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The Perturbation Method in Elastic Wave Scattering

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Abstract—Methods of theoretical study in seismic wave scattering are reviewed with the emphasis on the perturbation method. Detailed analysis for weak scattering using Born approximation is given. For elastic random media, the mean square amplitudes of scattered waves are derived using a new approach by working directly in the spectrum domain. The conditions for the scalar wave approximation are obtained. The problem of sensitivity of fore- and backscattering to heterogeneities with different scales and properties (velocity or impedance) is discussed.

Key words: Scattering, elastic wave, perturbation method, heterogeneity.

1. Formulations of Elastic Wave Scattering Problems and the Solutions

In a source-free linear heterogeneous elastic medium, the equation of motion for displacement **u** is (see AKI and RICHARDS, 1980)

$$\rho \ddot{u}_i = \tau_{ij,j} = (c_{ijpq} u_{p,q})_{,j} \tag{1}$$

where ρ is density, u_i and τ_{ij} are the components of the displacement vector and the stress tensor respectively, "$\cdot$" stands for $\partial/\partial t$, "$_{,j}$" for $\partial/\partial x_j$, and repeated index implies the summations with respect to the index over the spatial dimensions, c_{ijpq} are the components of the tensor of elastic constant **c** of the medium, which are functions of the position.

In the case of isotropic heterogeneous media, (1) can be written as

$$\rho(\mathbf{x})\ddot{u}_i = (\lambda(\mathbf{x})u_{j,j})_{,i} + [\mu(\mathbf{x})(u_{i,j} + u_{j,i})]_{,j} \tag{2}$$

where λ and μ are the Lamé constants of the medium. All parameters ρ, λ, μ change with position **x**. Heterogeneities in the medium can be treated as two different classes: continuous and discontinuous. Discontinuous heterogeneities are inclusions in the medium. Both inside and outside the inclusions in the media are homogeneous. Sharp discontinuities occur on the boundaries. The scattering problem for a single

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inclusion can be formulated as a boundary value problem (either a system of partial differential equations with boundary conditions or a system of boundary integral equations). The problem for a complex medium with multiple inclusions can be attacked by a multiple scattering theory based on the solution of the single inclusion scattering problem. Another approach is the perturbation method, which can be used for both discontinuous (weak heterogeneities) and continuous media and is more often used in seismology. We first discuss briefly the boundary matching approach and then concentrate on the perturbation method.

2. Boundary Matching Approach

Outside the inclusion, the medium is homogeneous. Therefore (2) becomes

$$\rho_0 \ddot{u}_i - (\lambda_0 + \mu_0)(u_{j,j})_{,i} - \mu_0 u_{i,jj} = 0. \tag{3}$$

Equation (3) can be written as a vector equation after some simple tensor algebra,

$$\rho_0 \ddot{\mathbf{u}} - (\lambda_0 + 2\mu_0)\nabla(\nabla \cdot \mathbf{u}) + \mu_0 \nabla \times (\nabla \times \mathbf{u}) = 0, \tag{4}$$

where "∇", "$\nabla \cdot$" and "$\nabla \times$" are the gradient, divergence and curl operators respectively, and ρ_0, λ_0 and μ_0 are the medium parameters outside the inclusion. The interior field $\mathbf{u}_1$, satisfies the same equation as (4) except the parameters ρ_0, λ_0 and μ_0 are replaced by ρ_1, λ_1 and μ_1 of the inclusion.

Taking divergence and curl of (4), we get a pair of equations,

$$\begin{aligned} \nabla \cdot \ddot{\mathbf{u}} - \frac{\lambda_0 + 2\mu_0}{\rho_0} \nabla^2(\nabla \cdot \mathbf{u}) &= 0, \\ \nabla \times \ddot{\mathbf{u}} - \frac{\mu_0}{\rho_0} \nabla^2(\nabla \times \mathbf{u}) &= 0. \end{aligned} \tag{5}$$

Equations (5) show that within the homogeneous region of the medium, the compressional wave (P wave) and the shear wave (S wave) are decoupled. The first equation in (5) is the P wave equation, the second is the S wave equation. The corresponding P wave and S wave propagation speeds are $\alpha_0 = \sqrt{(\lambda_0 + 2\mu_0)/\rho_0}$ and $\beta_0 = \sqrt{\mu_0/\rho_0}$, respectively.

The inclusion scattering problem can be attacked by solving equation (4) for both the exterior and interior fields and matching the boundary conditions (continuity of displacement and normal stresses) at the boundary. Equivalently, we can combine equations (5) and the boundary conditions to formulate a system of boundary integral equations. This formulation for discontinuously heterogeneous media, such as in the inclusion scattering problem, and curved interface scattering problem, etc., is widely used in the field of engineering seismology and seismic exploration. However, exact solutions exist only for the case of a uniform sphere or a uniform circular cylinder in an infinite homogeneous medium (MORSE and

FESHBACH, 1953; PAO and MOW, 1973). These "exact" solutions are in the form of infinite series. Only for long wavelengths these series can be calculated with a few terms. When the wavelength is much shorter than the radius of the sphere or cylinder, the series converge very slowly. The other closed form solution is for a semi-infinite crack, derived by using the functional theoretical technique (the Wiener-Hopf technique) (ACHENBACH, 1973). Therefore, for most cases of complex objects we have to rely on the numerical methods. The boundary-integral-equation approach is very suitable for applying different numerical techniques.

3. *Perturbation Method*

In the perturbation method the heterogeneous medium is decomposed into a reference medium (the background medium) and the perturbations. The scattering problem can be therefore transferred into a radiation problem by considering the response of the perturbations to the incident wave as the excitation of secondary sources. MILES (1980) formulated the problem into a volume integral equation and derived the explicit expressions in the case of Rayleigh scattering (for small heterogeneities) using Born approximations. HERRERA and MAL (1965) considered the problem of interface discontinuity of the heterogeneities and applied the method to the scattering of thin inclusions with strong contrast (dikes, lenses, cracks, etc.). This approach was also used by HADDON and CLEARY (1974) for the P-wave scattering near the mantle-core boundary and by HUDSON for the scattering by granular media (1968) and for the scattered waves in the coda of P (1977). HUDSON (1977) also gave a unified expression of the first-order approximation for several scattering problems (weak heterogeneities, thin inclusions and slightly rough surface). GUBERNATIS *et al.* (1977a) also formulated the scattering problem of a homogeneous inclusion into an integral equation and obtained solutions for the case of Born approximation in the whole frequency range. SATO (1984), WU and AKI (1985a,b) applied this approach to the coda generation and the seismogram envelope problem. WU and AKI (1985a) expressed the Rayleigh scattering in terms of equivalent point sources with different forces and force moments, and decomposed the general elastic wave scattering into "velocity type" and "impedance type" and pointed out their different scattering characteristics. SATO (1984) expressed the results of Born approximation in terms of density, P velocity and S velocity. SATO (1984), WU and AKI (1985b) have derived also the scattering coefficients for elastic random media with different types of correlation function. In the following, we give a brief derivation of the basic formulas and then present the main results and their geophysical applications.

In this paper we mainly deal with the large-angle scattering problem. For small-angle scattering (forward scattering) problem, such as the transmission fluctuation problem, the scalar wave approximation can be applied. The perturbation method for that case will be treated in another paper of this issue (WU and FLATTÉ, 1989). Also we restrict ourself in this paper to the problem of volume heterogeneities and leave the boundary perturbation method, which is used in the core-mantle-boundary scattering problem, to the other paper in this issue (BATAILLE *et al.*, 1989).

Suppose in equations (1) or (2) the medium parameters can be expressed as

$$\begin{aligned} \rho(\mathbf{x}) &= \rho_0 + \delta\rho(\mathbf{x}) \\ \mathbf{c}(\mathbf{x}) &= \mathbf{c}^0 + \delta\mathbf{c}(\mathbf{x}) \end{aligned} \tag{6}$$

or

$$\lambda(\mathbf{x}) = \lambda_0 + \delta\lambda(\mathbf{x}), \quad \mu(\mathbf{x}) = \mu_0 + \delta\mu(\mathbf{x}),$$

where ρ_0, $\mathbf{c}^0$ (or λ_0 and μ_0 in the case of isotropic medium) are the parameters of the reference medium, which is either a homogeneous or a slowly varying continuous medium, and $\delta\rho$, $\delta\mathbf{c}$ or $\delta\lambda$ and $\delta\mu$ are the deviations (perturbations) from their reference values. We decompose the total field as

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}^0(\mathbf{x}, t) + \mathbf{U}(\mathbf{x}, t) \tag{7}$$

where u^0 is the primary field, i.e., the field when there is no perturbation in the reference medium, and U is the scattered field. Substituting (6) and (7) into (1) yields

$$\begin{aligned} \rho_0 U_i - (c^0_{ijpq} U_{p,q})_{,j} &= Q_i, \\ Q_i &= -\{\delta\rho \ddot{u}_i - (\delta c_{ijpq} u_{p,q})_{,j}\} \end{aligned} \tag{8}$$

where $\mathbf{Q}(\mathbf{x}, t)$ are the equivalent body forces due to the interaction of the heterogeneities with the wave field.

