into the model in Eq. (2.9) by means of the weighting vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)$, which is a p-dimensional vector defined in the region

$$\alpha \in A = \{\alpha \mid \alpha_p \in [0,1],\ \sum_p \alpha_p = 1,\ p \in I\}. \tag{2.10}$$

The optimal solution is then found on the basis of an optimization model of the form [2]

$$\operatorname{opt} y_3 \equiv \min_{y \in Y^c} \prod_p y_p^{\alpha_p}. \tag{2.11}$$

Note that in most cases in practice it is impossible to make a well-founded choice of weighting factors. In those cases when the weighting factors cannot be chosen in the form in Eq. (2.10), the most expedient choice of optimal model, rather than those above, is the solution approaching the utopic point in critical space in the sense

$$\sum_{j=1}^{p} (y_j - y_{\ \ j})^2 \to \min, \tag{2.12}$$

where y_{utj} is the optimum value of the criterion y_j, which corresponds to a control belonging to the impermissible control region [3].

It should be emphasized that the minimization of Eqs. (2.8), (2.9), (2.11), and (2.12) gives points in control space and the solutions obtained are Pareto-optimal, i.e., belong to the set of unimprovable solutions (Fig. 1).

As an illustration, two examples — the optimization of a sphere–rod system and a thin-walled framed plate — will be considered.

3. Example 1. The outline and structure of a system is given, together with its load (Fig. 2), the material (St-3) with the calculated drags of tensioned (R^+ = 210 MPa) and compressed (R^- = 160 MPa) elements, and also the shape of the rod cross section (a T made from two equal-sided angle brackets), and the limiting elasticity of tensioned and compressed elements ($[\lambda^+] = 400$, $[\lambda^-] = 200$); h = 3 m; d = 4 m; a = 2.45; b = 0.5; $E = 2 \cdot 10^5$ MPa; $K = \psi_T c k_T = 3.4$ [7]; and

$$P_{\mathrm{I}} = \begin{Vmatrix} 400 \\ 0{,}0 \end{Vmatrix} \text{ (kN)}; \quad P_{\mathrm{II}} = \begin{Vmatrix} 0{,}0 \\ 400 \end{Vmatrix} \text{ (kN)}.$$

The following groups of rods of identical areas are established

$$q = 2, \quad \{I_1 = 1,\ 2,\ 3,\ 4,\ 5,\ 8,\ 9,\ 10;\ I_2 = 6,\ 7\};$$
$$q = 3, \quad \{I_1 = 1,\ 2,\ 5,\ 10;\ I_2 = 3,\ 4,\ 8,\ 9;\ I_3 = 6,\ 7\};$$
$$q = 4, \quad \{I_1 = 1,\ 2,\ 5,\ 10;\ I_2 = 3,\ 9;\ I_3 = 4,\ 8;\ I_4 = 6,\ 7\}.$$

The minimum radii of inertia are as follows: for rods 5 and 10

$$i_{\min}^{(+)} = 0.75 \text{ cm}, \quad i_{\min}^{(-)} = 1.5 \text{ cm};$$

for rods 1, 2, 6, and 7

$$i_{\min}^{(+)} = 1.0 \text{ cm}, \quad i_{\min}^{(-)} = 2.0 \text{ cm};$$

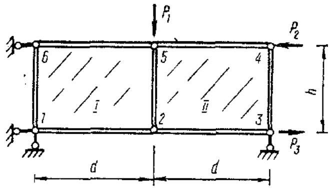

Fig. 3

for rods 3, 4, 8, and 9

$$i^{(+)}_{\min} = 1.25\ \text{cm}, \quad i^{(-)}_{\min} = 2.5\ \text{cm}.$$

The optimal solution with respect to each of the criteria in Eq. (2.1) is now found: with respect to the weight

$$G_{\min} = 7.36\ (\text{kN})\ (100\%), \quad q = 10,$$

$$F_G = \{F_i^*\}^2 = \{20.2;\ 12.4;\ 7.72;\ 34.4;\ 5.92;\ 38.4;\ 54.6;\ 21.1;\ 17.26;\ 6.96\}\ (\text{cm}^2):$$

with respect to difficulty of construction

$$T_{\min} = 1.4 \cdot \sqrt{G_1 m q_1} = 5.58\ \ (\text{man-h})\ \ (100\%), \quad q = 1, \ F_T = \{49.4\}\ (\text{cm}^2);$$

with respect to rigidity

$$\delta_{\min} = 1.961\ (\text{cm})\ (100\%), \quad q = 1.$$

An iterational method of the type in [6, 8] is used for optimization of the construction with respect to a single criterion, since methods of nonlinear programing are difficult to use when the constraints in Eqs. (2.2)-(2.6) are taken into account; this does not exclude optimization using other methods of the type in [9].

