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The Loading Problem for a Linear Viscoelastic Earth: I. Compressible, Non-Gravitating Models

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Summary – A Legendre mode solution is given for deformation of a solid isotropic linear viscoelastic sphere under applied surface stresses. Under the simplifying assumptions that the sphere is elastic in compression and standard linear solid in shear two relaxation times appear; one the creep relaxation time of the material, the other depending on mode. It is shown formally how to reduce the case of a layered viscoelastic sphere to an equivalent unlayered one.

1. Introduction

In order to elucidate various geodynamical processes, the geodynamicist often resorts to various simple earth models. Using such models, rough calculations may be made of the deformation due to a particular geodynamical process. Processes of interest in this connection are: a) decay of the non-hydrostatic bulge, b) isostatic adjustments due to loads of various time-histories and areal extents, c) decay of the Chandler wobble, d) solid-earth tidal deformation and e) eigenvibrations following a large earthquake.

Spherical earth models (as opposed to flat layered ones) are clearly needed to analyze processes a) through e). We expect the earth to be affected to considerable depth by these long-period processes; on the other hand there should be operative in these large-scale processes a smoothing effect which averages the physical properties of any given depth with those nearby. Since delayed response effects are an important part of such geodynamical phenomena, the model must show these.

We shall be interested in constructing spherical earth models to analyze phenomena a) and b). As a constraint, the models must show at least the general features of c), d), and e). The observed time-delayed responses will be accommodated by taking the models to be isotropic and linear viscoelastic. For simplicity, the model earth will be either uniform (unlayered) or constructed of a small number of concentric viscoelastic shells. We shall call such a model a 'net earth', under the supposition (and hope!) that each shell averages the net viscoelastic properties of its corresponding region of the real earth.

Problems a) and b) both involve earth loads of various extents and time-histories on the earth's surface. Analysis using net-earth models thus calls for access to certain

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results involving loads on uniform and layered viscoelastic spheres. In principle, such results can be derived using the so-called 'correspondence principle' (GURTIN and STERNBERG [3]) from solutions developed for elastic spheres (e.g. SLICHTER and CAPUTO [6], LONGMAN [5], ALTERMAN *et al.* [1]). In practice, however, the work only begins there, especially if one wishes to develop the results for viscoelastic constitutive rules of any more complexity than simple Maxwell or Voigt ones. In this paper, therefore, we show our results in some detail, listing in full all equations which are apt to be of use to the geodynamicist.

This is the first of three papers on the viscoelastic loading problem. In this paper we display the machinery needed to treat loaded compressible but non-gravitating viscoelastic spheres. In the second paper the loaded self-gravitating but incompressible sphere will be treated. The third paper will compare the two approaches, and will show that the general problem of compressible and self-gravitating spheres is much harder to solve than either of the two more limited problems.

2. *Equation of motion*

The general equation of motion for an isotropic linear viscoelastic body under quasi-static deformation is given by GURTIN and STERNBERG [3] as

$$\nabla^2 \mathbf{u} * dG_1 + \nabla(\nabla \cdot \mathbf{u}) * d\frac{\Lambda}{3} + 2\mathbf{F} = 0 \tag{1a}$$

where

$$\Lambda = G_1 + 2G_2. \tag{1b}$$

In this equation G_1 and G_2 are time functions characteristic of the material giving its stress relaxation after unit steps in shear and compressive strain, respectively. The notation $A * dB$ stands for the Stieltjes convolution

$$A * dB = \int_{-\infty}^{t} A(t-\tau)\frac{dB(\tau)}{d\tau}\,d\tau. \tag{2}$$

The displacement vector is $\mathbf{u}$ and the body force $\mathbf{F}$. Note that these equations reduce to the familiar elastostatic case by making the substitutions

$$*dG_1 \leftarrow 2\mu \tag{3a}$$

$$*dG_2 \leftarrow 3k \tag{3b}$$

where μ is rigidity and k incompressibility.

