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Automated Production of Traditional Proofs in Solid Geometry*

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Abstract. This paper presents a method of producing readable proofs for theorems in solid geometry. The method is for a class of constructive geometry statements about straight lines, planes, circles, and spheres. The key idea of the method is to eliminate points from the conclusion of a geometric statement using several (fixed) high-level basic propositions about the signed volumes of tetrahedrons and Pythagorean differences of triangles. We have implemented the algorithm, and more than 80 examples from solid geometry have been used to test the program. Our program is efficient and the proofs produced by it are generally short and readable.

Key words: automated theorem proving, Euclidean traditional proofs, volume method, constructive geometry statements.

1. Introduction

Since the pioneering work of Wen-Tsün Wu in 1977 [12], highly successful algebraic methods for automated proving of geometry theorems have been developed. Computer programs based on these methods have been used to prove hundreds of non-trivial geometry theorems [1, 2, 8]. Algebraic methods, which are very different from the traditional proof methods used by geometers since Euclid, generally can only tell whether a statement is true or not. If we want to look at the proofs, we only have to see tedious computations of polynomials. The traditional method usually can give elegant proofs. Researchers have been studying automated generation of traditional proofs using computer programs since the work by H. Gelernter, J. R. Hanson, and D. W. Loveland [6]. In spite of the enormous amount of efforts and great improvements, see, e.g., [9, 11], the successes in this direction have been limited in the sense that no computer program has been developed which can prove non-trivial geometry theorems efficiently. In spite of the great successes of the algebraic methods, the research in the automated generation of traditional proofs is still a very attractive and challenging topic.

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Recently, we presented a method which can produce short and readable proofs for a class of geometric statements in plane Euclidean geometry [15, 3, 4]. [5] is a collection of 400 geometry theorems proved by our program, including the complete machine proofs of 100 theorems. This method is based on properties of the *signed area* and the *Pythagorean difference* which have been studied extensively in [14] for the purpose of geometry education.

This paper is an extension of our area method to solid geometry. We present a theorem-proving method for geometry statements whose hypotheses can be described constructively and whose conclusions are polynomial equations of several geometry quantities, such as volumes, ratios of line segments, ratios of areas, and Pythagorean differences. We call this method *the volume method* which is an extension of the area method. The key idea of the method is to eliminate points from the conclusion of a geometry statement using several basic propositions about volumes. The automatically produced proofs are "traditional" in the sense that they are generally short, readable, and each step of the proof has a clear geometric meaning. The proofs are of a shape that a student could write with pencil and paper in a few lines.

The volume method is complete for the class of constructive geometry statements which covers a large portion of the equational geometry theorems about lines, planes, circles, and spheres. The idea of constructive type and the associated automated proving may be resorted to Hilbert [7] and was first pointed out by Wu in [13]. The volume method works not only for Euclidean solid geometries but also for metric solid geometries associated with any number field with characteristic zero. Certain geometry problems such as those involving inequalities are beyond the scope of the volume method.

We have implemented the method and our program* has proved more than 80 examples from solid geometry, including Ceva's theorem, Menelaus' theorem, Desargues' theorem, Monge's theorem, etc. The proofs produced by the program are generally short and readable. The algorithm is also very efficient. (See the statistic table in Section 6.)

In Section 2, we prove the basic propositions which will serve as the basis of our method. In Section 3, we define the constructive geometry statements. In Section 4, we present the method for the Hilbert intersection statements in affine geometry. In Section 5, we present the general volume method. In Section 6, we give some experiment results.

2. The Signed Volumes

We need some notions and results from plane geometry whose formal definition and proofs can be found in [3, 15].

We use capital English letters to denote points in the Euclidean space. We denote by $\overline{AB}$ the signed length of the oriented segment from A to B; denote

* The prover (euc.tar.Z) is available via ftp at emcity.cs.twsu.edu: pub/geometry/.

by S_{ABC} the signed area of the oriented triangle ABC. The area of an oriented quadrilateral $ABCD$ is $S_{ABCD} = S_{ABC} + S_{ACD} = S_{ABD} - S_{CBD}$. When we mention a *line* AB or a *plane* PQR, we always assume $A \neq B$ or P, Q, and R are not collinear.

PROPOSITION 2.1. **The Co-side Theorem.** *Let M be the intersection of two nonparallel lines AB and PQ and $Q \neq M$. Then* $\frac{\overline{PM}}{\overline{QM}} = \frac{S_{PAB}}{S_{QAB}}$; $\frac{\overline{PM}}{\overline{PQ}} = \frac{S_{PAB}}{S_{PAQB}}$; $\frac{\overline{QM}}{\overline{PQ}} = \frac{S_{QAB}}{S_{PAQB}}$.

PROPOSITION 2.2. *Let R be a point on line PQ. Then for any two points A and B in the same plane*

$$S_{RAB} = \frac{\overline{PR}}{\overline{PQ}} S_{QAB} + \frac{\overline{RQ}}{\overline{PQ}} S_{PAB}.$$

Two lines in the same plane are said to be *parallel* if they do not have a common point. We use the notation $AB \parallel CD$ to denote the fact that A, B, C, and D satisfy one of the following conditions (1) line AB and line CD are parallel; or (2) $A = B$ or $C = D$; or (3) A, B, C and D are collinear.

PROPOSITION 2.3. *$PQ \parallel AB$ iff $S_{PAB} = S_{QAB}$, i.e., iff $S_{PAQB} = 0$.*

2.1. CO-FACE THEOREM

In this subsection, we will formally define the signed volume and derive some of its properties which will serve as the deductive basis of the volume method.

DEFINITION 2.4. For any points A, B, C, and D in the space, the signed volume V_{ABCD} of the tetrahedron $ABCD$ is a real number* which satisfies the following properties

V.1 When two neighbor vertices of the tetrahedron are interchanged, the signed volume of the tetrahedron will change signs, e.g., $V_{ABCD} = -V_{ABDC}$.

V.2 Points A, B, C, and D are coplanar iff $V_{ABCD} = 0$.

V.3 There exist at least four points A, B, C, and D such that $V_{ABCD} \neq 0$.

V.4 For five points A, B, C, D, and O, we have $V_{ABCD} = V_{ABCO} + V_{ABOD} + V_{AOCD} + V_{OBCD}$.

* Here, we can use any number field and the results in this paper are still valid.

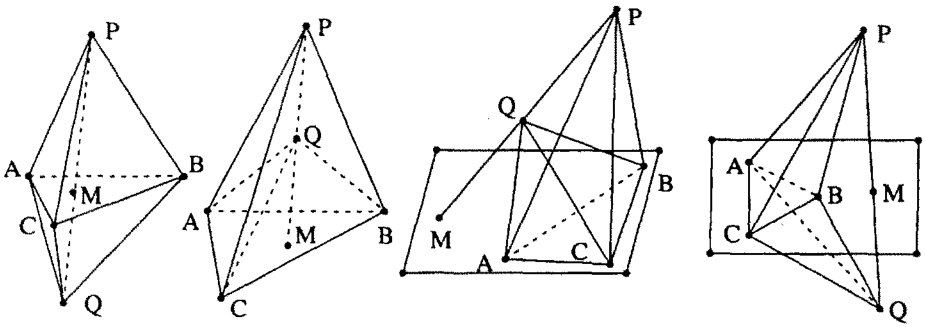

Fig. 1.

V.5 If A, B, C, D, E, and F are six coplanar points and $S_{ABC} = \lambda S_{DEF}$ then for any point T we have $V_{TABC} = \lambda V_{TDEF}$.

Note that we do not use the concept altitude in the definition of volume.

We denote by $PABCQ$ the polyhedron with faces PAB, PBC, PAC, QAB, QBC, and QAC. The volume of $PABCQ$ is defined to be

$$V_{PABCQ} = V_{PABC} - V_{QABC}$$

By V.4 of Definition 2.4, we have $V_{PABQ} = V_{PABQ} + V_{PCAQ} + V_{PBCQ}$.

PROPOSITION 2.5 (**The Co-face Theorem).** *A line PQ and a plane ABC meet in M. If $Q \neq M$, we have*

$$\frac{\overline{PM}}{\overline{QM}} = \frac{V_{PABC}}{V_{QABC}}, \qquad \frac{\overline{PM}}{\overline{PQ}} = \frac{V_{PABC}}{V_{PABCQ}}; \qquad \frac{\overline{QM}}{\overline{PQ}} = \frac{V_{QABC}}{V_{PABCQ}}.$$

Proof. Figure 1 shows several possible configurations of this proposition. By V.5 of Definition 2.4 and Proposition 2.1

$$\begin{aligned}\frac{V_{PABC}}{V_{QABC}} &= \frac{V_{PABC}}{V_{PABM}} \cdot \frac{V_{PABM}}{V_{QABM}} \cdot \frac{V_{QABM}}{V_{QABC}} = \frac{S_{ABC}}{S_{ABM}} \cdot \frac{S_{BPM}}{S_{BQM}} \cdot \frac{S_{ABM}}{S_{ABC}} \\ &= \frac{S_{BPM}}{S_{BQM}} = \frac{\overline{PM}}{\overline{QM}}.\end{aligned}$$

□

PROPOSITION 2.6. *Let R be a point on a line PQ and ABC be a triangle (Figure 2). Then we have*

$$V_{RABC} = \frac{\overline{PR}}{\overline{PQ}} V_{QABC} + \frac{\overline{RQ}}{\overline{PQ}} V_{PABC}.$$

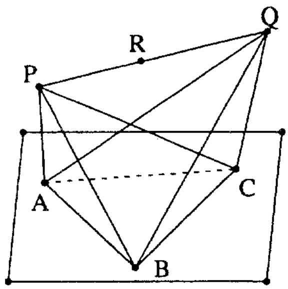

Fig. 2.

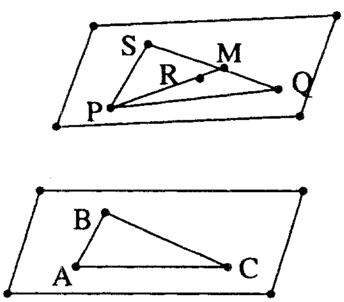

Fig. 3.