According to the principle in Eq. (2.9), the optimal solution corresponds to the value q = 2, i. e., introducing a measure of the relative decrease in quality of the solution with respect to each of the criteria — the "reduction cost" $\varkappa$ [4] — the following result is obtained: As $q_1 \to q_2$

$$\varkappa_1 > \varkappa_2 + \varkappa_3 \quad (0.813 > 0.39);$$

as $q_2 \to q_3$

$$\varkappa_1 < \varkappa_2 + \varkappa_3 \quad (0.14 < 0.157);$$

as $q_3 \to q_4$

$$\varkappa_1 < \varkappa_2 + \varkappa_3 \quad (0.066 < 0.128).$$

This solution corresponds to G_{opt} = 9.38 (kN) (127.44%); also

$$T_{opt} = 6.0195\ \ (\text{man-h})\ \ (108\%); \quad \delta_{opt} = 2.568\ (\text{cm})\ (131\%):$$

$$F^*_{opt} = \{23;\ 23;\ 23;\ 23;\ 23;\ 52.6;\ 52.6;\ 23;\ 23;\ 23;\}\ (\text{cm}^2).$$

The principle in Eq. (2.8) then gives the result

$$|\bar{Y}_1 - \bar{Y}_2| = \varepsilon^{(1)}_{\min} = 0.253 \to q = 2;$$

$$|\bar{Y}_1 - \bar{Y}_3| = \varepsilon^{(2)}_{\min} = 0.053 \to q = 3,$$

where $\bar{Y}_1$, $\bar{Y}_2$, and $\bar{Y}_3$ are the normalized values of the criteria according to the method of [4].

Comparing the sums of the differences in level of the criteria

$$P_1 = 0.253 + 0.074\ \ (q = 2) \text{ and } P_2 = 0.34 + 0.053\ \ (q = 3)$$

or calculating the total error

$$\Pi_1 = (27\% + 31\% + 8\%) = 64\%\ \ (q = 2); \quad \Pi_2 = (14\% + 35\% + 25\%) = 74\%\ \ (q = 3),$$

the result is again q_{opt} = 2.

According to Eq. (2.12)

$$\sum_{j=1}^{3} (\bar{y}_j - y_{utj})^2 \to \min$$

and this sum → **1.44** for q = 1, → 0.1754 for q = 2, → **0.209** for q = 3, i. e., $q_{opt} = 2$.

Example 2. It is required to find a framed plate of reasonable weight and rigidity, subjected to two types of load. The system is shown in Fig. 3. The following information is also given: The material is AK4-1; the calculated drag of the elements of the longitudinal and transverse sets is $R^{(+)} = 160$ MPa, $R^{(-)} = 100$ MPa, and that of the wall is $[\tau] = \bar{K}_i R^+ = 80$ MPa, $\bar{K}_i = 0.5$ [11]; E/G = 2.5; $E = 7.1 \cdot 10^4$ MPa, h = 200 mm, d = 400 mm. The first type of load corresponds to a force $P_1 = 40$ kN and the second to $P_2 = P_3 = 30$ kN. The areas of elements of the longitudinal and transverse sets within the limits of the panel are taken to be constant and to belong to one of the given groups of identical areas

$$q=2\colon \{I_1=1{,}11;\ I_2=1-2;\ 2-3;\ 3-4;\ 4-5;\ 5-6;\ 1-6;\ 2-5;\};$$

$$q=3\colon \{I_1=1{,}11;\ I_2=1-2;\ 2-3;\ 4-5;\ 5-6;\ I_3=1-6;\ 2-5;\ 3-4;\};$$

$$q=4\colon \{I_1=1{,}11;\ I_2=1-2;\ 4-5;\ 5-6;\ I_3=1-6;\ 2-5;\ I_4=2-3;\ 3-4;\};$$

$$q=5\colon \{I_1=1{,}11;\ I_2=1-2;\ 4-5;\ 5-6;\ I_3=2-3;\ 3-4;\ I_4=2-5;\ I_5=1-6\}, \text{ etc.}$$