In the present paper, we shall suppose $\mathbf{F}=0$. In spherical polar coordinates (r,θ,φ) and assuming axial symmetry $(\partial/\partial\varphi=0)$, the field equation (1) then becomes

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, 2rw) * dG_1 - \frac{\partial\Delta}{\partial r} * d\left(G_1+\frac{\Lambda}{3}\right)=0 \tag{4a}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(2rw) * dG_1 + \frac{1}{r}\frac{\partial\Delta}{\partial\theta} * d\left(G_1+\frac{\Lambda}{3}\right)=0 \tag{4b}$$

where

$$\Delta=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, u_\theta) \tag{4c}$$

and

$$2rw=\frac{\partial}{\partial r}(ru_\theta)-\frac{\partial}{\partial\theta}u_r. \tag{4d}$$

Following the well-known elastostatic analogue of this problem (Slichter and Caputo [6]), we separate variables by

$$u_r=U(r,t)P_n(\cos\theta) \tag{5a}$$

$$u_\theta=V(r,t)\frac{\partial P_n}{\partial\theta}(\cos\theta) \tag{5b}$$

where $P_n(\cos\theta)$ is the Legendre polynomial of order n. The result is the set of four equations in four unknowns, U, V, W, and X:

$$n(n+1)\frac{W}{r^2} * dG_1 + \frac{\partial X}{\partial r} * d\left(G_1+\frac{\Lambda}{3}\right)=0 \tag{6a}$$

$$\frac{\partial W}{\partial r} * dG_1 + X * d\left(G_1+\frac{\Lambda}{3}\right)=0 \tag{6b}$$

$$X=\frac{\partial U}{\partial r}+\frac{2U}{r}-n(n+1)\frac{V}{r} \tag{6c}$$

$$\frac{W}{r}=\frac{\partial V}{\partial r}+\frac{V-U}{r}. \tag{6d}$$

These four unknowns are functions of time t, radius r, and Legendre order n, but are independent of polar angle θ.

3. General solution

Solving equations (6), we have

$$U = A_1 r^{n+1} + A_2 r^{-n} + A_3 r^{n-1} + A_4 r^{-n-2} \tag{7a}$$

$$V = A_1 * dMr^{n+1} + A_2 * dNr^{-n} + \frac{1}{n} A_3 r^{n-1} - \frac{1}{n+1} A_4 r^{-n-2} \tag{7b}$$

$$X = A_1 * d[(n+3)h - n(n+1)M]r^n$$
$$+ A_2 * d[(-n+2)h - n(n+1)N]r^{-n-1} \tag{7c}$$

where

$$n(n+1)M = (n+3)h(t) + 6(2n+3)G_1 * d[(n-6)G_1 + 2nG_2]^{-1} \tag{8a}$$

and

$$-n(n+1)N = (n-2)h(t) - 6(2n-1)G_1 * d[(n+7)G_1 + 2(n+1)G_2]^{-1}. \tag{8b}$$

Notice that for the Legendre mode $n = 6$ the function M takes a particularly simple form, with the second term involving $G_1 * dG_2^{-1}$.

The solution for W involves a complicated function times r^{-n} plus another times r^{n+1}. Since we shall not need them in this paper, we do not bother to write these complicated functions out explicitly.

In the above equations, $h(t)$ is the Heaviside unit step,

$$h(t) = \begin{matrix} 1 & t > 0 \\ 0 & t < 0. \end{matrix} \tag{9}$$

The inverse $(\ )^{-1}$ is defined in such a way that

$$A * dA^{-1} = h \tag{10a}$$

for any function $A(t)$. From a computational standpoint, inverse functions are found using the property that

$$\hat{A}^{-1} = \frac{1}{p^2} \hat{A} \tag{10b}$$

where the hat denotes Laplace transform of variable p. Thus some sort of Laplace and inverse Laplace transform scheme proves useful for calculation of functions like M and N.

Note that the functions A_i in equation (7) depend on time t and Legendre order n, but not on radius r. These functions are determined by the boundary conditions of the problem at hand. Clearly, this solution is the general one, appropriate for a spherical shell. If the sphere is solid all the way to the center, finiteness conditions there will require that $A_2 = A_4 = 0$. Similarly $A_1 = A_3 = 0$ for the problem of a spherical hole in an infinite medium.