Introducing the Green's function for the reference medium $G^0_{ij}(\mathbf{x}, t \mid x', t')$ and applying the representation theory to (8), we can express the scattered field as (HERRERA and MAL, 1965)

$$U_i = \int_V G^0_{ij} * Q_j \, d^3\mathbf{x}' + \int_S (G^0_{ij} * \tau_{jk} - \sigma^0_{ijk} * u_j) n_k \, d^2\mathbf{x}' + \int_{S_1} G^0_{ij} * n^1_k [\delta c_{jkpq} u_{p,q}] \, d^2\mathbf{x}' \tag{9}$$

where "$*$" stands for the temporal convolutions, V is the volume containing heterogeneities, S is its boundary surface, $\mathbf{x}'$ is the position on the surface, $\mathbf{n}$ is the normal on the surface towards outside, $\mathbf{u}$ and τ are the displacement and stress on the surface, σ^0 is the stress produced on the surface due to $\mathbf{G}^0$; S_1 is the surface across which exists a jump discontinuity of the normal stress $n^1_k[\delta c_{jkpq} u_{p,q}]$, where $\mathbf{n}^1$ is the normal of S_1 and [ ] stands for the jump of a quantity across the surface. Formula (9) is a general equation valid for both continuous and discontinuous heterogeneities and for a general reference medium.

If the reference medium is unbounded, we can set S to be the surface on which both $\mathbf{G}$ and $\mathbf{u}$ are zero. We can further combine S and S_1 into a closed surface system so that we can apply the divergence theory to it

$$\int_{S+S_1} G^0_{ij} * n_k [\delta c_{jkpq} u_{p,q}] \, d^2\mathbf{x}^1 = \int_V \{G^0_{ij} * [\delta c_{jkpq} u_{p,q}]\}_{,k} \, d^3 x'. \tag{10}$$

In the case of weak scattering, $\mathbf{U} \ll \mathbf{u}^0$, so that

$$[\delta c_{jkpq} u_{p,q}] \approx [\delta c_{jkpq}] u_{p,q} = -\delta c_{jkpq} u_{p,q}. \tag{11}$$

Substituting (11), (10) and Q_j (8) into (9), we obtain

$$U_i = \int_V \{-\delta\rho \ddot{u}_j * G^0_{ij} - \delta c_{jkpq} u_{p,q} * G^0_{ij,k}\} \, d^3\mathbf{x}'. \tag{12}$$

Because the cancellations of the term in volume integration and the term in surface integration in (9), it concludes with a simple equation (12) which does not have the term involving with spatial derivatives of $\delta\mathbf{c}$. Therefore, equation (12) is valid for both continuous and discontinuous heterogeneities provided the scattering is weak. This point is first illustrated by HUDSON (1968).

In the case of isotropic media, (12) becomes

$$U_i = \int_V \{-\delta\rho \ddot{u}_j G^0_{ij} - [\delta_{jk}\, \delta\lambda u_{qq} + \delta\mu(u_{j,k} + u_{k,j}) * G^0_{ij,k}\} \, d^3\mathbf{x}'. \tag{13}$$

4. *Born Approximation for Weak Scattering*

Now let us concentrate on the case of isotropic media. When the scattered field U is weak compared to the primary field u^0, we can approximate the total field u in (12) or (13) by the primary field u^0. This is the Born approximation. Equation (13) thus becomes an explicit formula for the scattered field

$$\begin{aligned} U_i(\mathbf{x}) = &-\int_V \delta\rho(\mathbf{x}')\ddot{u}^0_j(\mathbf{x}') * G_{ij}(\mathbf{x}, \mathbf{x}') \, dV(\mathbf{x}') \\ &- \int_V \{\delta_{jk}\delta\lambda(\mathbf{x}')[\nabla \cdot \mathbf{u}^0(\mathbf{x}')] + \delta\mu(\mathbf{x}')[u^0_{j,k}(\mathbf{x}') + u^0_{k,j}(\mathbf{x}')]\} * G_{ij,k}(\mathbf{x}, \mathbf{x}') \, dV(\mathbf{x}') \\ = &\int_V [F_j(\mathbf{x}') * G_{ij}(\mathbf{x}, \mathbf{x}') + M_{jk}(\mathbf{x}') * G_{ij,k}(\mathbf{x}, \mathbf{x}')] \, dV(\mathbf{x}'), \end{aligned} \tag{14}$$

where $F_j(\mathbf{x}')$ is the equivalent single force and $M_{jk}(\mathbf{x}')$ is the moment tensor of the equivalent force couples at point $\mathbf{x}'$ of the scatterer. As no confusion can happen, we drop the '0' for the Green's function. The Born approximation is valid under the condition

$$\frac{\delta p}{p_0} kR \ll 1, \tag{15}$$

where k is wavenumber of the wave field, R is size of the heterogeneity volume (such as the diameter of a sphere), and $\delta p/p_0$ is the average parameter perturbation of the heterogeneities,

$$\frac{\delta p}{p_0}=\frac{(\delta p)_{\mathrm{rms}}}{p_0}+\frac{(\delta\lambda)_{\mathrm{rms}}+2(\delta\mu)_{\mathrm{rms}}}{\lambda_0+2\mu_0}, \tag{16}$$

where $(\ )_{\mathrm{rms}}$ stands for the root mean square value. Roughly speaking, (15) means that the total "phase fluctuation" caused by scattering must be less then one radian. This restriction can be satisfied for two cases: (1) when the size of the heterogeneity volume is very small compared with the wavelengths, i.e., $kR \ll 1$. This is the case of Rayleigh scattering. In this case, the heterogeneities can be strong, namely $\delta p/p_0$ is not necessarily small. (2) When the size of the heterogeneity volume is not small, but the heterogeneities are weak such that (15) is satisfied. In optics it is called the Rayleigh-Gans scattering.

Elastic Wave Rayleigh Scattering

When the size of the scatterer is much smaller than the wavelength, phase differences between the scattered far fields from the different parts of the scatterer can be neglected. The whole heterogeneous body can be considered as a point scatterer. The right-hand side of (14) can be integrated out. Therefore (14) becomes

$$U_i(\mathbf{x})=F_j(\mathbf{x}^0)^*G_{ij}(\mathbf{x},\mathbf{x}^0)+M_{jk}(\mathbf{x}^0)^*G_{ij,k}(\mathbf{x},\mathbf{x}^0), \tag{17}$$

where $\mathbf{x}^0$ is the position of the center of the scatterer and

$$\begin{aligned} F_j(\mathbf{x}^0)&=-\overline{\delta\rho}V\ddot{u}_j^0(\mathbf{x}^0),\\ M_{jk}(\mathbf{x}^0)&=-\delta_{jk}\overline{\delta\lambda}V[\nabla\cdot u^0(\mathbf{x}^0)]-\overline{\delta\mu}V[u^0_{j,k}(\mathbf{x}^0)+u^0_{k,j}(\mathbf{x}^0)], \end{aligned} \tag{18}$$

where V is the volume of the scatterer, $\overline{\delta\rho}$, $\overline{\delta\lambda}$ and $\overline{\delta\mu}$ are the average values of the corresponding perturbations.

If the incident wave is a plane harmonic P-wave in x_1 direction, i.e.,

$$u_j^0=\delta_{j1}\exp[-i\omega(t-x_1/\alpha_0],$$

where α_0 is the P-wave speed in the background medium, then the equivalent single force is (assuming $\mathbf{x}^0$ is at the origin)

$$\mathbf{F}=\hat{x}\omega^2\overline{\delta\rho}Ve^{-i\omega t} \tag{19}$$

where $\hat{x}$ is unit vector in x_1 direction and the equivalent force moment tensor is

$$\mathbf{M}=-i\frac{\omega}{\alpha_0}V\begin{bmatrix}\overline{\delta\lambda}+2\overline{\delta\mu} & 0 & 0\\ 0 & \overline{\delta\lambda} & 0\\ 0 & 0 & \overline{\delta\lambda}\end{bmatrix}e^{-i\omega t}. \tag{20}$$

Substituting (19), (20) and the far field expressions for G_{ij} and $G_{ij,k}$ (see AKI and RICHARDS, 1980, chapter 4) into (17), we have

$$U_i = \frac{\omega^2 V}{4\pi\alpha_0^2}\frac{1}{r}e^{-i\omega(t-r/\alpha_0)}\left\{\frac{\overline{\delta\rho}}{\rho_0}\gamma_i\gamma_1 - \frac{\overline{\delta\lambda}}{\lambda_0+2\mu_0}\gamma_i - \frac{2\overline{\delta\mu}}{\lambda_0+2\mu_0}\gamma_i\gamma_1^2\right\}$$
$$+\frac{\omega^2 V}{4\pi\beta_0^2}\frac{1}{r}e^{-i\omega(t-r/\beta_0)}\left\{\frac{\overline{\delta\rho}}{\rho_0}(\delta_{i1}-\gamma_i\gamma_1) - 2\left[\frac{\beta_0}{\alpha_0}\right]\frac{\overline{\delta\mu}}{\mu_0}(\delta_{i1}\gamma_1-\gamma_i\gamma_1^2)\right\}, \qquad (21)$$

where $r = |\mathbf{x} - \mathbf{x}^0|$ and γ_i is direction cosine of the scattering directions with respect to the i-th axis, which is the i-th component of the unit vector $\hat{\mathbf{r}}$.

If we take spherical coordinates having their polar axis in the incident direction x_1 (i.e., in the direction of particle motion) (Figure 1a), we can write the scattered P-wave ${}^PU_r^P$ and S-wave ${}^PU_{\text{mer}}^S$ as

$${}^PU_r^P = \frac{V}{4\pi}\frac{\omega^2}{\alpha_0^2}\left\{\frac{\overline{\delta\rho}}{\rho_0}\cos\theta - \frac{\overline{\delta\lambda}}{\lambda_0+2\mu_0} - \frac{2\overline{\delta\mu}}{\lambda_0+2\mu_0}\cos^2\theta\right\}\frac{1}{r}e^{-i\omega(t-r/\alpha_0)}, \qquad (22)$$

and

$${}^PU_{\text{mer}}^S = \frac{V}{4\pi}\frac{\omega^2}{\alpha_0^2}\left[\frac{\alpha_0^2}{\beta_0^2}\right]\left\{\frac{\overline{\delta\rho}}{\rho_0}\sin\theta + \left[\frac{\beta_0}{\alpha_0}\right]\frac{\overline{\delta\mu}}{\mu_0}\sin 2\theta\right\}\frac{e^{-i\omega(t-r/\beta_0)}}{r}, \qquad (23)$$

where the subscript r stands for the r-component, and "mer" stands for the meridian component. Because of the symmetry of the problem with respect to the polar axis, there is no latitudinal component of the S-wave, i.e., ${}^PU_{\text{lat}}^S = 0$.