Proof. By Proposition 2.5, we have

$$\frac{V_{PRBC}}{V_{PQBC}} = \frac{\overline{PR}}{\overline{PQ}} \cdot \frac{V_{PARC}}{V_{PAQC}} = \frac{\overline{PR}}{\overline{PQ}} \cdot \frac{V_{PABR}}{V_{PABQ}} = \frac{\overline{PR}}{\overline{PQ}}.$$

By V.4 $V_{RABC} = V_{PABC} - V_{PRBC} - V_{PARC} - V_{PABR} = V_{PABC} - \frac{\overline{PR}}{\overline{PQ}}(V_{PQBC} + V_{PAQC} + V_{PABQ}) = V_{PABC} - \frac{\overline{PR}}{\overline{PQ}} V_{PABCQ} = \frac{\overline{PR}}{\overline{PQ}} V_{QABC} + \frac{\overline{RQ}}{\overline{PQ}} V_{PABC}$. □

PROPOSITION 2.7. *Let R be a point in the plane PQS. Then for three points A, B, and C, we have*

$$V_{RABC} = \frac{S_{PQR}}{S_{PQS}} V_{SABC} + \frac{S_{RQS}}{S_{PQS}} V_{PABC} + \frac{S_{PRS}}{S_{PQS}} V_{QABC}.$$

Proof. For any point X, let $V_X = V_{XABC}$. Without loss of generality, let M be the intersection of PR and QS (Figure 3). By proposition 2.6,

$$V_R = \frac{\overline{PR}}{\overline{PM}} V_M + \frac{\overline{RM}}{\overline{PM}} V_P = \frac{\overline{PR}}{\overline{PM}} \left(\frac{\overline{QM}}{\overline{QS}} V_S + \frac{\overline{MS}}{\overline{QS}} V_Q \right) + \frac{\overline{RM}}{\overline{PM}} V_P. \quad (1)$$

By the co-side theorem (2.1),

$$\frac{\overline{RM}}{\overline{PM}} = \frac{S_{RQS}}{S_{PQS}}, \qquad \frac{\overline{QM}}{\overline{QS}} = \frac{S_{PQR}}{S_{PQRS}},$$

$$\frac{\overline{MS}}{\overline{QS}} = \frac{S_{PRS}}{S_{PQRS}}, \qquad \frac{\overline{PR}}{\overline{PM}} = \frac{S_{PQRS}}{S_{PQS}}.$$

Substituting these into (1), we obtain the result. □

2.2. Parallels

Two planes, or a straight line and a plane are said to be *parallel* if they have no point in common. The notation $PQ \parallel ABC$ means that A, B, C, P, and Q satisfy one of the following conditions: (1) $P = Q$, (2) A, B, and C are collinear, or (3) A, B, C, P, and Q are on the same plane, or (4) line PQ and plane ABC are parallel. According to the above definition, if $PQ \not\parallel ABC$ then line PQ and plane ABC have a normal intersection. For six points A, B, C, P, Q, and R, $ABC \parallel PQR$ iff $AB \parallel PQR$, $BC \parallel PQR$, and $AC \parallel PQR$.

PROPOSITION 2.8. *$PQ \parallel ABC$ iff $V_{PABC} = V_{QABC}$ or equivalently*

$$V_{PABCQ} = 0.$$

Proof. If $V_{PABC} \neq V_{QABC}$, let O be a point on line PQ such that $\frac{\overline{PO}}{\overline{PQ}} = \frac{V_{PABC}}{V_{PABCQ}}$. Thus $\frac{\overline{OQ}}{\overline{PQ}} = -\frac{V_{QABC}}{V_{PABCQ}}$. By Proposition 2.6,

$$V_{OABC} = \frac{\overline{PO}}{\overline{PQ}} V_{QABC} + \frac{\overline{OQ}}{\overline{PQ}} V_{PABC} = 0.$$

By V.2, point O is also in plane ABC, i.e., line PQ is not parallel to ABC. Conversely, if $PQ \not\parallel ABC$ let O be the intersection of PQ and ABC. By Proposition 2.5,

$$\frac{\overline{OP}}{\overline{OQ}} = \frac{V_{PABC}}{V_{QABC}} = 1,$$

i.e., $P = Q$ which is a contradiction. □

PROPOSITION 2.9. *$PQR \parallel ABC$ iff $V_{PABC} = V_{QABC} = V_{RABC}$.*
Proof. This is a consequence of Proposition 2.8. □

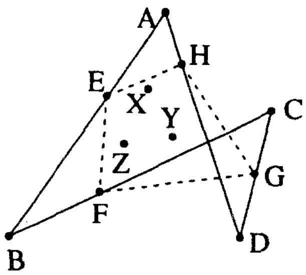

Fig. 4.

PROPOSITION 2.10. *Let $PQTS$ be a parallelogram. Then for points A, B, and C, we have*

$$V_{PABC} + V_{TABC} = V_{QABC} + V_{SABC} \quad or \quad V_{PABCQ} = V_{SABCT}.$$

Proof. This is a consequence of Proposition 2.6, because both sides of the equation are equal to $2V_{OABC}$ where O is the intersection of PT and SQ. □

PROPOSITION 2.11. *Let triangle ABC be a parallel translation of triangle DEF. Then for any point P we have $V_{PABC} = V_{PDEFA}$.*

Proof. By Proposition 2.10, $V_{PABC} = V_{PAEC} - V_{PADC} = V_{PAEF} - V_{PAED} - V_{PADC} = V_{PAEF} - V_{PAED} - V_{PADF} = V_{PDEFA}$. □

PROPOSITION 2.12. *Let triangle ABC be a parallel translation of triangle DEF. Then for two points P and Q we have*

$$V_{PABC} + V_{QDEF} = V_{QABC} + V_{PDEF} \quad or \quad V_{PABCQ} = V_{PDEFQ}.$$

Proof. By Proposition 2.11, $V_{PABC} = V_{PDEF} - V_{ADEF}$; $V_{QABC} = V_{QDEF} - V_{ADEF}$ from which we obtain the result immediately. □

2.3. Working Examples

Before presenting the method, we use several examples to show how to use the above properties about volumes to prove theorems. The proofs given below are actually modifications of the proofs produced by our program.

EXAMPLE 2.13 (**Menelaus' theorem**). If the sides AB, BC, CD, and DA of any skew quadrilateral are cut by a plane XYZ in the points E, F, G, and H, respectively, then

$$\frac{\overline{AE}}{\overline{EB}} \cdot \frac{\overline{BF}}{\overline{FC}} \cdot \frac{\overline{CG}}{\overline{GD}} \cdot \frac{\overline{DH}}{\overline{HA}} = 1.$$

Proof. By the co-face theorem

$$\frac{\overline{DH}}{\overline{AH}} = \frac{V_{DXYZ}}{V_{AXYZ}}, \qquad \frac{\overline{CG}}{\overline{DG}} = \frac{V_{CXYZ}}{V_{DXYZ}},$$

$$\frac{\overline{BF}}{\overline{CF}} = \frac{V_{BXYZ}}{V_{CXYZ}}, \qquad \frac{\overline{AE}}{\overline{BE}} = \frac{V_{AXYZ}}{V_{BXYZ}}.$$

Then it is clear that

$$\frac{\overline{AE}}{\overline{EB}} \cdot \frac{\overline{BF}}{\overline{FC}} \cdot \frac{\overline{CG}}{\overline{GD}} \cdot \frac{\overline{DH}}{\overline{HA}} = 1.$$

For the non-degenerate conditions of this example, see Section 3.3. □

EXAMPLE 2.14. Let $A_1B_1C_1$ be the parallel projection of any triangle ABC in any plane. Show that the tetrahedral $ABCA_1$ and $A_1B_1C_1A$ are equal in volume (Figure 5).

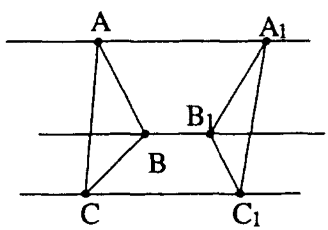

Fig. 5.

Proof. Since CC_1 is parallel to plane AA_1B_1, by Proposition 2.8, $V_{AA_1B_1C_1} = V_{AA_1B_1C}$. Similarly, $V_{AA_1B_1C} = V_{AA_1BC}$. □

3. Constructive Geometry Statements

Our volume method is for constructive geometry statements defined as follows.

3.1. CONSTRUCTIVE GEOMETRY STATEMENTS

By a *geometry quantity*, we mean one of the following three quantities: (i) the ratio of the lengths of two oriented segments on one line or on two parallel lines; (ii) the ratio of the areas of two oriented triangles in the same plane or in two parallel planes; or (iii) the signed volume of a tetrahedron.

DEFINITION 3.1. A construction is one of the following ways of introducing new points in the space.

C1 (POINTS $A_1, \ldots, A_l$). Take arbitrary points $A_1, \ldots, A_l$ in the space. Each A_i has three degrees of freedom.

C2 (PRATIO $AWUVr$). Take a point A on the line passing through W and parallel to line UV such that $\overline{WA} = r\overline{UV}$, where r could be a rational number, a rational expression in some geometric quantities, or a variable.

If r is a fixed quantity, A is a fixed point; if r is a variable, A has one degree of freedom. The non-degenerate (ndg) condition is $U \neq V$.

C3 (ARATIO $ALMNr_1r_2r_3$), where

$$r_1 = \frac{S_{AMN}}{S_{LMN}}, \quad r_2 = \frac{S_{LAN}}{S_{LMN}}, \quad \text{and} \quad r_3 = \frac{S_{LMA}}{S_{LMN}}$$

are the area coordinates of point A with respect to LMN. The r_1, r_2 and r_3 could be rational numbers, rational expressions in geometric quantities, or indeterminates satisfying $r_1 + r_2 + r_3 = 1$. The ndg condition is that L, M, N are not collinear. The degree of freedom of A is equal to the number of indeterminates in $\{r_1, r_2, r_3\}$.

C4 (INTER A(LINE UV)(LINE PQ)). Point A is the intersection of line PQ and line UV which are in the same plane. The ndg condition is $PQ \nparallel UV$. Point A is a fixed point.

C5 (INTER A(LINE UV)(PLANE LMN)). Take the intersection of a line UV and a plane LMN. The ndg condition is that $UV \nparallel LMN$. Point A is a fixed point.

C6 (FOOT2LINE $APUV$). Point A is the foot from point P to line UV. The ndg condition is $U \neq V$. Point A is a fixed point.

DEFINITION 3.2. A constructive statement is a list $S = (C_1, C_2, \ldots, C_k, G)$ where

1. Each C_i, introduces a new point from the points introduced by the previous constructions.
2. $G = (E_1, E_2)$ where E_1 and E_2 are polynomials in some geometric quantities about the points introduced by the C_i and $E_1 = E_2$ is the conclusion of S.

The non-degenerate (ndg) condition of S is the set of ndg conditions of the C_i plus the condition that the geometry quantities in E_1 and E_2 have geometry meanings, i.e., their denominators could not be zero.

The set of all constructive statement is denoted by S_C. If the constructions are limited to C1–C5, the corresponding statements are called *the Hilbert intersection point statements* in the space and are denoted by S_H.

3.2. The Predicate Form

The constructive description of geometry statements can be transformed into the commonly used predicate form. We first introduce several basic predicates.

1. Point (POINT P): P is a point in the space.
2. Collinear (COLL P_1, P_2, P_3): points P_1, P_2, and P_3 are on the same line. It is equivalent to $S_{P_1P_2P_3} = 0$.
3. Co-plane (COPL P_1, P_2, P_3, P_4): P_1, P_2, P_3, and P_4 are in the same plane. It is equivalent to $V_{P_1P_2P_3P_4} = 0$.
4. Parallel between two lines (PRLL P_1, P_2, P_3, P_4): $P_1P_2 \parallel P_3P_4$. It is equivalent to $V_{P_1P_2P_3P_4} = 0$ and $S_{P_1P_3P_2P_4} = 0$.
5. Parallel between a line and a plane (PRLP P_1, P_2, P_3, P_4, P_5): $P_1P_2 \parallel P_3P_4P_5$. It is equivalent to $V_{P_1P_3P_4P_5P_2} = 0$.