4. Stresses

Radial and tangential stresses σ_{rr} and $\sigma_{r\theta}$ are given by

$$\sigma_{rr} = \tfrac{1}{3}\Delta * d(G_2 - G_1) + \frac{\partial u_r}{\partial r} * dG_1 \tag{11a}$$

$$\sigma_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right) * dG_1. \tag{11b}$$

It will prove useful to bring the present notation in line with that of ALTERMAN *et al.* [1]. To that end we define the following functions of r, n, and t:

$$y_1 = U \tag{12a}$$

$$y_2 = \tfrac{1}{3}X * d(G_2 - G_1) + \frac{\partial U}{\partial r} * dG_1 \tag{12b}$$

$$y_3 = V \tag{12c}$$

$$y_4 = \frac{1}{2}\left(\frac{\partial V}{\partial r} + \frac{U - V}{r}\right) * dG_1. \tag{12d}$$

It is easily verified that

$$u_r(r, \theta, t, n) = y_1(r, t, n) P_n(\cos\theta) \tag{13a}$$

$$u_\theta(r, \theta, t, n) = y_3(r, t, n) \frac{\partial P_n}{\partial \theta}(\cos\theta) \tag{13b}$$

$$\sigma_{rr}(r, \theta, t, n) = y_2(r, t, n) P_n(\cos\theta) \tag{13c}$$

$$\sigma_{r\theta}(r, \theta, t, n) = y_4(r, t, n) \frac{\partial P_n}{\partial \theta}(\cos\theta). \tag{13d}$$

Thus the functions y_i, where $i = 1, 2, 3, 4$, represent the r-dependent parts of displacement and stress in axially symmetric spherical polar coordinates. Note that these '*AJP* variables', as we shall call the y_i, are precisely those which are continuous at a welded contact between two concentric spherical shells.

Substituting (7) into (12), we see that

$$y_2 = r^n A_1 * dH_1 + r^{-n-1} A_2 * dH_2 + r^{n-2} A_3 * dH_3 + r^{-n-3} A_4 * dH_4 \tag{14a}$$

$$y_4 = r^n A_1 * dK_1 + r^{-n-1} A_2 * dK_2 + r^{n-2} A_3 * dK_3 + r^{-n-3} A_4 * dK_4 \tag{14b}$$

where

$$H_1 = (n+1)G_1 - 2(2n+3)G_1 * d[(n-6)G_1 + 2nG_2]^{-1} * d[G_2 - G_1] \quad (15a)$$

$$H_2 = -nG_1 - 2(2n-1)G_1 * d[(n+7)G_1 + 2(n+1)G_2]^{-1} * d[G_2 - G_1] \quad (15b)$$

$$(n+1)K_1 = (n+2)G_1 + 3(2n+3)G_1 * dG_1 * d[(n-6)G_1 + 2nG_2]^{-1} \quad (15c)$$

$$nK_2 = (n-1)G_1 - 3(2n-1)G_1 * dG_1 * d[(n+7)G_1 + 2(n+1)G_2]^{-1} \quad (15d)$$

$$H_3 = (n-1)G_1 \quad H_4 = -(n+2)G_1 \quad (15e, f)$$

$$nK_3 = (n-1)G_1 \quad (n+1)K_4 = (n+2)G_1. \quad (15g, h)$$

Note that N may be derived from M, H_2 from H_1, K_2 from K_1, H_4 from H_3, and K_4 from K_3 by substituting $-n-1$ for n.

The solution given in equations (7) and (14) is sufficient in case stress relaxation functions $G_1(t)$ and $G_2(t)$ are known for the material in question. In certain situations, however, it may be more convenient to use creep curves $J_1(t)$ and $J_2(t)$ instead. Analytically, it can be shown (GURTIN and STERNBERG [3]) that

$$J_1(t) = G_1^{-1}(t) \quad (16a)$$

and

$$J_2(t) = G_2^{-1}(t) \quad (16b)$$

where the inverse is taken in the convolution sense of (10a). Physically, $J_1(t)$ represents shear creep upon application of a Heaviside unit shear stress, and $J_2(t)$ represents dilitational creep upon application of a Heaviside unit hydrostatic stress.

