Propagation of Love Waves in a Non-Homogeneous Stratum of Finite Depth Sandwitched Between Two Semi-Infinite Isotropic Media

By Nirmal Kumar Sinha [1])

Summary – The possibility of propagation of Love-waves in a non-homogeneous internal stratum internal stratum of finite depth sandwitched between two semi-infinite isotropic media has been studied in this paper. The density and rigidity both are considered to be variable, density being taken as

$$\varrho_2 = \frac{\varrho_0}{(1 + \alpha z)} \quad \text{and rigidity} \quad \mu_2 = \frac{\mu_0}{(1 + \alpha z)},$$

ϱ_0 and μ_0 are constants. The velocity $\beta = c_2$ is taken here as constant $= \sqrt{\mu_2/\varrho_2} = \sqrt{\mu_0/\varrho_0}$. It is found that the phase velocity c exists between the limit $c_2 < c \leqq (1.121\ c_2)$. The variability of KH (where K, the wave number and H, the depth of the internal medium) with the change of c/c_2 and C/c_2 (where C is the group velocity) is shown graphically.

Introduction

Dutta [1] [2]) showed that the Love-type of wave in a non-homogeneous internal stratum of finite depth lying between two semi-infinite isotropic media exists for a sufficiently large value of KH (where K, the wave number and H, the depth of sandwitched layer) in three different cases:

(i) when $\quad \mu_2 \propto \cosh^2 \frac{z}{\lambda}, \quad \varrho_2 \propto \cosh^2 \frac{z}{\lambda}$

(ii) when $\quad \mu_2 \propto \left(1 + \frac{z}{\lambda}\right)^2, \quad \varrho_2 \propto \left(1 + \frac{z}{\lambda}\right)^2$

(iii) when $\quad \mu_2 \propto e^{mz}, \quad \varrho_2 \propto e^{mz}.$

In the previous paper author [2] discussed the problem regarding Love-type of wave in a non-homogeneous intermediate layer lying between two semi-infinite media, when the rigidity of the medium varies as e^{2pz}, density being constant, distortional

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[2]) Numbers in brackets refer to References, page 70.

velocity also varies as e^{pz}. He showed that for $c > c_1 < c_3$ Love-type wave is possible, where c_1 and c_3 are the distortional wave velocity in the upper and lower media respectively.

DUTTA [3] studied the problem of propagation of Love-waves in a non-homogeneous thin layer lying over a semi-infinite medium. In that paper he considered the rigidity $\mu_2 = \mu_0/(1+\alpha z)$ and density $\varrho_2 = \varrho_0/(1+\alpha z)$, μ_0 and ϱ_0 are constants and showed that the existence of Love-type of wave in possible.

In this paper taking the variation of rigidity and density same i.e. $\mu_2 = \mu_0/(1+\alpha z)$, $\varrho_2 = \varrho_0/(1+\alpha z)$ (where μ_0 and ϱ_0 are constants) a problem is studied in the case of a non-homogeneous stratum sandwitched between two semi-infinite media.

Mathematical derivation

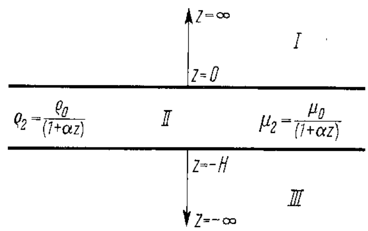

Figure 1

Let us assume that the medium I is extended from $z=0$ to $z=\infty$, the medium II is extended from $z=0$ to $z=-H$ and the medium III is extended from $z=-H$ to $z=-\infty$ (Figure 1).

The component of displacement (u, v, w) in a plane wave travelling in the direction x increasing in any medium may be assumed to be the real part of $(0, V, 0)\, e^{ik(x-ct)}$, where V is a function of z only. For Love-wave we consider $u=w=0$.

The Equation of motion is (EWING, JARDETZKY and PRESS [4])

$$\varrho \frac{\delta^2 v}{\delta t^2} = \frac{\delta p_{xy}}{\delta x} + \frac{\delta p_{yz}}{\delta z}. \tag{1}$$

for Love wave we assume that all displacements are independent of y-coordinates.

The non-zero stress-strain relations are

$$p_{xy} = \mu \frac{\delta v}{\delta x} \quad \text{and} \quad p_{yz} = \mu \frac{\delta v}{\delta z}. \tag{2}$$

(EWING, JARDETZKY and PRESS [4]).

Let v_j $(j=1, 3)$ be the displacement components in medium I and III respectively. Putting (2) in (1) we have,

$$\frac{\delta}{\delta x}\left(\frac{\delta v_j}{\delta x}\right) + \frac{\delta}{\delta z}\left(\frac{\delta v_j}{\delta z}\right) = \frac{\varrho_j}{\mu_j} \frac{\delta^2 v_j}{\delta t^2}, \quad (j = 1, 3) \tag{3}$$

where (μ_j, ϱ_j), $(j=1, 3)$ are the constant rigidity and density of medium I and III respectively.