Figure 1 gives the equivalent forces and the scattered P and S waves produced by $\overline{\delta\rho}$, $\overline{\delta\lambda}$ and $\overline{\delta\mu}$ in the case of plane P-wave incidence. We see that $\overline{\delta\rho}V$ generates a point single force, $\overline{\delta\lambda}V$ behaves like a point negative explosion (or contraction), and $\overline{\delta\mu}V$ behaves like a point crack (collapse).

Similarly for a plane S-wave incident in x_1-direction and having its particle motion in x_2-direction,

$$u_j^0 = \delta_{j2}\exp[i\omega(t - x_1/\beta_0)], \qquad (24)$$

the equivalent forces and scattered field are

$$\mathbf{F} = \hat{y}\omega^2\overline{\delta\rho}Ve^{-i\omega t}$$

$$\mathbf{M} = -i\frac{\omega}{\beta_0}V\begin{bmatrix} 0 & \overline{\delta\mu} & 0 \\ \overline{\delta\mu} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}e^{-i\omega t}, \qquad (25)$$

where $\hat{y}$ is the unit director in the x_2-direction, and

$$U_i = \frac{\omega^2\overline{\delta\rho}V}{4\pi\rho_0}\left\{\frac{\gamma_i\gamma_2}{\alpha_0^2}\frac{1}{r}e^{-i\omega(t-r/\alpha_0)} - \frac{(\gamma_i\gamma_2-\delta_{i2})}{\beta_0^2}\frac{1}{r}e^{i\omega(t-r/\beta_0)}\right\} + \frac{\omega^2}{\beta_0}\frac{\overline{\delta\mu}V}{4\pi\rho_0}$$
$$\times\left\{\frac{-2\gamma_i\gamma_1\gamma_2}{\alpha_0^3}\frac{1}{r}e^{-i\omega(t-r/\alpha_0)} + \frac{(2\gamma_i\gamma_1\gamma_2-\delta_{i1}\gamma_2-\delta_{i2}\gamma_1)}{\beta_0^3}\frac{1}{r}e^{-i\omega(t-r/\beta_0)}\right\}. \qquad (26)$$

If we take the direction of particle motion of the incident field (y-axis) as the polar axis of the spherical coordinates (Figure 2a), the scattered P-wave ${}^{S}U_r^P$ and the scattered S-wave ${}^{S}U_{\text{mer}}^S$ and ${}^{S}U_{\text{lat}}^S$ can be written as

$$ {}^{S}U_r^P = \frac{V}{4\pi}\frac{\omega^2}{\alpha_0^2}\left\{\frac{\overline{\delta\rho}}{\rho_0}\cos\theta - \left[\frac{\beta_0}{\alpha_0}\right]\frac{\overline{\delta\mu}}{\mu_0}\sin 2\theta \sin\phi\right\}\frac{1}{r}e^{-i\omega(t-r/\alpha_0)}, \tag{27}$$

$$ {}^{S}U_{\text{mer}}^S = -\frac{V}{4\pi}\frac{\omega^2}{\alpha_0^2}\left[\frac{\alpha_0}{\beta_0}\right]^2\left\{\frac{\overline{\delta\rho}}{\rho_0}\sin\theta + \frac{\overline{\delta\mu}}{\mu_0}\cos 2\theta \sin\phi\right\}\frac{1}{r}e^{-i\omega(t-r/\beta_0)}, \tag{28}$$

and

$$ {}^{S}U_{\text{lat}}^S = \frac{V}{4\pi}\frac{\omega^2}{\alpha_0^2}\left[\frac{\alpha_0}{\beta_0}\right]^2\frac{\overline{\delta\mu}}{\mu_0}\cos\theta\cos\phi\frac{1}{r}e^{-\omega(t-r/\beta_0)}. \tag{29}$$

Figure 2 shows the equivalent forces and the scattered fields by $\overline{\delta\rho}$ and $\overline{\delta\mu}$. Note that $\overline{\delta\lambda}$ does not have any effect on S-wave scattering as expected. The density perturbation $\overline{\delta\rho}V$ acts still like a single force in the direction of particle motion. The scattered waves due to $\overline{\delta\mu}V$ are equivalent to the radiation field of a point double force couple, which is equivalent to a point dislocation (shear motion along a crack). In Figure 2, we decompose the scattered S-wave due to $\overline{\delta\mu}$ into two parts, each of which corresponds to a pattern due to a single couple.

From Figures 1, 2 and the related formulas we can see several important features of elastic wave scattering.

(1) The amplitudes of the scattered waves have a frequency dependence of ω^2, so the scattered power is proportional to ω^4. This is the characteristic of Rayleigh scattering, which is the same for acoustic wave, electromagnetic wave and elastic wave scattering.

(2) When $\mu = 0$, (22) becomes the same as for acoustic wave scattering. In general, the scattering patterns for elastic waves are much more complicated than that for acoustic waves.

(3) From Figures 1 and 2 we see that the cross-coupled scattered waves (P-S coupling, $S_{\text{mer}} - S_{\text{lat}}$ depolarization coupling) are always apart from the incident direction. Their maxima are in the directions perpendicular to the incident direction. Therefore for a pure forward scattering problem, the cross-coupled waves can be neglected.

(4) From (22) we see that in the forward direction, $\cos\theta = 1$, the combination of the parameter perturbations inside the bracelets becomes

$$\frac{\delta\rho}{\rho_0} - \frac{\delta\lambda + 2\delta\mu}{\lambda_0 + 2\mu_0} = -\frac{2\delta\alpha}{\alpha_0}.$$

Therefore in the forward direction, the scattering strength is only proportional to the velocity perturbations. On the other hand, in the backward direction, $\cos\theta = 1$, the combination of the parameter perturbations becomes

$$-\frac{\delta\rho}{\rho_0} - \frac{\delta\lambda + 2\delta\mu}{\lambda_0 + 2\mu_0} = -\frac{2\delta Z_p}{Z_{p0}},$$