In terms of creep functions J_1 and J_2 we have that

$$(n+1)M = [(n+9)J_2 + 2(n+3)J_1] * d[2nJ_1 + (n-6)J_2]^{-1} \quad (17a)$$

$$-nN = [(n-8)J_2 + 2(n-2)J_1] * d[2(n+1)J_1 + (n+7)J_2]^{-1} \quad (17b)$$

whence

$$(n-6)H_1 = n(n-1)J_1^{-1} - 6(2n+3)(n-2)[(n-6)J_2 + 2nJ_1]^{-1} \quad (18a)$$

$$(n+7)H_2 = -(n+1)(n+2)J_1^{-1} - 6(2n-1)(n+3)[(n+7)J_2 + 2(n+1)J_1]^{-1} \quad (18b)$$

$$(n+1)(n-6)K_1 = (n-1)(n+3)J_1^{-1} - 6n(2n+3)[(n-6)J_2 + 2nJ_1]^{-1} \quad (18c)$$

$$n(n+7)K_2 = (n^2-4)J_1^{-1} + 6(n+1)(2n-1)[(n+7)J_2 + 2(n+1)J_1]^{-1}. \quad (18d)$$

In deriving equations (17) and (18), we have used the following properties of Stieltjes convolutions:

$$A * dB * d(aA + bB)^{-1} = (bA^{-1} + aB^{-1})^{-1} \quad (19a)$$

and

$$A * dB^{-1} * d(aA + bB)^{-1} = \frac{1}{a}B^{-1} - \frac{b}{a}(aA + bB)^{-1}. \quad (19b)$$

Here A and B represent arbitrary time functions, while a and b represent scalars (independent of time). These properties are readily derived (FUNG [2], p. 414) from the group property of the Stieltjes operator '$*d$'.

For the mode $n = 6$ equations (18a) and (18c) do not hold since in their derivation we have divided by a quantity which vanishes for that mode. Reverting to equations (15a) and (15c) we see that for this case

$$H_1|_{n=6} = \tfrac{9}{2} J_1^{-1} + \tfrac{5}{2} J_1^{-1} * dJ_1^{-1} * dJ_2 \tag{20a}$$

$$K_1|_{n=6} = \tfrac{8}{7} J_1^{-1} + \tfrac{15}{28} J_1^{-1} * dJ_1^{-1} * dJ_2. \tag{20b}$$

Example: Solid sphere surface loaded by P_n stress

As a simple example using the above results we calculate the deformation of a solid linear viscoelastic sphere loaded at its surface by a stress system consisting of a single Legendre mode. Any given axial stress system can, of course, be built up by superposing such modes in the appropriate fashion, and, since the model is linear, the total deformation due to it will be the corresponding superposition of deformation modes. We will treat the cases of pure radial and pure shear stresses separately, since by the same reasoning the two results may be superposed to give the general case.

The finiteness condition at the center of the solid sphere gives

$$A_2 = A_4 = 0. \tag{21}$$

We first consider the case of pure radial stress. The boundary conditions at the surface of the sphere $r = a$ are approximately

$$\sigma_{rr}|_a = \beta_n P_n(\cos\theta) \tag{22a}$$

$$\sigma_{r\theta}|_a = 0. \tag{22b}$$

(These conditions are approximate since as deformation proceeds, the sphere's surface departs from the surface $r = a$ where the stress system (22) is taken to apply. Thus we are neglecting u_r $(r = a)$ in comparison with a. The assumption seems in line, given that the formulation of equations (1) has used an infinitesimal strain measure only.)