We first need to transform the construction into predicate forms.

C2 (PRATIO $AWUVr$) is equivalent to (PRLL $AWUV$), $r = \frac{\overline{WA}}{\overline{UV}}$, and $U \neq V$.

C3 (ARATIO $ALMNr_1r_2r_3$) is equivalent to (COPL $ALMN$),

$$r_1 = \frac{S_{AMN}}{S_{LMN}}, \qquad r_2 = \frac{S_{LAN}}{S_{LMN}}, \qquad r_3 = \frac{S_{LMA}}{S_{LMN}},$$

and $\neg$(COLL LMN).

C4 (INTER A(LINE UV)(LINE PQ)) is equivalent to (COLL AUV), (COLL APQ), and $\neg$(PRLL $UVPQ$).

C5 (INTER A(LINE UV)(PLANE LMN)) is equivalent to (COLL AUV), (COPL $ALMN$), and $\neg$(PRLP $UVLMN$).

C6 (FOOT2LINE $APUV$) is equivalent to (COLL AUV), (PERP $APUV$), and $U \neq V$.

Now a constructive statement $S = (C_1, \ldots, C_r, (E, F))$ can be transformed into the following predicate form

$$\forall P_1 \cdots \forall P_r((P(C_1) \wedge \cdots \wedge P(C_r)) \Rightarrow E = F)$$

where P_i is the point introduced by C_i and $P(C_i)$ is the predicate form of C_i.

3.3. MORE CONSTRUCTIONS

Constructions C1–C6, though simple, can be used to describe almost all the configurations about lines, planes, circles, and spheres. To see that, we first introduce more geometry objects.

- We consider three kinds of lines: (LINE PQ); (PLINE RPQ): the line passing through R and parallel to PQ; (OLINE $SPQR$): the line passing through point S and perpendicular to the plane PQR.
- We consider six kinds of planes: (PLANE LMN); (PPLANE $WOUV$): the plane passing through a point W and parallel to the plane OUV; (TPLANE WUV): the plane passing through a point W and perpendicular to the line UV; (BPLANE UV): the perpendicular-bisector of line UV;(CPLANE $ABPQR$): the plane passing through line AB and perpendicular to plane PQR; and (DPLANE $ABPQ$): the plane passing through line AB and parallel to line PQ.

Now we can introduce more constructions: taking an arbitrary point on a line or in a plane; taking the intersections of two lines, the intersections of lines with planes, and the intersections of three planes. From all possible combinations, we have 41 new constructions. For convenience, we introduce the following often used constructions.

(MIDPOINT AUV). A is the midpoint of UV, i.e., (PRATIO $AUUV$ $1/2$).
(LRATIO $AUVr$), i.e., (PRATIO $AUUVr$).
(ON $A\,ln$). Take an arbitrary point A on line ln. For instance,
(ON A(LINE UV)) is equivalent to (PRATIO $AUUVr$)
for an indeterminate r.

EXAMPLE 3.3. Example 2.13 can be described in the following constructive way.

$$\begin{aligned}
\big(&(\text{POINTS}\ ABCDXYZ)\\
&(\text{INTER}\ E(\text{LINE}\ AB)(\text{PLANE}\ XYZ))\\
&(\text{INTER}\ F(\text{LINE}\ BC)(\text{PLANE}\ XYZ))\\
&(\text{INTER}\ G(\text{LINE}\ CD)(\text{PLANE}\ XYZ))\\
&(\text{INTER}\ H(\text{LINE}\ AD)(\text{PLANE}\ XYZ))\\
&\Big(\frac{\overline{AE}}{\overline{BE}}\frac{\overline{CG}}{\overline{CF}}\frac{\overline{CG}}{\overline{DG}}\frac{\overline{DH}}{\overline{AH}}=1\Big)\big)
\end{aligned}$$

The ndg conditions: $AB \nparallel XYZ$, $BC \nparallel XYZ$, $CD \nparallel XYZ$, $AD \nparallel XYZ$, $B \neq E$, $C \neq F$, $D \neq G$, $A \neq H$.

The predicate form of this example is:

$$\forall A, B, \ldots, H(HYP \Rightarrow CONC)$$

where

HYP = ((COLL EB) $\wedge$ (COPL $EXYZ$) $\wedge$ (COLL FCB) $\wedge$ (COPL $FXYZ$)
$\wedge$(COLL GCD) $\wedge$ (COPL $GXYZ$) $\wedge$ (COLL HAD) $\wedge$ (COPL $HXYZ$)
$\wedge\neg$(PRLP $ABXYZ$) $\wedge$ $\neg$(PRLP $CBXYZ$) $\wedge$ $\neg$(PRLP $CDXYZ$)
$\wedge\neg$(PRLP $DAXYZ$) $\wedge\, B \neq E \wedge C \neq F \wedge D \neq G \wedge$ and $A \neq H$);
CONC $= \left(\frac{\overline{AE}}{\overline{BE}}\frac{\overline{BF}}{\overline{CF}}\frac{\overline{CG}}{\overline{DG}}\frac{\overline{DH}}{\overline{AH}} = 1.\right)$

We may also consider circles and spheres. We define (CIR OPQ) to be the circle in the plane OPQ which has O as its center and passes through point P. We define (SPHERE OP) to be the sphere with center O and passing through point P. Then we can use the following new constructions.

(ON a(CIR OUV)). Take an arbitrary point on the circle.
(ON A(SPHERE OU)). Take an arbitrary point on the sphere.
(INTER $A\, ln$(CIR OWP)). Take the intersection of line ln and circle (CIR OWP) which is different from W. We assume that line ln and the circle are in the same plane. Line ln could be (LINE WV), (PLINE WUV), and(OLINE $WLMN$).
(INTER $A\, ln$(SPHERE OW)). Take the intersection of line ln and sphere (SPHERE OW) which is different from W. Line ln could be (LINE WV), (PLINE WUV), and (OLINE $WRPQ$).
(INTER A(CIR O_1WU)(CIR O_2WV)). Take the intersection of circle (CIR O_1WU) and circle (CIR O_2WV) which is different from V. We assume that the two circles are in the same plane.
(INTER A(CIR O_1UV)(SPHERE O_2U)). Take the intersection of circle (CIR O_1UV) and sphere (SPHERE O_2U) which is different from U.

Here, we introduce another 10 new constructions. Thus, totally we have 51 constructions. The following fact can be proved without much difficulty.

PROPOSITION 3.4. *All the* 51 *constructions introduced in this subsection can be reduced to constructions* C1–C6.

4. Automated Theorem Proving for Class S_H

The volume method is to eliminate points from the conclusion of a geometry statement. More precisely, we need to eliminate points from geometry quantities.

4.1. Eliminating Points from Volumes

The method of eliminating points from volumes is the basis of the volume method. In this subsection, we will discuss four constructions C2–C5. C1 will be treated in Section 4.4.

LEMMA 4.1. *Let Y be introduced by* (PRATIO $YWUVr$). *Then we have*

$$V_{ABCY} = \begin{cases} \left(\frac{\overline{UW}}{\overline{UV}} + r\right)V_{ABCV} + \left(\frac{\overline{WV}}{\overline{UV}} - r\right)V_{ABCU} & \text{if } W \text{ is on line } UV. \\ V_{ABCW} + r(V_{ABCV} - V_{ABCU}) & \text{otherwise.} \end{cases}$$

Proof. Let $G = V_{ABCY}$. If W, U, V are collinear, by Proposition 2.6 we have

$$\begin{aligned} G &= \frac{\overline{UY}}{\overline{UV}} V_{ABCV} + \frac{\overline{YV}}{\overline{UV}} V_{ABCU} \\ &= \left(\frac{\overline{UW}}{\overline{UV}} + r\right)V_{ABCV} + \left(\frac{\overline{WV}}{\overline{UV}} - r\right)V_{ABCU}. \end{aligned}$$

Otherwise, take a point S such that $\overline{WS} = \overline{UV}$. Then we have

$$G = \frac{\overline{WY}}{\overline{WS}} V_{ABCS} + \frac{\overline{YS}}{\overline{WS}} V_{ABCW} = rV_{ABCS} + (1-r)V_{ABCW}.$$

By Proposition 2.10, we have $V_{ABCS} = V_{ABCW} + V_{ABCV} - V_{ABCU}$. Substituting this into the above equation, we obtain the result. Note that the ndg condition $U \neq V$ is needed. □

LEMMA 4.2. *Let Y be introduced by* (ARATIO $YLMNr_1r_2r_3$). *Then we have*

$$V_{ABCY} = r_1V_{ABCN} + r_2V_{ABCM} + r_3V_{ABCL}.$$

Proof. This lemma is a direct consequence of Proposition 2.7. □

LEMMA 4.3. *Let Y be introduced by* (INTER Y(LINE UV)(LINE IJ)). *Then we have*

$$V_{ABCY} = \frac{S_{UIJ}}{S_{UIVJ}} V_{ABCV} - \frac{S_{VIJ}}{S_{UIVJ}} V_{ABCU}.$$

Proof. By Propositions 2.6 and 2.1,

$$V_{ABCY} = \frac{\overline{UY}}{\overline{UV}} V_{ABCV} + \frac{\overline{YV}}{\overline{UV}} V_{ABCU} = \frac{S_{UIJ}V_{ABCV} - S_{VIJ}V_{ABCU}}{S_{UIVJ}}.$$

Since $UV \nparallel IJ$, we have $S_{UIVJ} \neq 0$. □

LEMMA 4.4. *Let Y be introduced by* (INTER Y(LINE UV)(PLANE LMN)). *Then*

$$V_{ABCY} = \frac{1}{V_{ULMNV}}(V_{ULMN}V_{ABCV} - V_{VLMN}V_{ABCU})$$

Proof. Let $G = V_{ABCY}$. By Proposition 2.6 and the co-face theorem,

$$\begin{aligned} V_{ABCY} &= \frac{\overline{UY}}{\overline{UV}} V_{ABCV} + \frac{\overline{YV}}{\overline{UV}} V_{ABCU} \\ &= \frac{V_{ULMN}V_{ABCV} - V_{VLMN}V_{ABCU}}{V_{ULMNV}}. \end{aligned}$$

Since $UV \not\parallel LMN$, we have $V_{ULMNV} \neq 0$. □

4.2. Eliminating points from area ratios

LEMMA 4.5. *Let Y be introduced by* (PRATIO $YWUVr$). *Then we have*

$$\frac{\overline{ABY}}{\overline{CDE}} = \begin{cases} \frac{rV_{UABWV}}{V_{WCDEA}} & \text{if } W \text{ is not in plane } ABY. \\ \frac{V_{UABW}+rV_{UABV}}{V_{UCDEA}} & \text{if } W \text{ is in plane } ABY \text{ but line } UV \text{ is not.} \\ \frac{S_{ABW}+r(S_{ABV}-S_{ABU})}{S_{CDE}} & \text{if } W, U, V, A, B, Y \text{ are coplanar.} \end{cases}$$

Proof. If $W \notin ABY$, let RPQ be a parallel translation of triangle CDE to plane ABY, and WS be a parallel translation of UV to line WY (Figure 6). By V.5,

$$G = \frac{S_{ABY}}{S_{RPQ}} = \frac{V_{WABY}}{V_{WRPQ}}.$$

By Proposition 2.12, $V_{RPQW} = V_{WCDEA}$. By Propositions 2.6 and 2.10,

$$V_{WABY} = \frac{\overline{WY}}{\overline{WS}} V_{WSAB} = rV_{UABWV}.$$

We prove the first case. The second case can be proved similarly as the first case: just replacing W by U. For the third case, see [3]. □

LEMMA 4.6. *Let Y be introduced by* (ARATIO $YLMNr_1r_2r_3$). *Then we have*

$$\frac{\overline{ABY}}{\overline{CDE}} = \begin{cases} \frac{r_2V_{LABM}+r_3V_{LABN}}{V_{LCDEA}} & \text{if one of } L, M, N, \text{ say } L, \text{ is not in } ABY. \\ \frac{r_1S_{ABL}+r_2S_{ABM}+r_3S_{ABN}}{S_{CDE}} & \text{if } L, M, N \text{ are in plane } ABY. \end{cases}$$

Proof. If L is not in ABY,

$$\frac{\overline{ABY}}{\overline{CDE}} = \frac{V_{LABY}}{V_{LCDEA}}.$$

Now the result comes from Lemma 4.2. The second case can be proved similarly as Proposition 2.7. □

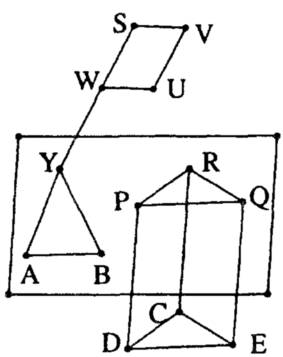

Fig. 6.