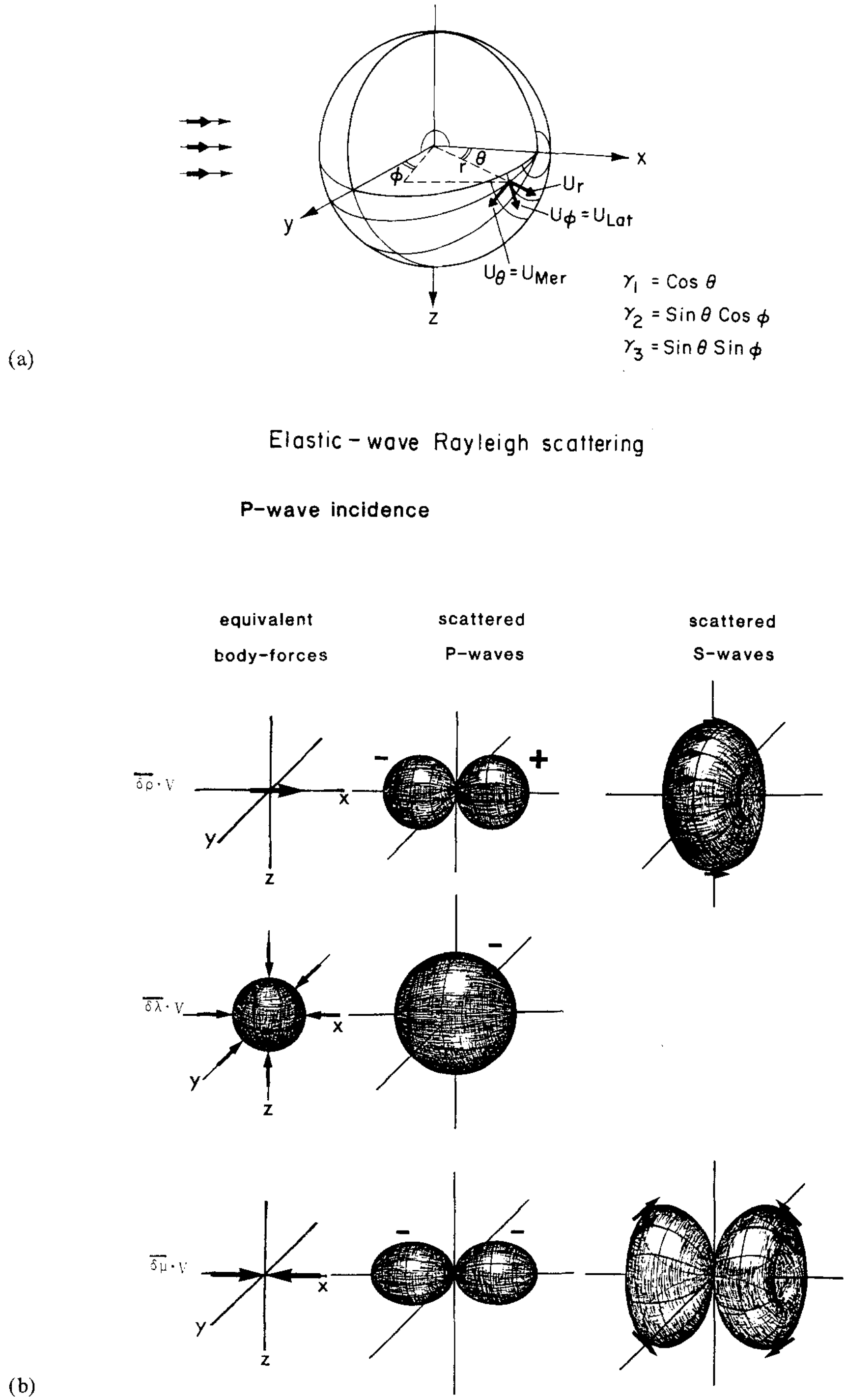
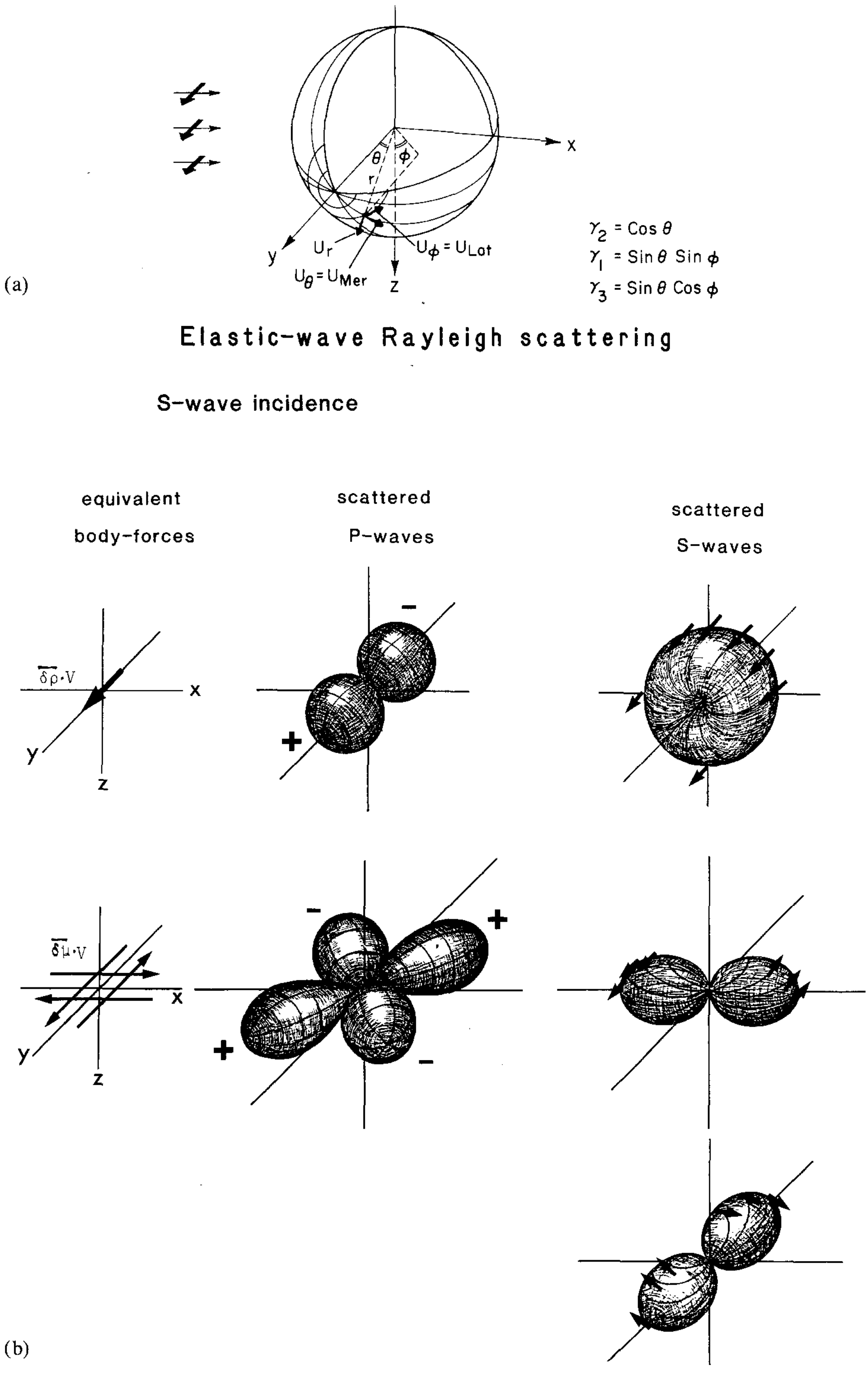


Figure 1

(a) Spherical coordinate system for P-wave incidence, and (b) the scattering patterns for different equivalent forces.


x
y
z
θ
φ
r
U_r
$U_\phi = U_{Lot}$
$U_\theta = U_{Mer}$
γ_2 = Cos θ
γ_1 = Sin θ Sin φ
γ_3 = Sin θ Cos φ
(a)
Elastic-wave Rayleigh scattering
S-wave incidence
equivalent body-forces
scattered P-waves
scattered S-waves
$\overline{\delta\rho}\cdot V$
$\overline{\delta\mu}\cdot V$
x
y
z
+
−
(b)

where Z_p is the P-wave impedance. Therefore, the strength of the back-scattered waves is only dependent on the impedance perturbations. We see that the forescattering and the backscattering reveal quite different characteristics of the medium, although the parameter dependences in both cases become simpler.

The same conclusion can be drawn for the common-mode S-S scattering (28). In the forward direction ($\theta = \phi = 90°$) the strength is dependent only on the S-wave velocity perturbations, while in the backward direction ($\theta = 90°$, $\phi = -90°$), only on the S-wave impedance perturbations.

(5) Comparing the strengths of the scattered P waves (22) and (27) and S waves (23, 28 and 29), we see that the scattering S waves are $[\alpha_0/\beta_0]$ times stronger than the scattered P waves for similar perturbations. That is because the S-wave impedance of the medium is always smaller than the P-wave impedance. Therefore, after propagating and scattering for a long distance, the scattered waves will be dominated by S-waves that agree with the observations on coda waves.

If we choose $\delta\rho/\rho_0$, $\delta\alpha/\alpha_0$ and $\delta\beta/\beta_0$ as free parameters, the scattering patterns (22), (23) and (27)–(29) can be also expressed as

$$
\begin{aligned}
{}^{P}E_r^P &= \left\{\overline{\frac{\delta\rho}{\rho_0}}\left(\cos\theta - 1 + 2\frac{\beta_0^2}{\alpha_0^2}\sin^2\theta\right) - 2\overline{\frac{\delta\alpha}{\alpha_0}} + 4\frac{\beta_0^2}{\alpha_0^2}\sin^2\theta\,\overline{\frac{\delta\beta}{\beta_0}}\right\}, \\
{}^{P}E_{\text{mer}}^S &= \left\{\overline{\frac{\delta\rho}{\rho_0}}\left(-1 + 2\frac{\beta_0}{\alpha_0}\cos\theta\right)\sin\theta + 4\frac{\beta_0}{\alpha_0}\overline{\frac{\delta\beta}{\beta_0}}\sin\theta\cos\theta\right\}, \\
{}^{S}E_r^P &= \left\{\overline{\frac{\delta\rho}{\rho_0}}\left(\cos\theta - \frac{\beta_0}{\alpha_0}\sin 2\theta\sin\phi\right) - 2\frac{\beta_0}{\alpha_0}\overline{\frac{\delta\beta}{\beta_0}}\sin 2\theta\sin\phi\right\}, \\
{}^{S}E_{\text{mer}}^S &= -\left\{\overline{\frac{\delta\rho}{\rho_0}}(\sin\theta + \cos 2\theta\sin\phi) + 2\overline{\frac{\delta\beta}{\beta_0}}\cos 2\theta\sin\phi\right\},
\end{aligned}
\tag{30}
$$

and

$$
{}^{S}E_{\text{lat}}^S = \left\{\left(\overline{\frac{\delta\rho}{\rho_0}} + 2\overline{\frac{\delta\beta}{\beta_0}}\right)\cos\theta\cos\phi\right\}.
$$

The scattering patterns can be written also in terms of velocity and impedence perturbations (see WU and AKI, 1985a).

Figure 2

(a) Spherical coordinate system for S-wave incidence, and (b) the scattering patterns for different equivalent forces.

Elastic Wave Rayleigh-Gans Scattering

When the wavelength is comparable to or smaller than the size of the inclusion, the phase differences of the incident field at different parts of the inclusion and of the scattered field from different parts of the inclusions can no longer be ignored. The equivalent source of scattering can no longer be regarded as a point source. Nevertheless, if the resulted scattered field is still weaker than the incident field, the Born approximation can still be applied to the problem. This is the case of Rayleigh-Gans scattering.

From (14) we see that the scattered field is a superposition of the scattered field by all the volume elements of the heterogeneity, each of which is of Rayleigh scattering type.