Using (21) and (22) in (14), we have

$$\beta_n = a^n A_1 * dH_1 + a^{n-2} A_3 * dH_3 \tag{23a}$$

$$0 = a^n A_1 * dK_1 + a^{n-2} A_3 * dK_3 \tag{23b}$$

whence

$$A_1 = a^{-n} \beta_n * d[H_1 - K_1 * dK_3^{-1} * dH_3]^{-1} \tag{24a}$$

$$A_3 = -a^{-n+2} \beta_n * d[H_1 - K_1 * dK_3^{-1} * dH_3]^{-1} * dK_1 * dK_3^{-1}. \tag{24b}$$

For the case of pure tangential stress we take the boundary conditions at the surface of the sphere to be

$$\sigma_{rr}|_a = 0 \tag{25a}$$

$$\sigma_{r\theta}|_a = \beta_n' \frac{\partial P_n(\cos\theta)}{\partial\theta}. \tag{25b}$$

Then

$$0 = a^n A_1' * dH_1 + a^{n-2} A_3' * dH_3 \tag{26a}$$

$$\beta_n' = a^n A_1' * dK_1 + a^{n-2} A_3' * dK_3 \tag{26b}$$

whence

$$A_1' = a^{-n} \beta_n' * d[K_1 - H_1 * dH_3^{-1} * dK_3]^{-1} \tag{27a}$$

$$A_3' = a^{-n+2} \beta_n' * d[K_1 - H_1 * dH_3^{-1} * dK_3]^{-1} * dH_1 * dH_3^{-1}. \tag{27b}$$

The primes here indicate pure tangential loading. Note the interchange of H's and K's between equations (24) and (27).

The above solutions include the possibility that β_n or β_n' may be a function of time. The displacement of the sphere's surface, $r = a$, under a P_n radial stress, for example, is given by substitution of (24) into (8)

$$U_n = a\beta_n * d[H_1 - K_1 * dK_3^{-1} * dH_3]^{-1} * d[h - K_1 * dK_3^{-1}] \tag{28a}$$

$$V_n = a\beta_n * d[H_1 - K_1 * dK_3^{-1} * dH_3]^{-1} * d\left[M - \frac{1}{n} K_1 * dK_3^{-1}\right]. \tag{28b}$$

Thus even in this simple problem the loading time function, β_n, convolves with some rather complicated time functions in order to give the time history of surface deformation.

The results of equation (28) are listed formally in terms of H_i and K_i for reasons to appear later. For the problem at hand, we see that by (15)

$$K_3^{-1} * dH_3 = nh(t) \tag{29a}$$

$$H_3^{-1} * dK_3 = h(t)/n. \tag{29b}$$

Using (29), (18) and (19) in (24) we find that

$$A_1 = -\frac{(n+1)}{2} a^{-n} \beta_n * dJ_1 * dL^{-1} * d[2nJ_1 + (n-6)J_2] \tag{30a}$$

$$A_3 = \frac{n}{2} \frac{a^{-n+2}}{(n-1)} \beta_n * dJ_1 * dL^{-1} * d[2n(n+2)J_1 + (n+3)(n-1)J_2] \tag{30b}$$

where

$$L = (2n^2 + 4n + 3)J_1 + n(n-1)J_2 \tag{30c}$$

a similar result holds for A_1' and A_3'. Specifically

$$A_1' = \frac{n(n+1)}{2} a^{-n} \beta' * dJ_1 * dL^{-1} * [2nJ_1 + (n-6)J_2] \tag{30d}$$

$$A_3' = \frac{n(n+1)}{2(n-1)} a^{-n+2} \beta_n' * dJ_1 * dL^{-1} * d[2(n^2 - n - 3)J_1 + n(n-1)J_2]. \tag{30e}$$

The above solution fails for the Legendre mode $n = 1$ because this mode represents the stress system which gives the sphere a rigid translation without deformation.

We remark that the solution for a P_n stress applied to the surface of a spherical hole in an infinite viscoelastic medium is obtained by substituting $-n-1$ for n in equations (30). The mode $n = 1$ remains valid for this latter problem.

5. *Possible simplifying assumptions*

In some situations the effect of shear creep may be of much more interest than that of compressional creep. The impenetrability of matter, for example, assures that any possible $J_2(t)$ functions must be bounded, often within limits which can be determined empirically. Meanwhile the $J_1(t)$ curves may be unbounded, or at any rate dominant over $J_2(t)$ with respect to their total variations. In such situations we may be justified in supposing our material to be essentially elastic in compression, but viscoelastic in shear. That is, in the foregoing equations we substitute

$$J_2(t) = h(t)/3k \quad \text{or} \quad G_2(t) = 3kh(t) \tag{31}$$

where k is some equivalent elastic incompressibility.