LEMMA 4.7. *Let Y be introduced by* $(\mathrm{INTER}\, Y (\mathrm{LINE}\, UV)(\mathrm{LINE}\, IJ))$. *Then we have*

$$\frac{\overline{ABY}}{\overline{CDE}} = \begin{cases} \frac{S_{UIJ}V_{UABV}}{S_{UIVJ}V_{UCDEA}} & \text{if one of } U, V, I, J, \text{ say } U, \\ & \quad \text{is not in } ABY. \\ \frac{S_{IUV}S_{ABJ} - S_{JUV}S_{ABI}}{S_{CDE}S_{IUJV}} & \text{if } U, V, I, J, A, B, Y \text{ are coplanar.} \end{cases}$$

Proof. If U is not in ABY,

$$\frac{\overline{ABY}}{\overline{CDE}} = \frac{V_{UABY}}{V_{UCDEA}} = \frac{\overline{UY}}{\overline{UV}} \frac{V_{UABV}}{V_{UCDEA}} = \frac{S_{UIJ}}{S_{UIVJ}} \frac{V_{UABV}}{V_{UCDEA}}.$$

The second case can be proved similarly as Lemma 4.3. □

LEMMA 4.8. *Let Y be introduced by*

$$C = (\mathrm{INTER}\, Y (\mathrm{LINE}\, UV)(\mathrm{PLANE}\, LMN)).$$

Then we have

$$\frac{\overline{ABY}}{\overline{CDE}} = \begin{cases} \frac{V_{ULMN}}{V_{ULMNV}} \frac{V_{UABV}}{V_{UCDEA}} & \text{if } U \text{ (or } V\text{) is not in } ABY. \\ \frac{V_{ULMN}S_{ABV} - V_{VLMN}S_{ABU}}{S_{CDE}V_{ULMNV}} & \text{if } U, V \text{ are in } ABY. \end{cases}$$

Proof. If U is not in ABY,

$$\frac{\overline{ABY}}{\overline{CDE}} = \frac{V_{UABY}}{V_{UCDEA}} = \frac{\overline{UY}}{\overline{UV}} \frac{V_{UABY}}{V_{UCDEA}} = \frac{V_{ULMN}}{V_{ULMNV}} \frac{V_{UABV}}{V_{UCDEA}}.$$

The second case is a consequence of Proposition 2.2 and the co-face theorem. □

4.3. Eliminating points from length ratios

In the following lemmas, point Y is introduced by construction C.

LEMMA 4.9. *Let* $G = \frac{\overline{DY}}{\overline{EF}}$, $C = (\text{PRATIO}\, YWUVr)$. *Then*

$$G = \begin{cases} \dfrac{\frac{\overline{DW}}{\overline{UV}}+r}{\frac{\overline{EF}}{\overline{UV}}} & \text{if } D \in WY. \\ \dfrac{V_{DWUV}}{V_{EWUVF}} & \text{if } D \notin WY,\ U \notin DWY. \\ -\dfrac{V_{UEDWV}}{V_{UEFWV}} & \text{if } D \notin WY,\ R \notin DWY. \\ \dfrac{S_{DUWV}}{S_{EUFV}} & \text{if all points are coplanar.} \end{cases}$$

Proof. If all points are collinear (the first and the last cases), see [3, 15]. If $U \notin DWY$, take a point S such that $\overline{DS} = \overline{EF}$ (Figure 7). By the co-face theorem

$$G = \frac{\overline{DY}}{\overline{DS}} = \frac{V_{DWUV}}{V_{DWUVS}} = \frac{V_{DWUV}}{V_{EWUVF}}.$$

If $E \notin DWY$, take a point T such that $\overline{WT} = \overline{UV}$. By the co-side and co-face theorems

$$G = \frac{\overline{DY}}{\overline{DS}} = \frac{S_{DWT}}{S_{DWST}} = \frac{V_{DWTE}}{V_{DWTES}}.$$

By Propositions 2.10 and 2.12

$$\begin{aligned} V_{DWTE} &= V_{DWVE} - V_{DWUE} = -V_{UEDWV}, \\ V_{DWTES} &= V_{EWTEF} = -V_{FWTE} = -V_{FWVE} + V_{FWUE} = V_{UEFWV} \end{aligned}$$

which prove the lemma. □

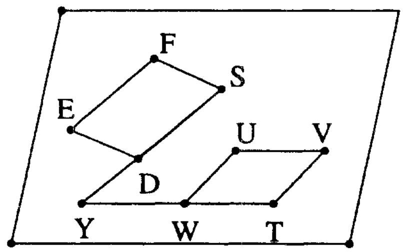

Fig. 7.

LEMMA 4.10. *Let* $G = \frac{\overline{DY}}{\overline{EF}}$, $C = (\mathrm{ARATIO}\,YLMNr_1r_2r_3)$. *Then we have*

$$G = \begin{cases} \frac{V_{DLMN}}{V_{ELMNF}} & \text{if } D \notin LMN. \\ -\frac{V_{DMNE}}{V_{FMNE}} & \text{if } D \in LMN,\ E \notin LMN, \text{ and } DY \not\parallel NM. \\ \frac{S_{DMN}}{S_{EMFN}} & \text{if all points are coplanar and } DY \not\parallel NM. \end{cases}$$

Proof. If $D \notin LMN$, the result is a direct consequence of Propositions 2.5 and 2.12. For the second case, take a point S such that $\overline{DS} = \overline{EF}$. Then

$$G = \frac{\overline{DY}}{\overline{DS}} = \frac{S_{DMN}}{S_{DMSN}} = \frac{V_{DMNE}}{V_{DMNES}} = \frac{V_{DMNE}}{V_{EMNEF}} = -\frac{V_{DMNE}}{V_{FMNE}}.$$

The third case can be proved similarly. □

LEMMA 4.11. *Let* $G = \frac{\overline{DY}}{\overline{EF}}$, $C = (\mathrm{INTER}\,Y(\mathrm{LINE}\,UV)(\mathrm{LINE}\,IJ))$. *Then we have*

$$G = \begin{cases} \frac{V_{DUVI}}{V_{EUVIF}} & D \notin UVIJ \text{ and } \neg(\mathrm{COLL}\,UVI). \\ \frac{V_{EDUV}}{V_{EFVU}} & D \in UVIJ,\ EF \notin UVIJ, \text{ and } D \notin UV. \\ \frac{S_{DUV}}{S_{EUFV}} & D, E, F \text{ are in } UVIJ, \text{ and } D \notin UV. \end{cases}$$

Proof. The first case is a consequence of the co-face theorem. For the second case, we assume $D \in UVIJ$. Take a point S such that $\overline{DS} = \overline{EF}$. Then we have

$$G = \frac{\overline{DY}}{\overline{DS}} = \frac{S_{DUV}}{S_{DUV} - S_{SUV}} = \frac{V_{DUVE}}{V_{DUVES}} = \frac{V_{DUVE}}{V_{EUVEF}} = -\frac{V_{DUVE}}{V_{FUVE}}.$$

The third case can be proved similarly. □

LEMMA 4.12. *Let* $G = \frac{\overline{DY}}{\overline{EF}}$, $C = (\text{INTER}\,Y(\text{LINE}\,UV)(\text{PLANE}\,LMN))$. *Then we have*

$$G = \begin{cases} \frac{V_{DLMN}}{V_{ELMN}-V_{FLMN}} & \text{if } D \text{ is not in plane } LMN. \\ \frac{V_{DUVL}}{V_{EUVL}-V_{FUVL}} & \text{if } D \in LMN \text{ and one of } L, M, N, \\ & \quad \text{say } L \notin DUV. \end{cases}$$

Proof. If D is not in plane LMN, the result is a direct consequence of the co-face theorem. For the second case, take a point S such that $\overline{DS} = \overline{EF}$. Then we have

$$G = \frac{\overline{DY}}{\overline{DS}} = \frac{V_{DUVL}}{V_{DUVLS}} = \frac{V_{DUVL}}{V_{EUVLF}}. \qquad \square$$

4.4. Free points and volume coordinates

After applying the above lemmas to any rational expression E in geometric quantities, we can eliminate the non-free points introduced by all constructions from E. Now the new E is a rational expression of indeterminates and volumes of *free points* in space. For more than five free points in the space, the volumes of the tetrahedra formed by them are not independent, e.g., see V.4 of Definition 2.4. To deal with this problem, we introduce the concept of *volume coordinates*.

DEFINITION 4.13. Let X be a point in the space. For four noncoplanar points O, W, U, and V, the volume coordinates of X w.r.t. $OWUV$ are

$$r_1 = \frac{V_{OWUX}}{V_{OWUV}}, \quad r_2 = \frac{V_{OWXV}}{V_{OWUV}}, \quad r_3 = \frac{V_{OXUV}}{V_{OWUV}}, \quad r_4 = \frac{V_{XWUV}}{V_{OWUV}}.$$

It is clear that $r_1 + r_2 + r_3 + r_4 = 1$.

The points in the space are in a one to one correspondence with the four-tuples (x, y, z, w) such that $x + y + z + w = 1$.