We can further simplify (14) by taking the far-field Fraunhofer approximation to $G_{ij}(\mathbf{x}, \mathbf{x}')$, i.e., (henceforth we use the frequency-domain expression and drop the factor $e^{-i\omega t}$)

$$G_{ij}(\mathbf{x}, \mathbf{x}') = G^P_{ij}(\mathbf{x}, \mathbf{x}') + G^S_{ij}(\mathbf{x}, \mathbf{x}')$$

$$\approx \frac{1}{4\pi\rho_0\alpha_0^2}\hat{\mathbf{o}}_i\hat{\mathbf{o}}_j g^\alpha(\mathbf{x}, \mathbf{x}') + \frac{1}{4\pi\rho_0\beta_0^2}(\delta_{ij} - \hat{\mathbf{o}}_i\hat{\mathbf{o}}_j)g^\beta(\mathbf{x}, \mathbf{x}') \tag{31}$$

where $\mathbf{G}^P$ and $\mathbf{G}^S$ are P-wave part and S-wave part of $\mathbf{G}$ respectively, and

$$g^v(\mathbf{x}, \mathbf{x}') = \frac{1}{r}e^{i(\omega/v_0)r} \approx \frac{1}{r^0}e^{i(\omega/v_0)[r^0 - \hat{\mathbf{o}}\cdot(\mathbf{x}' - \mathbf{x}^0)]} = g^v(\mathbf{x}, \mathbf{x}^0)\psi^v(\mathbf{x}), \tag{32}$$

where

$$g^v(\mathbf{x}, \mathbf{x}^0) = \frac{1}{r^0}e^{i(\omega/v_0)r^0}, \quad \psi^v(\mathbf{x}') = e^{-i(\omega/v_0)\hat{\mathbf{o}}\cdot(\mathbf{x}' - \mathbf{x}^0)}$$

$$r = |\mathbf{x} - \mathbf{x}'|, \quad r^0 = |\mathbf{x} - \mathbf{x}^0|,$$

and $\mathbf{x}^0$ is the center of the inclusion, $\hat{\mathbf{o}} = (\mathbf{x} - \mathbf{x}^0)/r^0$ is the unit vector in the outgoing direction of the scattered wave (scattering direction).

Substituting (30) and (31) into (14), we obtain

$$U_i(\mathbf{x}) = U_i^P(\mathbf{x}) + U_i^S(\mathbf{x}),$$

$$U_i^P(\mathbf{x}) = \frac{1}{4\pi\rho_0\alpha_0^2}g^\alpha(\mathbf{x}, \mathbf{x}^0)\int_V \hat{\mathbf{o}}_i\hat{\mathbf{o}}_j\left\{F_j(\mathbf{x}') - i\frac{\omega}{\alpha}\hat{\mathbf{o}}_k M_{jk}(\mathbf{x}')\right\}\psi^\alpha(\mathbf{x}')\,dV(\mathbf{x}')$$

$$U_i^S(\mathbf{x}) = \frac{1}{4\pi\rho_0\beta_0^2}g^\beta(\mathbf{x}, \mathbf{x}^0)\int_V (\delta_{ij} - \hat{\mathbf{o}}_i\hat{\mathbf{o}}_j)\left\{F_j(\mathbf{x}') - i\frac{\omega}{\beta}\hat{\mathbf{o}}_k M_{jk}(\mathbf{x}')\right\}\psi^\beta(\mathbf{x}')\,dV(\mathbf{x}'). \tag{33}$$

In this section we will use a more general form of the incident wave (primary field). If the source is far from the inclusion, the primary field at $\mathbf{x}'$ can be approximated locally by a plane wave

$$u_i^0(\mathbf{x}') = A_i\Phi(\mathbf{x}') = A_i\exp[i\mathbf{k}^{in}\cdot(\mathbf{x}' - \mathbf{x}^0) - i\omega t] \tag{34}$$

where $\mathbf{A} = A\hat{\mathbf{a}}$ is the amplitude of the incident wave, $\hat{\mathbf{a}}$ is a unit vector, $\mathbf{k}^{in}$ is the wave number vector of the incident wave, $\mathbf{k}^{in} = k^{in}\hat{\mathbf{i}} = (\omega/v_0)\hat{\mathbf{i}}$, where v_0 is P-wave or S-wave velocity of the reference medium depending on the wave type of the incident wave, and $\hat{\mathbf{i}}$ is the unit vector in the incident direction. Thus we can derive F_j and M_{jk} (see equation 18) as follows

$$\begin{aligned} F_j(\mathbf{x}') &= \delta\rho(\mathbf{x}')\omega^2 A_j \Phi(\mathbf{x}'), \\ M_{jk}(\mathbf{x}') &= -i\{\delta\lambda(\mathbf{x}')\delta_{jk}(\mathbf{A}\cdot\mathbf{k}^{in}) + \delta\mu(\mathbf{x}')(A_j k_k^{in} + A_k k_j^{in})\}\Phi(\mathbf{x}'). \end{aligned} \tag{35}$$

Therefore, from (33) we have (expressing in vector form)

$$\begin{aligned} U^P(\mathbf{x}) &= \frac{1}{4\pi\rho_0\alpha_0^2} g^\alpha \int_V \hat{\mathbf{o}}\left\{(\hat{\mathbf{o}}\cdot\mathbf{A})\omega^2\delta\rho(\mathbf{x}') - \frac{\omega}{\alpha_0}\delta\lambda(\mathbf{x}')(\mathbf{A}\cdot\mathbf{k}^{in})\right. \\ &\quad \left. - 2\frac{\omega}{\alpha_0}\delta\mu(\mathbf{x}')(\hat{\mathbf{o}}\cdot\mathbf{A})(\hat{\mathbf{o}}\cdot\mathbf{k}^{in})\right\} e^{i(\mathbf{k}^{in} - k_\alpha\hat{\mathbf{o}})\cdot(\mathbf{x}' - \mathbf{x}^0)}\, dV(\mathbf{x}') \\ U^S(\mathbf{x}) &= \frac{1}{4\pi\rho_0\beta_0^2} g^\beta \int_V \left\{[\mathbf{A} - \hat{\mathbf{o}}(\hat{\mathbf{o}}\cdot\mathbf{A})]\omega^2\delta\rho(\mathbf{x}') - \frac{\omega}{\beta_0}\delta\mu(\mathbf{x}')[(\hat{\mathbf{o}}\cdot\mathbf{k}^{in})(\mathbf{A} - \hat{\mathbf{o}}(\hat{\mathbf{o}}\cdot\mathbf{A}))\right. \\ &\quad \left. + (\hat{\mathbf{o}}\cdot\mathbf{A})(\mathbf{k}^{in} - \hat{\mathbf{o}}(\hat{\mathbf{o}}\cdot\mathbf{k}^{in}))]\right\} e^{i(\mathbf{k}^{in} - k_\beta\hat{\mathbf{o}})\cdot(\mathbf{x}' - \mathbf{x}^0)}\, dV(\mathbf{x}') \end{aligned} \tag{36}$$

where $k_\alpha = \omega/\alpha_0$ and $k_\beta = \omega/\beta_0$.

Writing explicitly in the forms of P-wave and S-wave incidence, (36) becomes

$$\begin{aligned} {}^P\mathbf{U}^P &= \frac{k_\alpha^2}{4\pi} g^\alpha \int_V \hat{\mathbf{o}}(\mathbf{x}')A(\mathbf{x}')\left\{(\hat{\mathbf{o}}\cdot\hat{\mathbf{i}})\frac{\delta\rho(\mathbf{x}')}{\rho_0} - \frac{\delta\lambda(\mathbf{x}')}{\lambda_0 + 2\mu_0} - (\hat{\mathbf{o}}\cdot\hat{\mathbf{i}})^2\frac{2\delta\mu(\mathbf{x}')}{\lambda_0 + 2\mu_0}\right\} \\ &\quad \cdot e^{i(k_\alpha\hat{\mathbf{i}} - k_\alpha\hat{\mathbf{o}})\cdot(\mathbf{x}' - \mathbf{x}^0)}\, dV(\mathbf{x}') \\ {}^P\mathbf{U}^S &= \frac{k_\beta^2}{4\pi} g^\beta \int_V [\hat{\mathbf{a}}(\mathbf{x}') - \hat{\mathbf{o}}(\mathbf{x}')(\hat{\mathbf{o}}\cdot\hat{\mathbf{a}})]A(\mathbf{x}')\left\{\frac{\delta\rho(\mathbf{x}')}{\rho_0} - (\hat{\mathbf{o}}\cdot\hat{\mathbf{i}})\left[\frac{\beta_0}{\alpha_0}\right]\frac{2\delta\mu(\mathbf{x}')}{\lambda_0 + 2\mu_0}\right\} \\ &\quad \cdot e^{i(k_\alpha\hat{\mathbf{i}} - k_\beta\hat{\mathbf{o}})\cdot(\mathbf{x}' - \mathbf{x}^0)}\, dV(\mathbf{x}') \\ {}^S\mathbf{U}^P &= \frac{k_\alpha^2}{4\pi} g^\alpha \int_V \hat{\mathbf{o}}(\mathbf{x}')A(\mathbf{x}')\left\{(\hat{\mathbf{o}}\cdot\hat{\mathbf{a}})\frac{\delta\rho(\mathbf{x}')}{\rho_0} - (\hat{\mathbf{o}}\cdot\hat{\mathbf{a}})(\hat{\mathbf{o}}\cdot\hat{\mathbf{i}})\left[\frac{\beta_0}{\alpha_0}\right]\frac{2\delta\mu(\mathbf{x}')}{\mu_0}\right\} \\ &\quad \cdot e^{i(k_\beta\hat{\mathbf{i}} - k_\alpha\hat{\mathbf{o}})\cdot(\mathbf{x}' - \mathbf{x}^0)}\, dV(\mathbf{x}') \\ {}^S\mathbf{U}^S &= \frac{k_\beta^2}{4\pi} g^\beta \int_V A(\mathbf{x}')\left\{(\hat{\mathbf{a}} - \hat{\mathbf{o}}(\hat{\mathbf{o}}\cdot\hat{\mathbf{a}}))\frac{\delta\rho(\mathbf{x}')}{\rho_0} - [(\hat{\mathbf{o}}\cdot\hat{\mathbf{i}})[\hat{\mathbf{a}} - \hat{\mathbf{o}}(\hat{\mathbf{o}}\cdot\hat{\mathbf{a}})]\right. \\ &\quad \left. + (\hat{\mathbf{o}}\cdot\hat{\mathbf{a}})[\hat{\mathbf{i}} - \hat{\mathbf{o}}(\hat{\mathbf{o}}\cdot\hat{\mathbf{i}})]\frac{\delta\mu(\mathbf{x}')}{\mu_0}\right\} e^{i(k_\beta\hat{\mathbf{i}} - k_\beta\hat{\mathbf{o}})\cdot(\mathbf{x}' - \mathbf{x}^0)}\, dV(\mathbf{x}'). \end{aligned} \tag{37}$$