Under assumption (31), the equations (28) for the displacement of the surface of a P_n normally loaded solid sphere becomes

$$U_n = a\beta_n * dJ_1 * d\left[(2n^2 + 4n + 3)J_1 + n(n-1)\frac{1}{3k}\right]^{-1}$$
$$* \, d\{n(2n+1)J_1/(n-1) + (4n+3)/3k\} \tag{32a}$$

$$V_n = 3a\,\beta_n * dJ_1 * d\left[(2n^2 + 4n + 3)J_1 + n(n-1)\frac{1}{3k}\right]^{-1} * d\{(J_1/(n-1) - J_2\}. \tag{32b}$$

As a final simplifying assumption we suppose the material's viscoelastic shear behavior is that of a 'standard linear solid'. Such a solid (FUNG [2], p. 23) is characterized by creep and stress relaxation curves of the following simple exponential type:

$$J_1(t) = \left[1 - \left(1 - \frac{\tau_G}{\tau_J}\right) e^{-t/\tau_J}\right] \Big/ 2\mu \tag{33a}$$

$$J_1^{-1}(t) = 2\mu \left[1 - \left(1 - \frac{\tau_J}{\tau_G}\right) e^{-t/\tau_G}\right]. \tag{33b}$$

The standard linear solid has the particular virtue that both J_1 and J_1^{-1} take an elementary form, a characteristic by no means true of certain other functions sometimes used to fit empirical creep and stress-relaxation data. It has only three adjustable parameters, the 'elastic rigidity' μ, and the creep and stress-relaxation time constants τ_J and τ_G. Various physical restrictions limit the ratio τ_G/τ_J to the range between zero and unity.

In general, using (10b) one may show that the inverse of a given function

$$J = a + b\, e^{-t/\tau} \tag{34a}$$

is the function

$$J^{-1} = \frac{1}{a} + \left(\frac{1}{(a+b)} - \frac{1}{a}\right) e^{-t/\tau'} \tag{34b}$$

where

$$\tau' = \frac{(a+b)\,\tau}{a}. \tag{34c}$$

It is also true that given two functions

$$A = a_1 + a_2\, e^{-t/\tau_a} \tag{35a}$$

$$B = b_1 + b_2\, e^{-t/\tau_b} \tag{35b}$$

their convolution is given by

$$A * dB = a_1 b_1 + \left(a_2 b_1 + \frac{a_2 b_2 \tau_b}{(\tau_b - \tau_a)}\right) e^{-t/\tau_a} + \left(a_1 b_2 + \frac{a_2 b_2 \tau_a}{(\tau_a - \tau_b)}\right) e^{-t/\tau_b} \tag{35c}$$

if $\tau_a \neq \tau_b$

$$= a_1 b_1 + \left(a_1 b_2 + a_2 b_1 + a_2 b_2 - a_2 b_2 \frac{t}{\tau}\right) e^{-t/\tau} \tag{35d}$$

if $\tau_a = \tau_b = \tau$.

Using (33), (34), (35) and (19) on equations (32), we arrive at the following result:

$$U_n = \frac{a}{2}\beta_n * d\left[a_1 b_1 + \left(a_2 b_1 + \frac{a_2 b_2 \tau_b}{(\tau_b - \tau_a)}\right)e^{-t/\tau_a} + \left(a_1 b_2 + \frac{b_2 a_2 \tau_a}{(\tau_a - \tau_b)}\right)e^{-t/\tau_b}\right] \quad (36a)$$

$$V_n = \frac{a}{2}\beta_n * d\left[a_1 c_1 + \left(a_2 c_2 + \frac{a_2 c_2 \tau_c}{(\tau_c - \tau_a)}\right)e^{-t/\tau_a} + \left(a_1 c_2 + \frac{c_2 a_2 \tau_a}{(\tau_a - \tau_c)}\right)e^{-t/\tau_c}\right] \quad (36b)$$