LEMMA 4.14. *Let* $G = V_{ABCY}$, *and* O, W, U, V *be four noncoplanar points. Then we have*

$$G = V_{ABCO} + \frac{V_{OABCV}V_{OWUY} + V_{OABCU}V_{OVWY} + V_{OABCW}V_{OUVY}}{V_{OWUV}}.$$

Proof. We have

$$V_{ABCY} = V_{ABCO} + V_{ABOY} + V_{AOCY} + V_{OBCY}. \tag{1}$$

Without loss of generality, we assume that YO meets plane WUV in X. (Otherwise, let YW meet plane OUV in X, and so on.) By Proposition 2.5, we have

$$V_{OABY} = \frac{\overline{OY}}{\overline{OX}} V_{OABX} = \frac{V_{OWUVY} V_{OABX}}{V_{OWUV}}. \tag{2}$$

By Proposition 2.7,

$$V_{OABX} = \frac{S_{WUX}}{S_{WUV}} V_{OABV} + \frac{S_{WXV}}{S_{WUV}} V_{OABU} + \frac{S_{XUV}}{S_{WUV}} V_{OABW}. \tag{3}$$

By Lemma 4.8, we have

$$\frac{S_{WUX}}{S_{WUV}} = \frac{V_{OWUY}}{V_{OWUVY}}; \qquad \frac{S_{WXV}}{S_{WUV}} = \frac{V_{OVWY}}{V_{OWUVY}}; \qquad \frac{S_{XUV}}{S_{WUV}} = \frac{V_{OUVY}}{V_{OWUVY}}.$$

Substituting them into (3) and (2), we have

$$V_{OABY} = \frac{V_{OWUY} V_{OABV} + V_{OVWY} V_{OABU} + V_{OUVY} V_{OABW}}{V_{OWUV}}. \tag{4}$$

Similarly, we have

$$\begin{aligned} V_{OBCY} &= \frac{V_{OWUY} V_{OBCV} + V_{OVWY} V_{OBCU} + V_{OUVY} V_{OBCW}}{V_{OWUV}} \\ V_{OCAY} &= \frac{V_{OWUY} V_{OCAV} + V_{OVWY} V_{OCAU} + V_{OUVY} V_{OCAW}}{V_{OWUV}}. \end{aligned}$$

Substituting them into (1) and noticing that $V_{OABV} + V_{OBCV} + V_{OCAV} = V_{OABCV}$, $V_{OABU} + V_{OBCU} + V_{OCAU} = V_{OABCU}$, and $V_{OABW} + V_{OBCW} + V_{OCAW} = V_{OABCW}$, we obtain the result. □

Now we can describe the volume method as follows: for a geometry statement in S_H: $S = (C_1, \ldots, C_r, (E_1, E_2))$, let the point introduced by C_i be P_i. Then we can use the above lemmas to eliminate points $P_r, P_{r-1}, \ldots, P_1$ respectively from E_1 and E_2. At last, we obtain two rational expressions R_1 and R_2 respectively. S is a correct geometry statement if R_1 is identical to R_2. For the formal description of the algorithm, see the next section.

5. Automated Theorem Proving for Class S_C

5.1. The Pythagorean Difference

The *Pythagorean difference* P_{ABC} is defined as

$$P_{ABC} = \overline{AB}^2 + \overline{CB}^2 - \overline{AC}^2.$$

It is easy to check that

1. $P_{AAB} = 0$; $P_{ABC} = P_{CBA}$; $P_{ABC} + P_{ACB} = 2\overline{BC}^2 = P_{BCB}$.

2. If A, B, and C are collinear, $P_{ABC} = 2\overline{BA} \cdot \overline{BC}$,

For four points A, B, C, and D, we define

$$P_{ABCD} = P_{ABD} - P_{CBD} = \overline{AB}^2 + \overline{CD}^2 - \overline{BC}^2 - \overline{DA}^2.$$

Then $P_{ABCD} = -P_{ADCB} = P_{BADC} = -P_{BCDA} = P_{CDAB} = -P_{CBAD} = P_{DCBA} = -P_{DABC}$.

The following properties of Pythagorean differences are taken for granted in our volume method.

PROPOSITION 5.1. (1) (*Pythagorean theorem*) $AB \perp BC$ *iff* $P_{ABC} = 0$.
(2) *If* $OW \perp OU$, $OW \perp OV$, *and* $OU \perp OV$, *then* $V_{OWUV}^2 = \frac{1}{36}\overline{OW}^2\overline{OU}^2\overline{OV}^2$.

In (2), we use the square of the volume, because the sign of the volume cannot be determined by the signs of the edges of the tetrahedron.

PROPOSITION 5.2. $AB \perp CD$ *iff* $P_{ACD} = P_{BCD}$ *or* $P_{ACBD} = 0$.

Proof. Let M and N be the orthogonal projections of A and B upon CD respectively. Then $\overline{AC}^2 = \overline{AM}^2 + \overline{CM}^2$, $\overline{AD}^2 = \overline{AM}^2 + \overline{DM}^2$, $\overline{BC}^2 = \overline{BN}^2 + \overline{CN}^2$, $\overline{BD}^2 = \overline{BN}^2 + \overline{DN}^2$. Therefore

$$P_{ACBD} = \overline{CM}^2 - \overline{DM}^2 + \overline{DN}^2 - \overline{CN}^2 = 2\overline{CD}(\overline{DM} - \overline{DN}).$$

Hence $P_{ACBD} = 0$ iff $\overline{DM} = \overline{DN}$, i.e., iff $M = N$. It is clear that $N = M$ iff $AB \perp CD$. □

PROPOSITION 5.3. *Let D be the foot of the perpendicular from point P to a line AB. Then we have*

$$\frac{\overline{AD}}{\overline{AB}} = \frac{P_{PAB}}{2\overline{AB}^2}, \qquad \frac{\overline{DB}}{\overline{AB}} = \frac{P_{PBA}}{2\overline{AB}^2}.$$

Proof. By Proposition 5.2, $P_{PAB} = P_{DAB} = 2\overline{AB} \cdot \overline{AD}$. The result is clear now. □

PROPOSITION 5.4. *Let R be a point on line PQ with position ratio*

$$r_1 = \frac{\overline{PR}}{\overline{PQ}}, \qquad r_2 = \frac{\overline{RQ}}{\overline{PQ}}$$

w.r.t. PQ. Then for any points A and B, we have

$$\begin{aligned} P_{RAB} &= r_1 P_{QAB} + r_2 P_{PAB} \\ P_{ARB} &= r_1 P_{AQB} + r_2 P_{APB} - r_1 r_2 P_{PQP}. \end{aligned}$$

Proof. We first assume

$$\overline{RA}^2 = r_1\overline{QA}^2 + r_2\overline{PA}^2 - r_1r_2\overline{PQ}^2 \quad (1)$$
$$\overline{RB}^2 = r_1\overline{QB}^2 + r_2\overline{PB}^2 - r_1r_2\overline{PQ}^2. \quad (2)$$

Then $P_{RAB} = \overline{RA}^2 + \overline{AB}^2 - \overline{RB}^2 = r_1(\overline{QA}^2 + \overline{AB}^2 - \overline{QB}^2) + r_1(\overline{PA}^2 + \overline{AB}^2 - \overline{PB}^2) = r_1P_{QAB} + r_2P_{PAB}$. The second one can be proved similarly. To prove (1), let us first notice that by Proposition 5.2,

$$\frac{P_{APR}}{P_{APQ}} = \frac{\overline{PR}}{\overline{PQ}} = r_1.$$

Then $r_1\overline{QA}^2 + r_2\overline{PA}^2 - r - 1r_2\overline{PQ}^2 = \overline{PA}^2 + \overline{PR}^2 - r_1P_{APQ} = \overline{PA}^2 + \overline{PR}^2 - P_{APR} = \overline{AR}^2$. □

PROPOSITION 5.5. *Let $ABCD$ be a parallelogram. Then for points P and Q in the same plane, we have*

$$P_{APQ} + P_{CPQ} = P_{BPQ} + P_{DPQ} \quad or \quad P_{APBQ} = P_{DPCQ}$$
$$P_{PAQ} + P_{PCQ} = P_{PBQ} + P_{PDQ} + 2P_{BAD}.$$

Proof. Let O be the intersection of AC and BD. By the first equation of Proposition 5.4, $2P_{OPQ} = P_{APQ} + P_{CPQ} = P_{BPQ} + P_{DPQ}$. By the second equation of Proposition 5.4,

$$2P_{POQ} = P_{PAQ} + P_{PCQ} - \frac{1}{2}P_{ACA} = P_{PBQ} + P_{PDQ} - \frac{1}{2}P_{BDB}.$$

We only need to show $2P_{BAD} = \frac{1}{2}(P_{ACA} - P_{BDB})$ which comes from Proposition 5.4. □

5.2. Methods of Eliminating Points

Since we have a new geometry quantity, the constructive statements can be enlarged in the following way: the conclusion of a statement can be equation of two polynomials of length ratios, area ratios, volumes and Pythagorean differences. The class of the enlarged constructive statements is still denoted by S_C.

Now we have five constructions C2–C6 and four geometry quantities. We need to give a method to eliminate a point introduced by each of the five constructions from each of the four quantities. This section deals with the classes which are not discussed in Section 4.

Let Y be introduced by one of the constructions C2–C6. By Proposition 5.4, to eliminate point Y from P_{ABY} or P_{AYB} we only need to find the position ratios $\frac{\overline{UY}}{\overline{UV}}$ and $\frac{\overline{YV}}{\overline{UV}}$, and this has been done in Section 4.1. (For C6, see Proposition 5.3.) Now there are only three cases left.

LEMMA 5.6. *If Y is introduced by* (FOOT2LINE PUV) *then*

$$V_{ABCY} = \frac{P_{PUV}}{P_{UVU}} V_{ABCQ} + \frac{P_{PVU}}{P_{UVU}} V_{ABCP}.$$

Proof. This is a consequence of Propositions 2.6 and 5.3. □

LEMMA 5.7. *Let Y be introduced by* (FOOT2LINE PUV). *Then*

$$\frac{\overline{DY}}{\overline{EF}} = \begin{cases} \frac{\overline{DU}}{\overline{EF}}\frac{P_{PDU}}{P_{DUD}} & \text{if } D \in UV \text{ and } D \neq U. \\ \frac{V_{DPUV}}{V_{EPUVF}} & \text{if } D \notin PUV. \\ \frac{V_{DUVE}}{V_{EUVF}} & \text{if } D \in PUV \text{ and } E \notin PUV. \\ \frac{S_{DUV}}{S_{EUFV}} & \text{if all points are coplanar.} \end{cases}$$

In all cases, we assume P is not on line UV; otherwise $P = Y$ and $\frac{\overline{DY}}{\overline{EF}} = \frac{\overline{DP}}{\overline{EF}}$.