The given problem may be formulated in the form in Eqs. (1.4)-(1.7) with the two optimality criteria

$$V=\sum_{i=1}^{m} F_i l_i + \sum_{j=1}^{c} f_j t_j; \tag{3.1}$$

$$\delta_{rs}=\sum_{i=1}^{m} \int_l \frac{N_{is}\bar{N}_{jr}}{EF_i}\,dl + \sum_{j=1}^{c} \int_f \frac{T_{js}\bar{T}_{jr}}{Gt_j}\,df, \tag{3.2}$$

where V is the volume of the system; δ_{rs}, displacement of a point r, characterizing the tendency to deformation of the system under a load s; m, number of elements of the longitudinal and transverse sets; F_i, cross-sectional area of the element i; f_j, area of wall j of height h and length d; t_j, wall thickness; $\bar{T}_{jr}$, shear force on the wall j due to a force P = 1 applied at the point r.

The structure is optimized with respect to one criterion using the algorithm of [6], which requires no more than 4-6 iterations to reach an optimal solution with respect to one criterion, satisfying the constraints specified in the conditions of the problem. The usual assumptions for thin-walled systems are made in the calculation [1, 12]. Questions of stability are not considered: It is assumed that the calculated drag of the set and the walls is chosen so that stability loss does not arise. Where required, stability of the thin-walled panels under shear may be verified, as in [10, 11], by calculating, and then making more accurate, the coefficient $\bar{K}_i$.

Optimizing the structure with respect to the volume for fixed q, and then maximizing the rigidity of the system according to [6] with fixed volume and q, the following values are obtained for the criteria:

$$q=2\to\bar{V}=1.4708;\quad \bar{\delta}=1;\quad q=3\to\bar{V}=1.295;\quad \bar{\delta}=1.099;$$

$$q=4\to\bar{V}=1.185;\quad \bar{\delta}=1.197;\quad q=5\to\bar{V}=1.155;\quad \bar{\delta}=1.216;$$

$$q=9\to\bar{V}=1.0;\quad \bar{\delta}=1.348, \text{ etc.}$$

These are normalized values of the criteria according to the method in [4]. According to the optimality principles in Eqs. (2.8) and (2.12), the best solution corresponds to $q_{opt} = 4$, $V^*_{opt} = 1048.48$ cm³ (118.5%), $\delta^*_{opt} = 0.1285$ cm (119.7%) with the following values of the local criteria (single-criterion optimization): $V_{opt} = 884.4$ cm³ (100%), q = 9; $\delta_{opt} = 0.107$ cm (100%), q = 2. The solution q = 4 corresponds to the following area vector (cm²) and wall thickness (cm): $F^*_{opt} = \{3;\ 1.5;\ 3;\ 4;\ 1.5;\ 0.2738;\ 0.2738\}$.

CONCLUSIONS

A structure of reasonable weight, rigidity, and ease of construction may be found using the theory of multicriterial optimization. For the given class of rod and thin-walled framed structures, the most expedient systems of compromises are those realized using the principles of quasiequality, valid reduction, and minimal deviation from the utopic point. The approach proposed for the solution of optimization problems for rod and thin-walled systems allows the factor of technological feasibility to be taken into account (F_i = const, $q < m$).

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EFFECT OF THE COMPLIANCE OF FOUNDATION SUPPORTS ON THE MOTION OF A CENTRIFUGAL PUMP IMPELLER

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This study deals with the motion of a centrifugal pump impeller. We consider the effect of the elasticity and damping characteristics of the foundation supports on the stability of synchronous precession of an out-of-balance impeller with a single-groove seal mounted on an elastic and inert base. For establishing the stability criteria, we apply the method of averaging to the periodic coefficients in the perturbation equations and then quasinormalize those equations.

The dynamic model of such an impeller is a flexible weightless shaft of stiffness c_1 with a disk of mass m_1 at the center of the span. The impeller is mounted on a perfectly rigid plate of mass m_2 resting on inertialess spring supports. All masses and stiffnesses of the system are assumed to be symmetric with respect to the plane of the disk, with the stiffness c_2 of the foundation supports the same in the horizontal plane and in the vertical plane.

The displacement of the disk, relative to the center of the seal, during flexure of the shaft produces a circumferential pressure gradient in the fluid around the disk and a hydrodynamic net friction force normal to the plane of disk flexure. This force can, under certain conditions, cause self-excited asynchronous precession of the impeller [2].

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