Let
$$v_j = V_j \, e^{ik(x-ct)}, \quad (j = 1, 3) \tag{4}$$
where V_j, $(j=1, 3)$ are functions of z only, the equation (3) reduces to
$$\frac{d^2 V_1}{dz^2} - s_1^2 \, V_1 = 0 \tag{5}$$
$$\frac{d^2 V_3}{dz^2} - s_3^2 \, V_3 = 0 \tag{6}$$
where
$$s_1 = k\left\{1 - \frac{c^2}{c_1^2}\right\}^{1/2} \quad \text{and} \quad s_3 = k\left\{1 - \frac{c^2}{c_3^2}\right\}^{1/2}. \tag{7}$$

The solutions of the equations (5) and (6) suitable for the problem are
$$V_1 = C_1 \, e^{-s_1 z} \quad \text{and} \quad V_3 = D_1 \, e^{s_3 z} \tag{8}$$
where C_1 and D_1 are constants.

The equation of motion in the second medium is
$$\frac{\delta}{\delta x}\left(\mu_2 \frac{\delta v_2}{\delta x}\right) + \frac{\delta}{\delta z}\left(\mu_2 \frac{\delta v_2}{\delta z}\right) = \varrho_2 \frac{\delta^2 v_2}{\delta t^2} \tag{9}$$
where v_2 is the displacement component in the second medium, ϱ_2, μ_2 the density and the rigidity of the material of the second medium, being function of z only and c_2 is constant.

Equation (9) can be written as
$$\mu_2 \frac{\delta^2 v_2}{\partial x^2} + \mu_2 \frac{\delta^2 v_2}{\delta z^2} + \frac{\delta \mu_2}{\delta z} \cdot \frac{\delta v_2}{\delta z} = \varrho_2 \frac{\delta^2 v_2}{\delta t^2}. \tag{10}$$

Putting $v^2 = V_2 \, ^{ik(x-ct)}$ in (10)
$$\frac{d^2 V_2}{dz^2} + \frac{1}{\mu_2} \cdot \frac{d\mu_2}{dz} \cdot \frac{dV_2}{dz} - k^2\left(1 - \frac{\varrho_2 \, c^2}{\mu_2}\right) V_2 = 0. \tag{11}$$
Again putting
$$\mu_2 = \frac{\mu_0}{(1 + \alpha z)}, \quad \varrho_2 = \frac{\varrho_0}{(1 + \alpha z)} \quad \text{and} \quad V_2 = \frac{V^1 (1 + \alpha z)^{1/2}}{\sqrt{\mu_0}}$$
for the medium II, the equation (11) reduces to
$$\frac{d^2 V^1}{dz^2} + \left[k^2\left(\frac{c^2}{c_2^2} - 1\right) - \frac{3\,\alpha^2}{4(1 + \alpha z)}\right] V^1 = 0. \tag{12}$$
Putting $1 + \alpha z = \xi$, (12) can be reduced to
$$\frac{d^2 V^1}{d\xi^2} + \left[\frac{k^2}{\alpha^2}\left(\frac{c^2}{c_2^2} - 1\right) - \frac{3}{4\,\xi^2}\right] V^1 = 0.$$

Hence the solution is

$$V^1 = A^1 \sqrt{\xi}\, J_1(r\,\xi) + B^1 \sqrt{\xi}\, Y_1(r\,\xi)$$

where

$$r = \frac{k}{\alpha}\left(\frac{c^2}{c_2^2} - 1\right)^{1/2} \tag{13}$$

substituting the value of V^1 in (13) we have

$$\frac{\sqrt{\mu_0}}{(1+\alpha z)^{1/2}}\, V_2 = A^1(1+\alpha z)^{1/2} J_1\{r(1+\alpha z)\} + B^1(1+\alpha z)^{1/2}\, Y_1\{r(1+\alpha z)\}$$

$$V_2 = A(1+\alpha z)\, J_1\{r(1+\alpha z)\} + B(1+\alpha z)\, Y_1\{r(1+\alpha z)\}$$

where

$$A = \frac{A^1}{\sqrt{\mu_0}} \quad \text{and} \quad B = \frac{B^1}{\sqrt{\mu_0}}, \tag{14}$$

A^1, B^1 being constants.

The boundary conditions are

$$\left.\begin{aligned} V_2 &= V_1 \quad \text{at} \quad z = 0 \\ V_2 &= V_3 \quad \text{at} \quad z = -H \\ \mu_1 \frac{dV_1}{dz} &= \mu_2 \frac{dV_2}{dz} \quad \text{at} \quad z = 0 \\ \mu_2 \frac{dV_2}{dz} &= \mu_3 \frac{dV_3}{dz} \quad \text{at} \quad z = -H. \end{aligned}\right\} \tag{15}$$

From the boundary conditions, we have,

$$A\, J_1(r) + B\, Y_1(r) = C_1$$

$$(1-\alpha H)\,[A\, J_1\{r(1-\alpha H)\} + B\, Y_1\{r(1-\alpha H)\}] = D_1\, e^{-s_3 H}$$

$$\mu_0\, r\, \alpha[A\, J_0(r) + B\, Y_0(r)] = -\mu_1\, s_1\, C_1$$

and

$$\mu_0\, r\, \alpha[A\, J_0\{r(1-\alpha H)] + B\, Y_0\{r(1-\alpha H)\}] = \mu_3\, s_3\, D_1\, e^{-s_3 H}.$$