Proof. For the first and last cases, see [3]. The second case is consequence of the co-face theorem. For the third case, let T be a point such that $\overline{DT} = \overline{EF}$. Then

$$\frac{\overline{DY}}{\overline{EF}} = \frac{\overline{DY}}{\overline{DT}} = \frac{S_{DUV}}{S_{DUTV}} = \frac{V_{DUVE}}{V_{DUVET}} = \frac{V_{DUVE}}{V_{EUVEF}} = -\frac{V_{DUVE}}{V_{FUVE}}. \qquad □$$

LEMMA 5.8. *Let Y be introduced by* (FOOT2LINE PUV). *Then*

$$\frac{S_{ABY}}{S_{CDE}} = \begin{cases} \frac{P_{PUV}V_{PABY}+P_{PVU}V_{PABU}}{2\overline{UV}^2 V_{PCDEA}} & \text{if } P \text{ is not in } ABY. \\ \frac{P_{PUV}V_{UABV}}{2\overline{UV}^2 V_{UCDEA}} & \text{if } U \text{ is not in } ABY. \\ \frac{P_{PVU}V_{VABU}}{2\overline{UV}^2 V_{VCDEA}} & \text{if } V \text{ is not in } ABY. \\ \frac{P_{PUV}S_{ABV}+P_{PVU}S_{ABU}}{2\overline{UV}^2 S_{CDE}} & \text{if } P, U, V \text{ are in } ABY. \end{cases}$$

Proof. If P is not in ABY, by V.5 $\frac{S_{ABY}}{S_{CDE}} = \frac{V_{PABY}}{V_{PCDEA}}$. Now the result comes from Lemma 5.6. The second and third cases can be proved similarly. For the last case, see [3]. □

5.3. THE ALGORITHM

In the preceding subsection, we gave elimination methods for points introduced by constructions C2–C6. Now we give the elimination method for free points. By Lemma 4.14, volumes of tetrahedrons can be reduced to volume coordinates w.r.t to four given points. The following lemma will also reduce the Pythagorean difference to volume coordinates.

LEMMA 5.9. *Let O, W, U, and V be four points not on the same plane such that $OW \perp OUV$, $OU \perp OWV$, and $OV \perp OWU$. Then*

(1) $P_{ABC} = \overline{AB}^2 + \overline{CB}^2 - \overline{AC}^2$.

(2) $\overline{AB}^2 = \overline{OW}^2\left(\frac{V_{AOUVB}}{V_{OWUV}}\right)^2 + \overline{OU}^2\left(\frac{V_{AOWVB}}{V_{OWUV}}\right)^2 + \overline{OV}^2\left(\frac{V_{AOWUB}}{V_{OWUV}}\right)^2$.

(3) $V_{OWUV}^2 = \frac{1}{36}\overline{OW}^2\overline{OU}^2\overline{OV}^2$.

Proof. (1) is the definition. (3) is from Proposition 5.1. For (2), let R, P, and Q be the orthogonal projections from the point B to the planes OUV, OWV, and OWU respectively, and D, E, and F be the orthogonal projections from the point A to the lines BR, BP, and BQ respectively. By the Pythagorean theorem

$$\overline{AB}^2 = \overline{OW}^2\left(\frac{\overline{BD}}{\overline{OW}}\right)^2 + \overline{OU}^2\left(\frac{\overline{BE}}{\overline{OU}}\right)^2 + \overline{OV}^2\left(\frac{\overline{BF}}{\overline{OV}}\right)^2.$$

Now the result comes from Lemma 4.9. □

Algorithm 5.10 (Volume)

INPUT: $S = (C_1, C_2, \ldots, C_k, (E, F))$ is a constructive geometric statement.

OUTPUT: The algorithm tells whether S is true or not, and if it is true, produces a proof for S.

S1. For $i = k, \ldots, 1$, do S2, S3, S4 and finally do S5.

S2. Check whether the ndg conditions of C_i are satisfied. The ndg condition of a construction has three forms: $A \neq B$, $PQ \nparallel UV$, $PQ \nparallel WUV$. For the first case, we check whether $P_{ABA} = 2\overline{AB}^2 = 0$. For the second case, we check whether $V_{OQUV} = 0$ and $S_{PUV} = S_{QUV}$. For the third case, we check whether $V_{PWUV} = V_{QWUV}$. If a ndg condition for a geometry statement is not satisfied, the statement is *trivially true*. The algorithm terminates.

S3. Let $G_1, \ldots, G_s$ be the geometric quantities occurring in E and F. For $j = 1, \ldots, s$ do S4.

S4. Let H_j be the result obtained by eliminating the point introduced by construction C_i from G_j using the lemmas in Sections 4 and 5. Replace G_j by H_j in E and F to obtain the new E and F.

S5. At least, E and F are rational expressions in independent variables. Hence if $E = F$, S is true under the ndg conditions. Otherwise S is false in the Euclidean plane geometry.

Proof. The last E and F are rational expressions in free parameters. If $E = F$, the statement is obviously true. Otherwise, we can find specific values for the free parameters in E and F such that when substituting them into E and F, we obtain two different values of E and F, i.e., we have found a counterexample. The ndg conditions of the statement ensures the validity of each step, because all the geometric quantities occurring in the proof have geometric meaning, i.e., their denominators will not vanish. □

For the complexity of the algorithm, let m and n be the number of free and non-free points in a statement, respectively. Then we will use the lemmas (except 4.14 and 5.9) for at most n times. Also note that each lemma will replace a geometric quantity by a rational expression with degree less than or equal to three. Then if the conclusion of the geometry statement is of degree d, the result after eliminating the nonfree points is at most degree $3^n d$. To eliminate the free points using Lemmas 4.14 and 5.9, the final result is at most degree $4 \cdot 3^n d$.

Remark 5.10. In the development of the volume method, no special property of the real number field has been used. As a result, the volume method works not only for Euclidean geometry but also for metric solid geometries associated with any field with characteristic zero.

6. Experiment Results and Comparisons

We have implemented the algorithm in Common Lisp on a NeXT workstation. The following is the machine produced proof for Example 2.13.

EXAMPLE 6.1. For the input like Example 3.3, our program produces the following machine proof (in Latex form) automatically.

The machine proof	The eliminants
$\frac{\overline{DH}}{\overline{AH}} \cdot \frac{\overline{CG}}{\overline{DG}} \cdot \frac{\overline{BF}}{\overline{CF}} \cdot \frac{\overline{AE}}{\overline{BE}}$	$\frac{\overline{DH}}{\overline{AH}} \stackrel{H}{=} \frac{V_{DXYZ}}{V_{AXYZ}}$
$\stackrel{H}{=} \frac{V_{DXYZ}}{V_{AXYZ}} \cdot \frac{\overline{CG}}{\overline{DG}} \cdot \frac{\overline{BF}}{\overline{CF}} \cdot \frac{\overline{AE}}{\overline{BE}}$	$\frac{\overline{CG}}{\overline{DG}} \stackrel{G}{=} \frac{V_{CXYZ}}{V_{DXYZ}}$
$\stackrel{G}{=} \frac{V_{DXYZ} \cdot V_{CXYZ}}{V_{AXYZ} \cdot V_{DXYZ}} \cdot \frac{\overline{BF}}{\overline{CF}} \cdot \frac{\overline{AE}}{\overline{BE}}$	$\frac{\overline{BF}}{\overline{CF}} \stackrel{F}{=} \frac{V_{BXYZ}}{V_{CXYZ}}$
$\stackrel{simplify}{=} \frac{V_{CXYZ}}{V_{AXYZ}} \cdot \frac{\overline{BF}}{\overline{CF}} \cdot \frac{\overline{AE}}{\overline{BE}}$	$\frac{\overline{AE}}{\overline{BE}} \stackrel{E}{=} \frac{V_{AXYZ}}{V_{BXYZ}}$
$\stackrel{F}{=} \frac{V_{CXYZ} \cdot V_{BXYZ}}{V_{AXYZ} \cdot V_{CXYZ}} \cdot \frac{\overline{AE}}{\overline{BE}}$	
$\stackrel{simplify}{=} \frac{V_{BXYZ}}{V_{AXYZ}} \cdot \frac{\overline{AE}}{\overline{BE}}$	
$\stackrel{E}{=} \frac{V_{BXYZ} \cdot V_{AXYZ}}{V_{AXYZ} \cdot V_{BXYZ}}$	
$\stackrel{simplify}{=} 1$	

In the above proof, the symbol $\stackrel{H}{=}$ means to eliminate point H. The *eliminants* are the separate elimination results by using the lemmas in Sections 4 and 5.

The following table contains the timing and proof-length statistics for the 80 examples proved by our program. Maxterm means the number of terms of the maximal polynomial occurring in a proof.

The Proof-Length		The Proving Time	
Maxterm	No. of Theorems	Time (secs)	No. of Theorems
$m = 1$	18	$t \leqslant 0.05$	28
$m = 2$	25	$0.05 < t \leqslant 0.1$	19
$2 < m \leqslant 5$	24	$0.1 < t \leqslant 0.5$	27
$5 < m \leqslant 10$	9	$0.5 < t \leqslant 1$	3
$10 < m \leqslant 140$	4	$1 < t \leqslant 100$	3

The key to the *volume* method presented here is a collection of powerful, high level theorems, such as the Co-face theorems about the signed volumes. This method can be contrasted with the earlier algebraic methods, which also proved astonishingly difficult theorems in geometry, but with low-level, mind-numbing polynomial manipulations. For more than eighty percent of the 80 theorems proved by the volume method, the maximal polynomials in their proofs have less than six terms. The maxterms of the proofs produced using algebraic methods are rarely less than six. On the other hand, the algebraic methods are more general, e.g., they can be used to prove theorems involving inequalities and theorems in differential geometry. Also see [10] for an interesting method based on the vector version of the Gröbner basis.

The previous methods based on the AI approach can also produce readable proofs for simple geometry theorems [6, 9]. The key tool in these methods are the congruent of triangles which prevents these method from going very far for two reasons. First, the congruent triangle techniques are used to prove some basic geometry facts and the proofs for most of the high level geometry theorems using other concepts besides the congruent triangles. Second, even in those proofs based on congruent triangles, auxiliary points or lines are often needed to form the required congruent triangles and these auxiliary points or lines are often added by the user instead of the computer program.

Appendix. Machine Produced Proofs for Several Examples

The proofs (in LaTeX form) of the following examples are produced *entirely* automatically by a program based on the Algorithm 5.10.

EXAMPLE 1 (**Ceva's Theorem for Skew Quadrilaterals).** The planes passing through a point O and the sides AB, BC, CD, and DA of any skew quadrilateral meet the opposite sides of the quadrilateral in G, H, E, and F respectively (Figure 8). Show that

$$\frac{\overline{AE}}{\overline{EB}} \cdot \frac{\overline{BF}}{\overline{FC}} \cdot \frac{\overline{CG}}{\overline{GD}} \cdot \frac{\overline{DH}}{\overline{HA}} = 1.$$

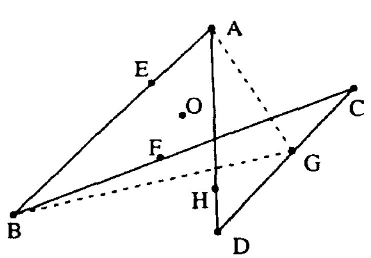

Fig. 8.

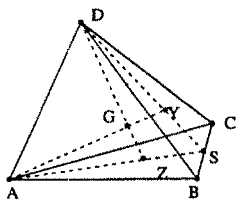

Fig. 9.