If the source is far enough from the inclusions such that $\mathbf{A}(\mathbf{x}')$ and $\hat{\mathbf{i}}(\mathbf{x}')$ are constant over the volume of the inclusions, then (37) becomes:

$$
\begin{aligned}
{}^{P}\mathbf{U}^{P} &= \frac{Ak_\alpha^2}{4\pi} g^\alpha \hat{\mathbf{o}} \left\{ (\hat{\mathbf{o}} \cdot \hat{\mathbf{i}}) \frac{\delta\rho(\tilde{\mathbf{k}})}{\rho_0} - \frac{\delta\lambda(\tilde{\mathbf{k}})}{\lambda_0 + 2\mu_0} - (\hat{\mathbf{o}} \cdot \hat{\mathbf{i}})^2 \frac{2\delta\mu(\tilde{\mathbf{k}})}{\lambda_0 + 2\mu_0} \right\} \\
{}^{P}\mathbf{U}^{S} &= \frac{Ak_\beta^2}{4\pi} g^\beta [\hat{\mathbf{a}} - \hat{\mathbf{o}}(\hat{\mathbf{o}} \cdot \hat{\mathbf{a}})] \left\{ \frac{\delta\rho(\tilde{\mathbf{k}})}{\rho_0} - (\hat{\mathbf{o}} \cdot \hat{\mathbf{i}}) \left[\frac{\beta_0}{\alpha_0} \right] \frac{2\delta\mu(\tilde{\mathbf{k}})}{\lambda_0 + 2\mu_0} \right\} \\
{}^{S}\mathbf{U}^{P} &= \frac{Ak_\alpha^2}{4\pi} g^\alpha \hat{\mathbf{o}} \left\{ (\hat{\mathbf{o}} \cdot \hat{\mathbf{a}}) \frac{\delta\rho(\tilde{\mathbf{k}})}{\rho_0} - \left[\frac{\beta_0}{\alpha_0} \right] (\hat{\mathbf{o}} \cdot \hat{\mathbf{a}})(\hat{\mathbf{o}} \cdot \hat{\mathbf{i}}) \frac{2\delta\mu(\tilde{\mathbf{k}})}{\mu_0} \right\} \qquad (38) \\
{}^{S}\mathbf{U}^{S} &= \frac{Ak_\beta^2}{4\pi} g^\beta \left\{ [\hat{\mathbf{a}} - \hat{\mathbf{o}}(\hat{\mathbf{o}} \cdot \hat{\mathbf{a}})] \frac{\delta\rho(\tilde{\mathbf{k}})}{\rho_0} - [(\hat{\mathbf{o}} \cdot \hat{\mathbf{i}})[\hat{\mathbf{a}} - \hat{\mathbf{o}}(\hat{\mathbf{o}} \cdot \hat{\mathbf{a}})] \right. \\
&\quad \left. + (\hat{\mathbf{o}} \cdot \hat{\mathbf{a}})[\hat{\mathbf{i}} - \hat{\mathbf{o}}(\hat{\mathbf{o}} \cdot \hat{\mathbf{i}})]] \frac{\delta\mu(\tilde{\mathbf{k}})}{\mu_0} \right\}
\end{aligned}
$$

where $\delta\rho(\mathbf{K})$, $\delta\lambda(\mathbf{K})$ and $\delta\mu(\mathbf{K})$ are the complex 3-D spatial spectra of $\delta\rho(\mathbf{x})$, $\delta\lambda(\mathbf{x})$ and $\delta\mu(\mathbf{x})$, respectively, and are defined as

$$\delta\rho(\mathbf{K}) = \iiint_{-\infty}^{\infty} \delta\rho(\mathbf{x}) e^{i\mathbf{K}\cdot\mathbf{x}}\, d^3x \qquad (39)$$

etc. In (38), $\tilde{\mathbf{k}} = \mathbf{k}^{in} - \mathbf{k}^{sc}$, where $\mathbf{k}^{sc}$ is the wave number of the scattered waves, which is $k_\alpha \hat{\mathbf{o}}$ or $k_\beta \hat{\mathbf{o}}$ depending on the wave type of the scattered waves.

Note that the vector $\mathbf{M} = \hat{\mathbf{a}} - \hat{\mathbf{o}}(\hat{\mathbf{o}} \cdot \hat{\mathbf{a}})$ is perpendicular to $\hat{\mathbf{o}}$ and lies in the plane determined by $\hat{\mathbf{o}}$ and $\hat{\mathbf{a}}$. We can define a unit vector in this direction (the meridian direction)

$$\hat{\mathbf{m}} = \frac{\mathbf{M}}{|\mathbf{M}|} = [\hat{\mathbf{a}} - \hat{\mathbf{o}}(\hat{\mathbf{o}} \cdot \hat{\mathbf{a}})]/|\hat{\mathbf{a}} - \hat{\mathbf{o}}(\hat{\mathbf{o}} \cdot \hat{\mathbf{a}})| \qquad (40)$$

and another unit vector perpendicular to the $(\hat{\mathbf{o}}, \hat{\mathbf{a}})$ plane (the latitudinal direction)

$$\hat{\mathbf{l}} = (\hat{\mathbf{m}} \times \hat{\mathbf{o}}), \qquad (41)$$

which is also perpendicular to the outgoing direction $\hat{\mathbf{o}}$. Therefore, $\hat{\mathbf{o}}$, $\hat{\mathbf{l}}$ and $\hat{\mathbf{m}}$ compose a right-hand coordinate system. We can decompose the scattered wave into two orthogonal components: a meridian component along $\hat{\mathbf{m}}$ (or in-plane component) and a latitudinal component along $\hat{\mathbf{l}}$ (or off-plane component). From (38) we see that ${}^{P}\mathbf{U}^{S}$ has only the meridian component. Whereas ${}^{S}\mathbf{U}^{S}$ has both meridian and latitudinal components,

$${}^{S}\mathbf{U}^{S} = \frac{Ak_\beta^2}{4\pi} g^\beta \left\{ \hat{\mathbf{m}}|\mathbf{M}| \left[\frac{\delta\rho(\tilde{\mathbf{k}})}{\rho_0} - (\hat{\mathbf{o}} \cdot \hat{\mathbf{i}}) \left[1 - \frac{(\hat{\mathbf{o}} \cdot \hat{\mathbf{a}})^2}{|\mathbf{M}|^2} \right] \frac{\delta\mu(\tilde{\mathbf{k}})}{\mu_0} \right] + \hat{\mathbf{l}}(\hat{\mathbf{o}} \cdot \hat{\mathbf{a}})(\hat{\mathbf{l}} \cdot \hat{\mathbf{i}}) \frac{\delta\mu(\tilde{\mathbf{k}})}{\mu_0} \right\}. \qquad (42)$$

Comparing (38) with the corresponding results of Rayleigh scattering (21 and 26) we see that the scattering formulas for the two cases have the same form except the total perturbation $\overline{\delta\rho V}$, $\overline{\delta\lambda V}$ and $\overline{\delta\mu V}$ are replaced by their corresponding spectral densities at the spatial frequency equal to the exchange wave number of the scattering process, $\tilde{\mathbf{k}} = \mathbf{k}^{in} - \mathbf{k}^{sc}$. In the case of Rayleigh scattering, since the wavelength is much longer than the size of the inclusion, the scattered waves can only detect the d.c. component of the heterogeneity; while in the case of Rayleigh-Gans scattering, the scattered waves respond to different spectral components of the heterogeneity depending on the scattering angles.

Composition Factor and Distribution Factor

If the perturbations are formed by a nonuniform distribution of heterogeneities with the same composition, then (38) can be further simplified. We introduce a parameter distribution function $D(\mathbf{x})$ such that

$$\delta\rho(\mathbf{x}) = \delta\rho_0 D(\mathbf{x}),$$

$$\delta\lambda(\mathbf{x}) = \delta\lambda_0 D(\mathbf{x}),$$

and (43)

$$\delta\mu(\mathbf{x}) = \delta\mu_0 D(\mathbf{x}),$$

where $\delta\rho_0$, $\delta\lambda_0$ and $\delta\mu_0$ are the parameter perturbations at the center of heterogeneity and satisfy

$$\delta\rho_0 \int_V D(\mathbf{x})\, dV(\mathbf{x}) = \overline{\delta\rho} V, \text{ etc.} \tag{44}$$

The scattered field in (38) then can be written generally as

$$\mathbf{U} = \mathbf{C}(\delta\rho_0, \delta\lambda_0, \delta\mu_0)\mathbf{D}(\tilde{\mathbf{k}}). \tag{45}$$

Here we call $\mathbf{C}$ the *composition factor*, which is the scattered field of elastic wave Rayleigh scattering by a unit volume of inclusion with perturbation $\delta\rho_0$, $\delta\lambda_0$ and $\delta\mu_0$, and $\mathbf{D}$, the *distribution factor* (or the *volume factor*). We see from (38) that ${}^P\mathbf{C}^P$, ${}^P\mathbf{C}^S$, ${}^S\mathbf{C}^P$ and ${}^S\mathbf{C}^S$ have the same expressions as in (38) for $\mathbf{U}$'s except $\delta\rho(\tilde{\mathbf{k}})$, $\delta\lambda(\tilde{\mathbf{k}})$ and $\delta\mu(\tilde{\mathbf{k}})$ are replaced by $\delta\rho_0$, $\delta\lambda_0$ and $\delta\mu_0$. The distribution factor (which is also called the "form factor" or "shape factor", etc. for uniform inclusions)

$$D(\tilde{\mathbf{k}}) = \int_V D(\mathbf{x}) e^{i\tilde{\mathbf{k}}\cdot\mathbf{x}}\, d^3x = D(\mathbf{K} = \tilde{\mathbf{k}}) \tag{46}$$

where $D(\mathbf{K})$ is 3-D spectrum of $D(\mathbf{x})$ and $\tilde{\mathbf{k}} = \mathbf{k}^{in} - \mathbf{k}^{sc}$. If the perturbation distribution is spherically symmetric, then

$$D(K) = -\frac{2\pi}{K}\frac{\partial}{\partial K} D_1(K) \tag{47}$$

where $D_1(K)$ is the corresponding 1-D spectrum and $K = |\mathbf{K}|$.