Eliminating A, B, C_1 and D_1 from the above equations and putting the value of r, we get the frequency equation

$$\frac{\mu_0\left(\frac{c^2}{c_2^2}-1\right)^{1/2} J_0\left\{\frac{k}{\alpha}\left(\frac{c^2}{c_2^2}-1\right)^{1/2}\right\} + \mu_1\left(1-\frac{c^2}{c_1^2}\right)^{1/2} J_1\left\{\frac{k}{\alpha}\left(\frac{c^2}{c_2^2}-1\right)^{1/2}\right\}}{\mu_0\left(\frac{c^2}{c_2^2}-1\right)^{1/2} J_0\left\{\frac{k}{\alpha}\left(\frac{c^2}{c_2^2}-1\right)^{1/2}(1-\alpha H) - \mu_3(1-\alpha H)\left(1-\frac{c^2}{c_3^2}\right)^{1/2} J_1\left\{\frac{k}{\alpha}\left(\frac{c^2}{c_2^2}-1\right)^{1/2}(1-\alpha H)\right\}}$$

$$= \frac{\mu_0\left(\frac{c^2}{c_2^2}-1\right)^{1/2} Y_0\left\{\frac{k}{\alpha}\left(\frac{c^2}{c_2^2}-1\right)^{1/2}\right\} + \mu_1\left(1-\frac{c^2}{c_1^2}\right)^{1/2} Y_1\left\{\frac{k}{\alpha}\left(\frac{c^2}{c_2^2}-1\right)^{1/2}\right\}}{\mu_0\left(\frac{c^2}{c_2^2}-1\right)^{1/2} Y_0\left\{\frac{k}{\alpha}\left(\frac{c^2}{c_2^2}-1\right)^{1/2}(1-\alpha H)\right\} - \mu_3(1-\alpha H)\left(1-\frac{c^2}{c_3^2}\right)^{1/2} Y_1\left\{\frac{k}{\alpha}\left(\frac{c^2}{c_2^2}-1\right)^{1/2}(1-\alpha H)\right\}}$$

The numerical calculations are performed by the help of 'The Table of Functions' (JAHNKE and EMDE [5]) and we take the following particular values

$$c_1 = 3.363\,, \quad c_3 = 4.362\,, \quad c_2 = 3.00 \quad \alpha H = .5$$
$$\mu_1 = 2.997\,, \quad \mu_0 = 2.385\,, \quad \mu_3 = 6.469$$

and plot the graph of KH against c/c_2 and C/c_2.

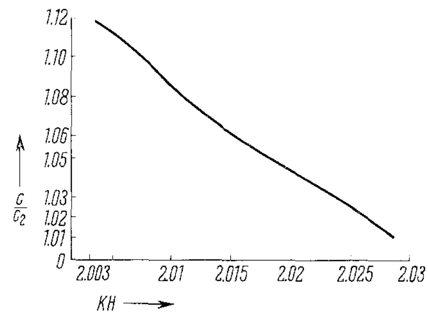

Figure 2

Table 1

$\frac{c}{c_2}$	1.01	1.02	1.04	1.06	1.08	1.12	1.121
KH	2.029	2.027	2.0212	2.0162	2.0117	2.00374	2.003738

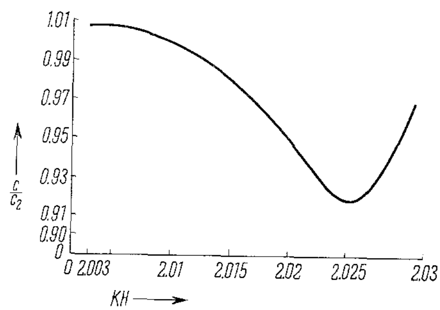

Figure 2

Table 2

$\frac{c}{c_2}$	1.01	1.02	1.04	1.06	1.08	1.12
$\frac{C}{c_2}$	.9676	.9254	.9407	.9718	.9921	1.072
KH	2.029	2.027	2.0212	2.0162	2.0117	2.00374

Acknowledgement

Finally I express my gratitude to Dr. S. DUTTA of Bangabasi College, Calcutta, for his guidance and encouragement at every stage of this work.

REFERENCES

[1] (i) S. DUTTA, Geophys. *XXVIII*, 2 (1963), 156–160.
(ii) S. DUTTA, Geofis. pura e appl., Milano *55* (1963/II).
[2] N. SINHA, In the press, Pure and Appl. Geophys.
[3] S. DUTTA, Geophys. J. Roy. Astron. Soc. *8* (1963), 231.
[4] M. EWING, W. JARDETZKY and F. PRESS, *Elastic Waves in Layered Media* (McGraw-Hill, New York 1957).
[5] JAHNKE and EMDE, *The table of Functions with Formulae and Curves* (Dover publication).

(Received 13th October 1966)