where

$$a_1 = 3k/[3k(2n^2 + 4n + 3) + 2\mu n(n-1)] \quad (36c)$$

$$a_2 = 3k\tau_G/[3k(2n^2 + 4n + 3)\tau_G + 2\mu n(n-1)\tau_J] \quad (36d)$$

$$b_1 = n(2n+1)/(n-1)\mu + (4n+3)/3k \quad (36e)$$

$$b_2 = n(2n+1)(\tau_J/\tau_G - 1)/(n-1)2\mu \quad (36f)$$

$$c_1 = 1/(n-1)2\mu - 1/3k \quad (36g)$$

$$c_2 = (\tau_J/\tau_G - 1)/(n-1)2\mu \quad (36h)$$

$$\tau_a = \frac{3k(2n^2 + 4n + 3)\tau_G + 2\mu n(n-1)\tau_J}{3k(2n^2 + 4n + 3) + 2\mu n(n-1)} = \tau_n \quad (36i)$$

$$\tau_b = \tau_c = \tau_J. \quad (36j)$$

An interesting feature of this solution is that two characteristic time constants occur. The first, as expected, is τ_J, the time constant of the material creep curve. The second, τ_n, is seen to depend on mode number n. Thus the general loading program, involving all Legendre orders n, will generate an infinite number of creep time constants τ_n. As can be seen from Table 1, the time constants so generated are a slowly monotonically increasing function of n and are larger than but roughly of the order of τ_G in magnitude.

In point of fact, of course, the time constants of the actual deformation are those generated by convolving $\beta_n(t)$ with e^{-t/τ_J} and e^{-t/τ_n}. The deformation itself has time constants τ_J and τ_n only if some linear term of $\beta_n(t)$ goes as $h(t)$. We see that according to (35d) a 'resonance' is possible with (exponentially damped) deformation linearly proportional to time when $\beta_n(t)$ itself goes as e^{-t/τ_J} or e^{-t/τ_n}. The reader is reminded that this result is approximate, however, since the boundary conditions we have used do not follow the deforming surface, but are taken to hold only at $r = a$.

6. *Formulation of solution for layered sphere*

We now wish to treat the deformation under a given program of surface stresses of a layered isotropic linear viscoelastic sphere. Such a sphere is assumed to be built up of concentric isotropic linear viscoelastic shells, welded at their interfaces. Each shell has its own characteristic compliance functions $J_1(t)$ and $J_2(t)$, presumably differing from those of neighboring shells.

Table 1

τ_n/τ_J versus n. The following values of τ_n/τ_J were calculated from equation (36i) assuming $3k/2\mu = 5/2$ (i.e., that Poisson's ratio $\nu = 1/4$)

n	$\tau_G/\tau_J = 0.5$	0.1	0.01
0	0.500	0.100	0.010
1	0.500	0.100	0.010
2	0.520	0.136	0.050
3	0.534	0.161	0.077
4	0.543	0.177	0.095
5	0.549	0.189	0.108
6	0.554	0.197	0.117
8	0.560	0.209	0.130
10	0.565	0.216	0.138
12	0.567	0.221	0.143
15	0.570	0.227	0.149
20	0.573	0.232	0.155
25	0.575	0.236	0.159
30	0.577	0.238	0.162
40	0.578	0.241	0.165
50	0.579	0.243	0.167
75	0.581	0.245	0.170
100	0.581	0.246	0.171
150	0.582	0.248	0.172
200	0.582	0.248	0.173
500	0.583	0.249	0.174
1000	0.583	0.250	0.175

We re-write the general solution (7) and (14) in the following subscripted form:

$$y_i(r) = T_{ij}(r) * dA_j \quad (i,j = 1,2,3,4) \tag{37a}$$

where

$$T_{ij} = \begin{Bmatrix} r^{n+1} & r^{-n} & r^{n-1} & r^{-n-2} \\ H_1 r^n & H_2 r^{-n-1} & H_3 r^{n-2} & H_4 r^{-n-3} \\ M r^{n+1} & N r^{-n} & \dfrac{1}{n} r^{n-1} & -\dfrac{1}{n+1} r^{-n-2} \\ K_1 r^n & K_2 r^{-n-1} & \dfrac{1}{n} H_3 r^{n-2} & -\dfrac{1}{n+1} H_4 r^{-n-3} \end{Bmatrix} \tag{37b}$$

In (37) the variables y_i, T_{ij}, K_i, H_i, M, N, and A_j are understood to depend on time t and Legendre order n, but the dependence on radius r is written out explicitly.