In the following, we list the distribution factors (i.e., the complex spectra) of some simple forms of distribution:

(1) For a uniform sphere of radius a,

$$D(K) = 4\pi a^3 \frac{1}{(Ka)^2}\left[\frac{\sin(Ka)}{Ka} - \cos(Ka)\right]. \tag{48}$$

(2) For a Gaussian spherical heterogeneity,

$$\begin{aligned} D(r) &= \exp(-r^2/a^2), \\ D(K) &= \pi\sqrt{\pi}a^3 \exp[-(Ka)^2/4]. \end{aligned} \tag{49}$$

(3) For a uniform cylinder of radius a, height $2b$ (in z-direction),

$$D(\mathbf{K}) = 4\pi a^2 b \frac{J_1(K_x a)}{K_x a} \frac{\sin(K_z b)}{K_z b}, \tag{50}$$

where K_z is the vertical component of $\mathbf{K}$ and K_x is the horizontal component of $\mathbf{K}$.

(4) For a Gaussian cylindrical heterogeneity,

$$\begin{aligned} D(x, z) &= \exp\left[-\frac{z^2}{b^2} - \frac{x^2}{a^2}\right], \\ D(\mathbf{K}) &= \sqrt{\pi}\pi a^2 b \exp\left[-\frac{(K_x a)^2 + (K_z b)^2}{4}\right]. \end{aligned} \tag{51}$$

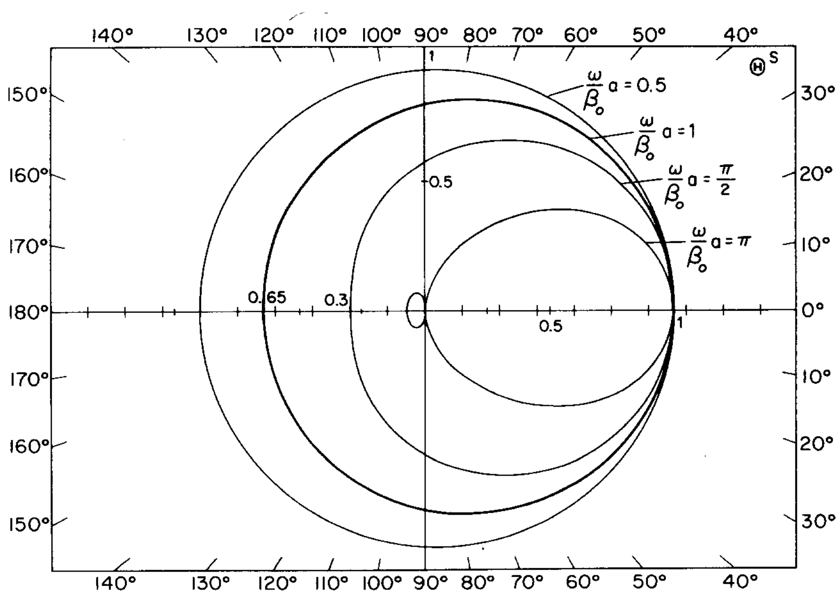
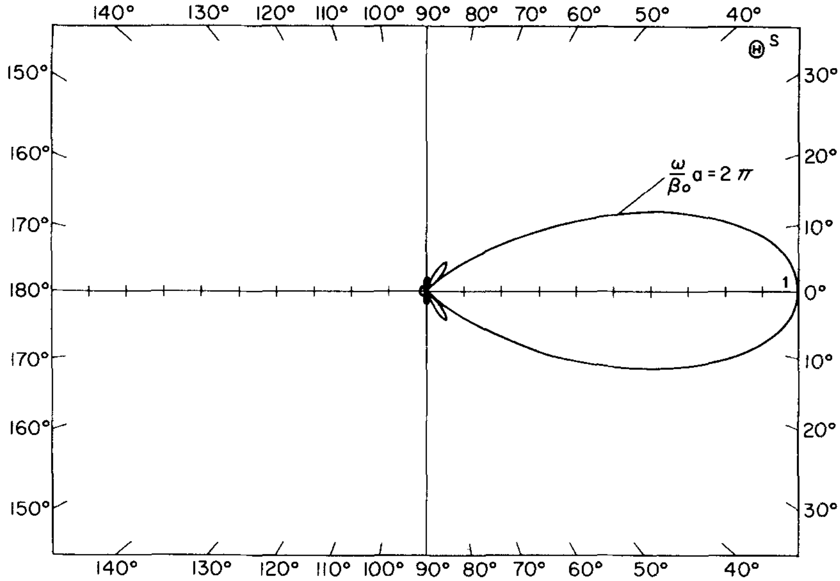
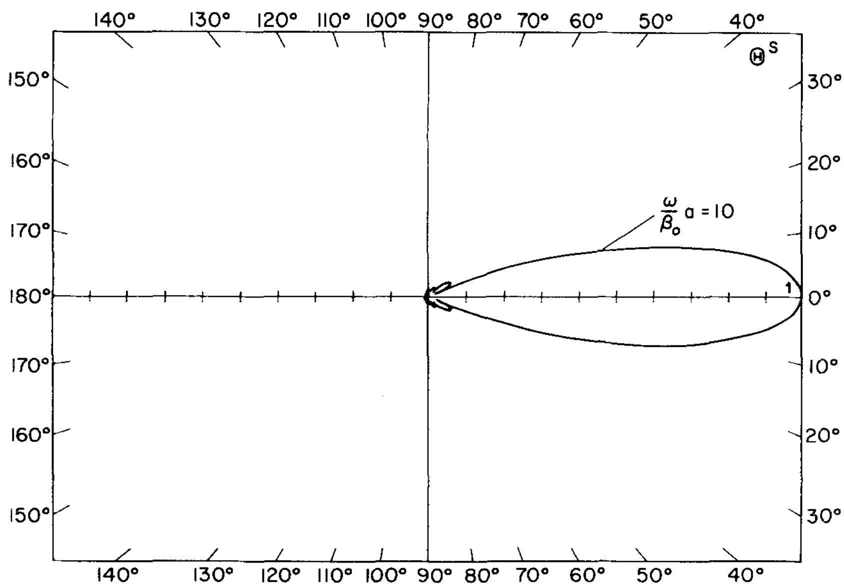


Figure 3(a)



(b)



(c)

Figure 3

(a) Distribution factors of S-wave scattering $D(\tilde{k})$, where

$$\tilde{k} = 2\frac{\omega}{\beta_0}\sin\frac{\theta}{2}$$

for a uniform sphere for different frequencies; (b) same as in (a), when $(\omega/\beta_0)a = 2\pi$; (c) same as in (a), when $(\omega/\beta_0)a = 10$.

(5) For an ellipsoid of axes a_1, a_2 and a_3 aligned along the x, y and z directions,

$$D(\mathbf{K}) = 4\pi a_1 a_2 a_3 \frac{1}{|Ka|^2}\left[\frac{\sin(|Ka|)}{|Ka|} - \cos(|Ka|)\right], \tag{52}$$

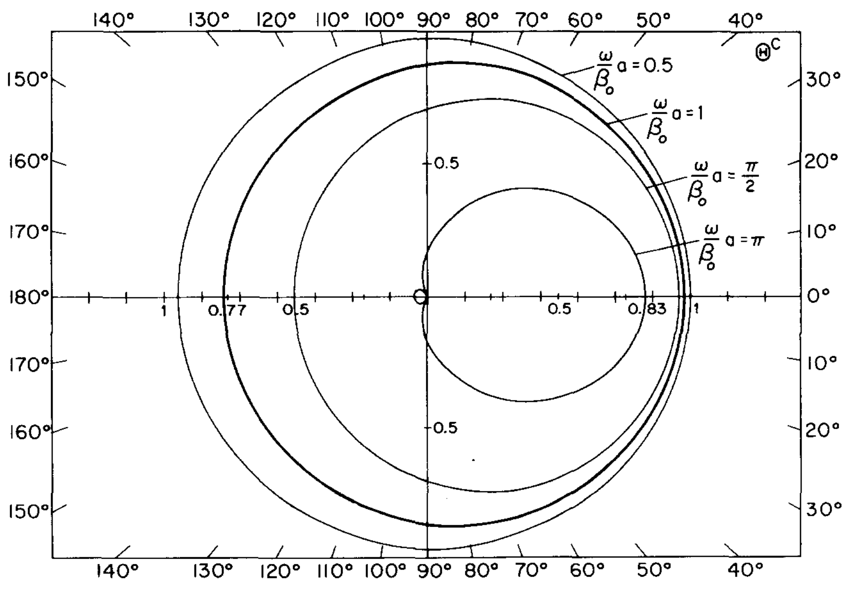
where $|Ka| = (K_x^2 a_1^2 + K_y^2 a_2^2 + K_z^2 a_3^2)^{1/2}$.