The input

$$\Big((\text{POINTS}\ ABCDO)$$
$$(\text{INTER}\ E(\text{LINE}\ AB)(\text{PLANE}\ OCD))$$
$$(\text{INTER}\ F(\text{LINE}\ BC)(\text{PLANE}\ OAD))$$
$$(\text{INTER}\ G(\text{LINE}\ CD)(\text{PLANE}\ OAB))$$
$$(\text{INTER}\ H(\text{LINE}\ DA)(\text{PLANE}\ OBC))$$
$$\Big(\frac{\overline{DH}}{\overline{AH}}\cdot\frac{\overline{CG}}{\overline{DG}}\cdot\frac{\overline{BF}}{\overline{CF}}\cdot\frac{\overline{AE}}{\overline{BE}}=1\Big)\Big)$$

The machine proof

$$\frac{\overline{DH}}{\overline{AH}}\cdot\frac{\overline{CG}}{\overline{DG}}\cdot\frac{\overline{BF}}{\overline{CF}}\cdot\frac{\overline{AE}}{\overline{BE}}$$
$$\stackrel{H}{=}\frac{V_{BCDO}}{V_{ABCO}}\cdot\frac{\overline{CG}}{\overline{DG}}\cdot\frac{\overline{BF}}{\overline{CF}}\cdot\frac{\overline{AE}}{\overline{BE}}$$
$$\stackrel{G}{=}\frac{V_{BCDO}\cdot V_{ABCO}}{V_{ABCO}\cdot V_{ABDO}}\cdot\frac{\overline{BF}}{\overline{CF}}\cdot\frac{\overline{AE}}{\overline{BE}}$$
$$\stackrel{simplify}{=}\frac{V_{BCDO}}{V_{ABDO}}\cdot\frac{\overline{BF}}{\overline{CF}}\cdot\frac{\overline{AE}}{\overline{BE}}$$
$$\stackrel{F}{=}\frac{V_{BCDO}\cdot(-V_{ABDO})}{V_{ABDO}\cdot(-V_{ACDO})}\cdot\frac{\overline{AE}}{\overline{BE}}$$
$$\stackrel{simplify}{=}\frac{V_{BCDO}}{V_{ACDO}}\cdot\frac{\overline{AE}}{\overline{BE}}$$
$$\stackrel{E}{=}\frac{V_{BCDO}\cdot V_{ACDO}}{V_{ACDO}\cdot V_{BCDO}}\stackrel{simplify}{=}1$$

The eliminants

$$\frac{\overline{DH}}{\overline{AH}}\stackrel{H}{=}\frac{V_{BCDO}}{V_{ABCO}}$$
$$\frac{\overline{CG}}{\overline{DG}}\stackrel{G}{=}\frac{V_{ABCO}}{V_{ABDO}}$$
$$\frac{\overline{BF}}{\overline{CF}}\stackrel{F}{=}\frac{V_{ABDO}}{V_{ACDO}}$$
$$\frac{\overline{AE}}{\overline{BE}}\stackrel{E}{=}\frac{V_{ACDO}}{V_{BCDO}}$$

The ndg conditions are $AB \nparallel OCD$; $BC \nparallel OAD$; $CD \nparallel OAB$; $AD \nparallel OBC$; $B \neq E$; $C \neq F$; $D \neq G$, $A \neq H$.

EXAMPLE 2 **(Centroid of a Tetrahedron).** The four medians of a tetrahedron meet in a point which divides each median in the ratio 3 : 1. The longer segment being on the side of the vertex of the tetrahedron (Figure 9).

The input

((POINTS $ABCD$)
(MIDPOINT SBC)
(LRATIO $ZAS2/3$)
(LRATIO $YDS2/3$)
(INTER G(LINE DZ)(LINE AY))
$\left(\frac{\overline{AG}}{\overline{GY}} = 3\right)$)

The machine proof

$$\frac{1}{3}\left(\frac{\overline{AG}}{\overline{GY}}\right)$$

$$\overset{G}{=} \frac{-\frac{S_{ADZ}}{S_{DZY}}}{3}$$

$$\overset{Y}{=} \frac{-\frac{S_{ADZ}}{S_{DSZ}}}{(3)\cdot\left(-\frac{2}{3}\right)}$$

$$\overset{Z}{=} \frac{\frac{2}{3}}{(2)\cdot(\frac{1}{3})}$$

$$\overset{simplify}{=} 1$$

The eliminants

$$\frac{\overline{AG}}{\overline{YG}} \overset{G}{=} \frac{S_{ADZ}}{S_{DZY}}$$

$$\frac{S_{ADZ}}{S_{DZY}} \overset{Y}{=} -\frac{3}{2}\left(\frac{S_{ADZ}}{S_{DSZ}}\right)$$

$$\frac{S_{ADZ}}{S_{DSZ}} \overset{Z}{=} 2(1)$$

The ndg conditions are $B \neq C$, $A \neq S$, $D \neq S$, $DZ \not\parallel AY$, $G \neq Y$.

EXAMPLE 3. If P, Q, R, and S are the feet of four cevains having the point O in common, we have

$$\frac{\overline{OP}}{\overline{AP}} + \frac{\overline{OQ}}{\overline{CR}} + \frac{\overline{OR}}{\overline{CR}} + \frac{\overline{OS}}{\overline{DS}} = 1$$

(see Figure 10).

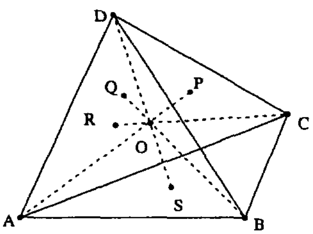

Fig. 10.

The input
((POINTS $ABCDO$))
(INTER P(LINE AO)(PLANE BCD))
(INTER Q(LINE BO)(PLANE ACD))
(INTER R(LINE CO)(PLANE ABD))
(INTER S(LINE DO)(PLANE ABC))
$\left(\frac{\overline{OS}}{\overline{DS}} + \frac{\overline{OR}}{\overline{CR}} + \frac{\overline{OQ}}{\overline{BQ}} + \frac{\overline{OP}}{\overline{AP}} = 1\right))$

The eliminants

$$\frac{\overline{OS}}{\overline{DS}} \stackrel{S}{=} \frac{V_{ABCO}}{V_{ABCD}}$$
$$\frac{\overline{OR}}{\overline{CR}} \stackrel{R}{=} \frac{-V_{ABDO}}{V_{ABCD}}$$
$$\frac{\overline{OQ}}{\overline{BQ}} \stackrel{Q}{=} \frac{V_{ACDO}}{V_{ABCD}}$$
$$\frac{\overline{OP}}{\overline{AP}} \stackrel{P}{=} \frac{-V_{ABCDO}}{V_{ABCD}}$$
$$V_{BCDO} = V_{ACDO} - V_{ABDO} + V_{ABCO} - V_{ABCD}$$

The machine proof

$$\frac{\overline{OS}}{\overline{DS}} + \frac{\overline{OR}}{\overline{CR}} + \frac{\overline{OQ}}{\overline{BQ}} + \frac{\overline{OP}}{\overline{AP}}$$
$$\stackrel{S}{=} \frac{-V_{ABCO} - V_{ABCD} \cdot \frac{\overline{OR}}{\overline{CR}} - V_{ABCD} \cdot \frac{\overline{OQ}}{\overline{BQ}} - V_{ABCD} \cdot \frac{\overline{OP}}{\overline{AP}}}{-V_{ABCD}}$$
$$\stackrel{R}{=} \frac{-V_{ABDO} \cdot V_{ABCD} + V_{ABCO} \cdot V_{ABCD} + V_{ABCD}^2 \cdot \frac{\overline{OQ}}{\overline{BQ}} + V_{ABCD}^2 \frac{\overline{OP}}{\overline{AP}}}{(V_{ABCD})^2}$$
$$\stackrel{simplify}{=} \frac{-\left(V_{ABDO} - V_{ABCO} - V_{ABCD} \cdot \frac{\overline{OQ}}{\overline{BQ}} - V_{ABCD} \cdot \frac{\overline{OP}}{\overline{AP}}\right)}{V_{ABCD}}$$
$$\stackrel{Q}{=} \frac{-\left(V_{ACDO} \cdot V_{ABCD} - V_{ABDO} \cdot V_{ABCD} + V_{ABCO} \cdot V_{ABCD} + V_{ABCD}^2 \cdot \frac{\overline{OP}}{\overline{AP}}\right)}{V_{ABCD} \cdot (-V_{ABCD})}$$
$$\stackrel{simplify}{=} \frac{V_{ACDO} - V_{ABDO} + V_{ABCO} + V_{ABCD} \cdot \frac{\overline{OP}}{\overline{AP}}}{V_{ABCD}}$$
$$\stackrel{P}{=} \frac{-V_{BCDO} \cdot V_{ABCD} + V_{ACDO} \cdot V_{ABCD} - V_{ABDO} \cdot V_{ABCO} + V_{ABCD} \cdot V_{ABCD}}{(V_{ABCD})^2}$$
$$\stackrel{simplify}{=} \frac{-(V_{BCDO} - V_{ACDO} + V_{ABDO} - V_{ABCO})}{V_{ABCD}}$$
$$= \frac{-(-V_{ABCD})}{V_{ABCD}}$$
$$\stackrel{simplify}{=} 1$$

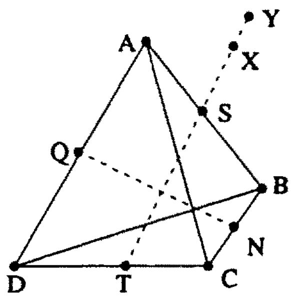

Fig. 11.

The ndg conditions are $AO \not\parallel BCD$, $BO \not\parallel ACD$, $CO \not\parallel ABD$, $DO \not\parallel ABC$, $A \neq P$, $B \neq Q$, $C \neq R$, $D \neq S$.

A line joining the midpoints of two opposite edges of a tetrahedron will be called a *bimedian* of the tetrahedron relative to the pair of edges considered. The common perpendicular to the two opposite edges of a tetrahedron is called the *bialtitude* of the tetrahedron relative to these edges.

EXAMPLE 4. The bialtitude relative to one pair of opposite edges of a tetrahedron is perpendicular to the two bimedians relative to the two other pairs of opposite edges (Figure 11).

The input to the program is

The input
((POINTS $XYAC$)
(FOOT2LINE $SAXY$)
(ON B(LINE SA))
(FOOT2LINE $TCXY$)
(ON D(LINE TC))
(MIDPOINT MAB)
(MIDPOINT NBC)
(MIDPOINT PDC)
(MIDPOINT QAD)
(PERPENDICULAR $NQXY$))

The ndg conditions: $X \neq Y$, $A \neq S$, $C \neq T$, $C \neq B$, $D \neq A$.