Note that the A_i are independent of r. In fact we may write

$$A_k = T_{ki}^{-1}(r_2) * dy_i(r_2) \tag{38}$$

where r_2 is some conveniently chosen value of radius r, and the 'inverse' of T_{kl} is taken in both the matrix algebra and the convolution sense. In other words, we define T_{kl}^{-1} by

$$T_{kl}^{-1}(r_2) * dT_{lj}(r_2) = h(t)\,\delta_{kj} \tag{39}$$

where δ_{kj} is the Kronecker delta. Under this definition (38) is seen to be the result of operating on (37a) with $T_{kl}^{-1} * d$.

Referring (37a) to radius r_1 and eliminating A_k by (38) gives

$$y_i(r_1) = T_{ik}(r_1) * dT_{kl}^{-1}(r_2) * dy_l(r_2). \tag{40}$$

Thus the '*AJP* vector' y_i at level $r = r_1$ may be derived from that at level $r = r_2$ by convolving with the matrix

$$\Pi_{il}(r_1, r_2) = T_{ik}(r_1) * dT_{kl}^{-1}(r_2). \tag{41}$$

We now let r_1 be the coordinate of the top of shell number I and r_2 be the coordinate of its bottom. The matrix Π_{il} then represents a layer matrix of the HASKELL [4] type, translating the *AJP* vector through the layer. But, by the comment after equations (13), the *AJP* vector is continuous at a welded boundary. Hence an iteration scheme is established which reduces the layered sphere to the unlayered case we have already considered.

To illustrate, consider a two-layered sphere having outer radius $r = a$ and an inner welded interface at $r = b$. By (40)

$$y_i(a) = \Pi_{im}(a, b) * dy_m(b). \tag{42}$$

Substituting for $y_m(b)$ by (37) we have

$$y_i(a) = \Pi_{im}(a, b) * dT_{mj}(b) * dA_j \tag{43}$$

where $\Pi_{im}\,(a, b)$ is evaluated with respect to the parameters of the outer shell (layer 1) and $T_{mj}(b)$ with respect to those of the inner sphere (layer 2). In case the two layers have identical parameters (no interface), (43) clearly reverts to (37). Otherwise, (43) is the form:

$$y_i(a) = S_{ij}(a, b) * dA_j \tag{44a}$$

where

$$S_{ij}(a, b) = \Pi_{im}(a, b) * dT_{mj}(h). \tag{44b}$$

For the inner core (21) continues to hold, so that solution (44) is left with only two free constants, A_1 and A_3; these are determined as usual by the surface boundary conditions.

The composite layer matrix S_{ij} has thus replaced the matrix T_{ij} of the solid sphere example, and equations (24) through (27) for A_i and A_i' continue to hold. It is only necessary to make the substitutions:

$$H_i \leftarrow S_{2i}\, a^{i-n-1} \tag{45a}$$

$$K_i \leftarrow S_{4i}\, a^{-n-1}. \tag{45b}$$

Equations (28), for example, are then replaced by:

$$U_n = \beta_n * d[S_{11} * dS_{43} - S_{13} * dS_{41}] * d[S_{21} * dS_{43} - S_{23} * dS_{41}]^{-1} \qquad (46a)$$

$$V_n = \beta_n * d[S_{31} * dS_{43} - S_{33} * dS_{41}] * d[S_{21} * dS_{43} - S_{23} * dS_{41}]^{-1} \qquad (46b)$$

where the inverse sign now indicates Stietjes convolution inverse only.

The generalization to an N-layered sphere is apparent. We comment that the deformation under prescribed surface stresses of a multiply cased spherical hole in an infinite viscoelastic medium is given by substituting $-n-1$ for n in the above solution.

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