The machine proof

$$\frac{P_{YXN}}{P_{YXQ}}$$

$$\stackrel{Q}{=} \frac{P_{YXN}}{\frac{1}{2}P_{YXD}+\frac{1}{2}P_{YXA}}$$

$$\stackrel{P}{=} \frac{(2)\cdot P_{YXN}}{P_{YXD}+P_{YXA}}$$

$$\stackrel{N}{=} \frac{(2)\cdot\left(\frac{1}{2}P_{YXB}+\frac{1}{2}P_{YXC}\right)}{P_{YXD}+P_{YXA}}$$

$$\stackrel{M}{=} \frac{P_{YXB}+P_{YXC}}{P_{YXD}+P_{YXA}}$$

$$\stackrel{D}{=} \frac{P_{YXB}+P_{YXC}}{-P_{YXT}\cdot\frac{\overline{TD}}{\overline{TC}}+P_{YXT}+P_{YXC}\cdot\frac{\overline{TD}}{\overline{TC}}+P_{YXA}}$$

$$= \frac{-(P_{YXB}+P_{YXC})}{-P_{YXC}-P_{YXA}}$$

$$\stackrel{B}{=} \frac{-P_{YXS}\cdot\frac{\overline{SB}}{\overline{SA}}+P_{YXS}+P_{YXC}+P_{YXA}\cdot\frac{\overline{SB}}{\overline{SA}}}{P_{YXC}+P_{YXA}}$$

$$= \frac{-(-P_{YXC}-P_{YXA})}{P_{YXC}+P_{YXA}}$$

$$\stackrel{simplify}{=} 1$$

The eliminants

$$P_{YXQ} \stackrel{Q}{=} \frac{1}{2}(P_{YXD} + P_{YXA})$$

$$P_{YXN} \stackrel{N}{=} \frac{1}{2}(P_{YXB} + P_{YXC})$$

$$P_{YXD} \stackrel{D}{=} -\left(P_{YXT} \cdot \frac{\overline{TD}}{\overline{TC}} - P_{YXT} - P_{YXC} \cdot \frac{\overline{TD}}{\overline{TC}}\right)$$

$$P_{YXT} = P_{YXC}$$

$$P_{YXB} \stackrel{B}{=} -\left(P_{YXS} \cdot \frac{\overline{SB}}{\overline{SA}} - P_{YXS} - P_{YXA} \cdot \frac{\overline{SB}}{\overline{SA}}\right)$$

$$P_{YXS} = P_{YXA}$$

EXAMPLE 5. * Let $ABCD$ be a tetrahedron and G the centroid of triangle ABC. The lines passing through points A, B, and C and parallel to line DG meet their opposite face in P, Q, and R, respectively. Show that $V_{GPQR} = 3V_{ABCD}$ (Figure 12).

* This is a problem from the 1964 International Mathematical Olympiad.

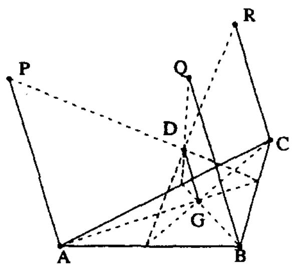

Fig. 12.

The input

((POINTS $ABCD$)
(CENTROID $GABC$)
(INTER P(PLINE ADG)(PLANE BCD))
(INTER Q(PLINE BDG)(PLANE ACD))
(INTER R(PLINE CDG)(PLANE ABD))
($3V_{ABCD} = V_{GPQR}$))

The eliminants

$$V_{CDGP} = -V_{ACDG}$$
$$V_{BDGP} = -V_{ABDG}$$
$$V_{BCGP} = V_{ABCD}$$
$$V_{DGPQ} = -V_{BDGP}$$
$$V_{CGPQ} = \frac{V_{CDGP} \cdot V_{ABCD} - V_{BCGP} \cdot V_{ACDG}}{V_{ACDG}}$$
$$V_{GPQR} = \frac{V_{DGPQ} \cdot V_{ABCD} - V_{CGPQ} \cdot V_{ABDG}}{V_{ABDG}}$$

The machine proof

$$\frac{(3) \cdot V_{ABCD}}{V_{GPQR}}$$
$$\stackrel{R}{=} \frac{(3) \cdot V_{ABCD} \cdot V_{ABDG}}{V_{DGPQ} \cdot V_{ABCD} - V_{CGPQ} \cdot V_{ABDG}}$$
$$\stackrel{Q}{=} \frac{(3) \cdot V_{ABCD} \cdot V_{ABDG} \cdot (V_{ACDG})^2}{-V_{CDGP} \cdot V_{ACDG} \cdot V_{ABDG} - V_{BDGP} \cdot V_{ACDG}^2 \cdot V_{ABCD} + V_{BCGP} \cdot V_{ACDG}^2 \cdot V_{ABDG}}$$
$$\stackrel{simplify}{=} \frac{(-3) \cdot V_{ABCD} \cdot V_{ABDG} \cdot V_{ACDG}}{V_{CDGP} \cdot V_{ABDG} \cdot V_{ABCD} + V_{BDGP} \cdot V_{ACDG} \cdot V_{ABCD} - V_{BCGP} \cdot V_{ACDG} \cdot V_{ABDG}}$$
$$\stackrel{P}{=} \frac{(-3) \cdot V_{ABCD} \cdot V_{ABDG} \cdot V_{ACDG} \cdot (V_{BCDG})^2}{-3V_{BCDG}^3 \cdot V_{ACDG} \cdot V_{ABDG} \cdot V_{ABCD}}$$
$$\stackrel{simplify}{=} 1$$

In the above proof the fact that G is the centroid of triangle ABC is not used. We thus have the following extension of Example 5.

EXAMPLE 6. Continue from Example 5. The result of Example 5 is still true if point G is any point in plane ABC.

We further ask whether the result of Example 5 is true or not if point G is an arbitrary point.

The input

$$\Big((\text{POINTS } ABCDG)$$
$$(\text{INTER } P(\text{PLINE } ADG)(\text{PLANE } BCD))$$
$$(\text{INTER } Q(\text{PLINE } BDG)(\text{PLANE } ACD))$$
$$(\text{INTER } R(\text{PLINE } CDG)(\text{PLANE } ABD))$$
$$\left(\frac{V_{GPQR}}{3V_{ABCD}}\right)\Big)$$

The eliminants

$$V_{GPQR} \stackrel{R}{=} \frac{V_{DGPQ}\cdot V_{ABCD}-V_{CGPQ}\cdot V_{ABDG}}{V_{ABDG}}$$

$$V_{CGPQ} \stackrel{Q}{=} \frac{V_{CDGP}\cdot V_{ABCD}-V_{BCGP}\cdot V_{ACDG}}{V_{ACDG}}$$

$$V_{DGPQ} \stackrel{Q}{=} -V_{BDGP}$$

$$V_{BCGP} \stackrel{P}{=} -(V_{ABCG} - V_{ABCD})$$

$$V_{BDGP} \stackrel{P}{=} -V_{ABDG}$$

$$V_{CDGP} \stackrel{P}{=} -V_{ACDG}$$

The machine proof

$$\frac{V_{GPQR}}{(3)\cdot V_{ABCD}}$$

$$\stackrel{R}{=} \frac{V_{DGPQ}\cdot V_{ABCD}-V_{CGPQ}\cdot V_{ABDG}}{(3)\cdot V_{ABCD}\cdot V_{ABDG}}$$

$$\stackrel{Q}{=} \frac{-V_{CDGP}\cdot V_{ABDG}\cdot V_{ABDG}-V_{BDGP}\cdot V_{ACDG}^2\cdot V_{ABCD}+V_{BCGP}\cdot V_{ACDG}^2\cdot V_{ABDG}}{(3)\cdot V_{ABCD}\cdot V_{ABDG}\cdot (V_{ACDG})^2}$$

$$\stackrel{simplify}{=} \frac{-(V_{CDGP}\cdot V_{ABDG}\cdot V_{ABCD}+V_{BDGP}\cdot V_{ACDG}\cdot V_{ABCD}-V_{BCGP}\cdot V_{ACDG}\cdot V_{ABDG})}{(3)\cdot V_{ABCD}\cdot V_{ABDG}\cdot (V_{ACDG})}$$

$$\stackrel{P}{=} \frac{-(V_{BCDG}^3\cdot V_{ACDG}\cdot V_{ABDG}\cdot V_{ABCG}-3V_{BCDG}^3\cdot V_{ACDG}\cdot V_{ABDG}\cdot V_{ABCD})}{(3)\cdot V_{ABCD}\cdot V_{ABDG}\cdot V_{ACDG}\cdot (V_{BCDG})^3}$$

$$= \frac{-(V_{ABCG}-3V_{ABCD})}{(3)\cdot V_{ABCD}}$$

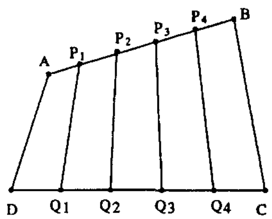

Fig. 13.

We thus obtain the following extension of Example 6:

EXAMPLE 7. $V_{GPQR} = 3V_{ABC}$ iff G is in plane ABC.

EXAMPLE 8. The sides AB and DC of a skew quadrilateral are cut into $2n+1$ equal segments by points $P_1, \ldots, P_{2n}$ and $Q_1, \ldots, Q_{2n}$ respectively (Figure 13). Show that $V_{P_nP_{n+1}Q_{n+1}Q_n} = \frac{1}{(2n+1)^2} V_{ABCD}$.

The following figure shows the case $n = 2$. Note that in the following machine proof for (1), we use some different names for points $P_n, P_{n+1}, Q_{n+1}, Q_n$.

Constructive description

$$\left((\text{POINTS } ABCD)\right.$$
$$(\text{LRATIO } XAB \tfrac{n}{2n+1})$$
$$(\text{LRATIO } YAB \tfrac{n+1}{2n+1})$$
$$(\text{LRATIO } UDC \tfrac{n}{2n+1})$$
$$(\text{LRATIO } VDC \tfrac{n+1}{2n+1})$$
$$\left.\left(\frac{V_{XYVU}}{V_{ABCD}}\right)\right)$$

The eliminants

$$V_{XYUV} \stackrel{V}{=} \frac{-(V_{DXYU} \cdot n + V_{CXYU} \cdot n + V_{CXYU})}{2n+1}$$
$$V_{CXYU} \stackrel{U}{=} \frac{(n+1) \cdot V_{CDXY}}{2n+1}$$
$$V_{DXYU} \stackrel{U}{=} \frac{-V_{CDXY} \cdot n}{2n+1}$$
$$V_{CDXY} \stackrel{Y}{=} \frac{-(V_{BCDX} \cdot n + V_{BCDX} + V_{ACDX} \cdot n)}{2n+1}$$
$$V_{ACDX} \stackrel{X}{=} \frac{V_{ABCD} \cdot n}{2n+1}$$
$$V_{BCDX} \stackrel{X}{=} \frac{-(n+1) \cdot V_{ABCD}}{2n+1}$$

The machine proof

$$\frac{-V_{XYUV}}{V_{ABCD}}$$
$$\stackrel{V}{=} \frac{-(-V_{DXYU} \cdot n - V_{CXYU} \cdot n - V_{CXYU})}{V_{ABCD} \cdot (2n+1)}$$
$$\stackrel{U}{=} \frac{4V_{CDXY} \cdot n^2 + 4V_{CDXY} \cdot n + V_{CDXY}}{V_{ABCD} \cdot (2n+1)^3}$$
$$\stackrel{simplify}{=} \frac{V_{CDXY}}{V_{ABCD} \cdot (2n+1)}$$
$$\stackrel{Y}{=} \frac{-V_{BCDX} \cdot n - V_{BCDX} - V_{ACDX} \cdot n}{V_{ABCD} \cdot (2n+1)^2}$$
$$\stackrel{X}{=} \frac{-(-4V_{ABCD} \cdot n^2 - 4V_{ABCD} \cdot n - V_{ABCD})}{V_{ABCD} \cdot (2n+1)^4}$$
$$\stackrel{simplify}{=} \frac{1}{(2n+1)^2}$$

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