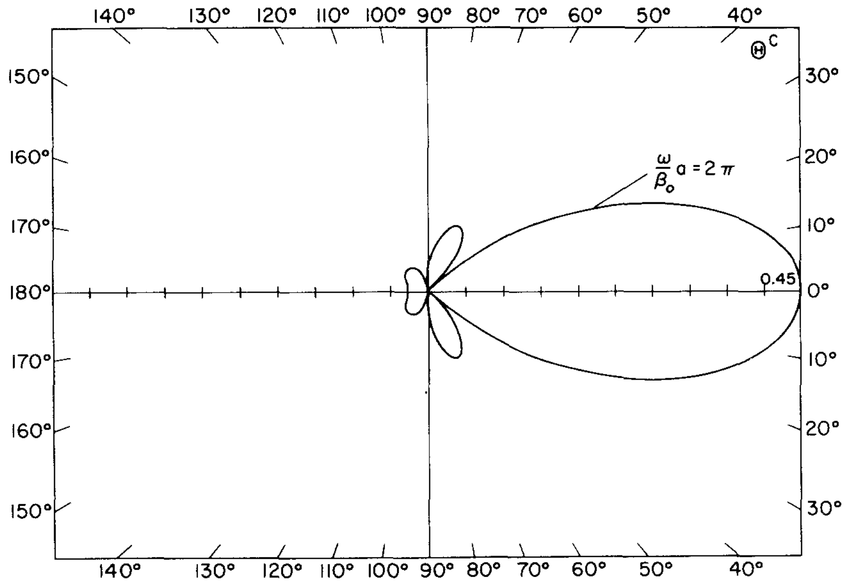
The value of $D(\tilde{\mathbf{k}})$ depends on $\tilde{\mathbf{k}} = \mathbf{k}^{in} - \mathbf{k}^{sc}$, which is a function of the scattering angle. In the case of spherically symmetric heterogeneities,

$$\begin{aligned} \tilde{k} = |\tilde{\mathbf{k}}| &= 2k \sin\frac{\theta}{2}, \quad \text{for common-type scattering} \\ &= \sqrt{k_\alpha^2 + k_\beta^2 - 2k_\alpha k_\beta \cos\theta}, \quad \text{for type-converting scattering,} \end{aligned} \tag{53}$$

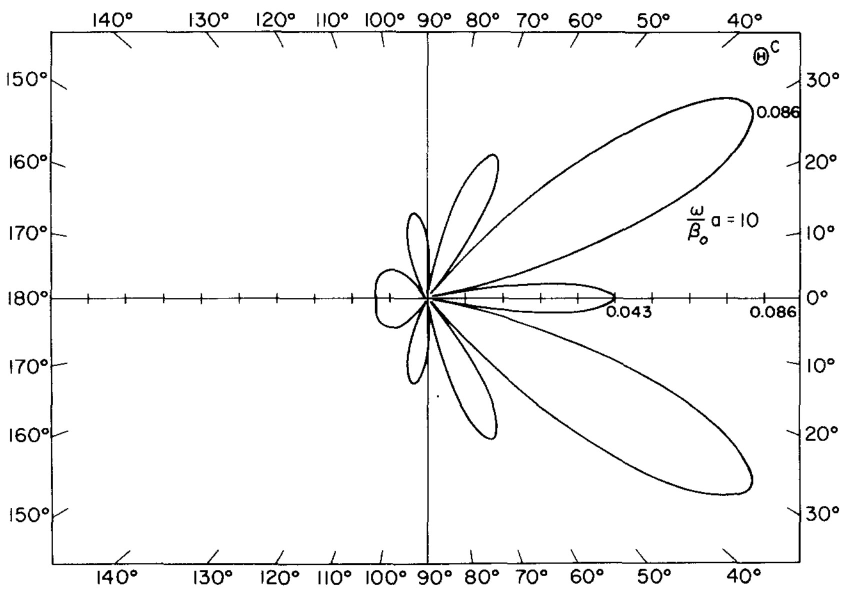
where θ is the scattering angle (the angle between the incident direction and the scattering direction), k_α and k_β are the P-wave number and S-wave number, respectively.

Figures 3 and 4 show the distribution factors of a uniform sphere as functions of scattering angles. Figure 3 shows the case of common-type scattering. In the forward direction, $\theta = 0$, $D(0)$ is the d.c. component of the inclusions and therefore is always the maximum. This is because all the scattered waves from different volume elements are always in-phase along the forward direction. For high frequencies, i.e., when $k_a \gg 1$, the main scattered energy is concentrated in the forward lobe and therefore the scattering characteristics are controlled by the velocity perturbations.





(b)



(c)

Figure 4

(a) Distribution factors for type-converting scattering $D(\tilde{k})$, where

$$\tilde{k} = \sqrt{\left[\frac{\omega}{\alpha_0}\right]^2 + \left[\frac{\omega}{\beta_0}\right]^2 + 2\left[\frac{\omega}{\alpha_0}\right]\left[\frac{\omega}{\beta_0}\right]\cos\theta},$$

for a uniform sphere for different frequencies; (b) same as in (a), when $(\omega/\beta_0)a = 2\pi$; (c) same as in (a), when $(\omega/\beta_0)a = 10$.

Figure 4 shows the D factors of type-converting scattering. In this case, however, $\tilde{k}$ does not go to zero in the forward direction ($\theta = 0$), but approaches a limited value. Therefore the D value oscillates in the forward direction and is not always the maximum.

The resultant scattering patterns for a Rayleigh-Gans scattering are the product of a Rayleigh scattering pattern and a D factor pattern.

5. *Elastic Wave Scattering by a Random Medium*

For very complex media, the deterministic approach is either intractable or impractical. The stochastic approach plays an important role in that case. If the rang-scale ratio $R/a \gg 1$, where a is the average scale length of the heterogeneities and R is the extension of the heterogeneous medium (or the propagation range), the case is more suitable for the stochastic approach.

Suppose in equation (6) $\delta\mathbf{c}$ or $\delta\rho$, $\delta\lambda$, and $\delta\mu$ are zero mean Gaussian random variables. Then the heterogeneous medium becomes a random medium. A random medium is a large family of innumerable heterogeneous media. Each member of the family, with a certain probability of existence, differs from the others in the detailed structure, but has some common statistical properties of the family, such as the average size and strength of the heterogeneities, etc. For a specific number of the random medium, called a realization, we can formally express the scattered far-field using the Born approximation as (37), (38) or (45). If we know the detailed spatial distribution of $\delta\rho(\mathbf{x}')$, $\delta\lambda(\mathbf{x}')$ and $\delta\mu(\mathbf{x}')$, we can calculate the scattered field. Often it is impossible to know these distributions and more often they are just what we are trying to infer from the measured scattered waves. The random medium approach deals with the statistical quantities of the medium and the wave field. One of the useful quantities is the mean-square amplitude of the scattered waves, which is proportional to the average scattered energy.

Mean-Square Amplitudes of Scattered Waves

If $\delta\rho$, $\delta\lambda$ and $\delta\mu$ are not totally correlated, $\langle|U|^2\rangle$ depends generally on the auto- and cross-power spectra of the perturbations. For instance, for the P-P scattering, taking ensemble average of the square amplitude of (38) results in

$$\langle|^P U^P|^2\rangle = \frac{A^2 k_\alpha^4 V}{(4\pi r)^2}\{(\hat{\mathbf{o}}\cdot\hat{\mathbf{i}})^2 W_\rho(\tilde{\mathbf{k}}) + W_\lambda(\tilde{\mathbf{k}}) + 4(\hat{\mathbf{o}}\cdot\hat{\mathbf{i}})^4 W_\mu(\tilde{\mathbf{k}}) - (\hat{\mathbf{o}}\cdot\hat{\mathbf{i}}) W_{\rho\lambda}(\tilde{\mathbf{k}})$$
$$- 2(\hat{\mathbf{o}}\cdot\hat{\mathbf{i}})^3 W_{\rho\mu}(\tilde{\mathbf{k}}) + 2(\hat{\mathbf{o}}\cdot\hat{\mathbf{i}})^2 W_{\lambda\mu}(\tilde{\mathbf{k}})\} \tag{54}$$

where $\langle\ \rangle$ stands for the ensemble average, W_ρ, W_λ and W_μ are the auto power-spectral densities and $W_{\rho\lambda}$, $W_{\rho\mu}$ and $W_{\lambda\mu}$ are the cross-power spectral densities of the random functions defined as

$$W_\rho(\mathbf{K}) = \frac{1}{V}\left\langle \frac{\delta\rho^*(\mathbf{K})}{\rho_0} \cdot \frac{\delta\rho(\mathbf{K})}{\rho_0} \right\rangle$$
$$W_\lambda(\mathbf{K}) = \frac{1}{V}\left\langle \frac{\delta\lambda^*(\mathbf{K})}{\lambda_0 + 2\mu_0} \cdot \frac{\delta\lambda(\mathbf{K})}{\lambda_0 + 2\mu_0} \right\rangle \tag{55}$$
$$W_{\rho\lambda}(\mathbf{K}) = \frac{1}{V}\left\langle \frac{\delta\rho^*(\mathbf{K})}{\rho_0} \cdot \frac{\delta\lambda(\mathbf{K})}{\lambda_0 + 2\mu_0} \right\rangle$$

etc., where * denotes the complex conjugate and V is the volume of the random medium and $\tilde{\mathbf{k}} = \mathbf{k}^{in} - \mathbf{k}^{sc}$.

If the size of the random medium is much larger than the scale of the heterogeneities, and the heterogeneities have a statistically uniform distribution, we can consider approximately the random medium as a stationary (or uniform) random field, so a correlation function may be defined for the medium. It is known that the power spectrum of a random medium is the Fourier transform of its correlation function, therefore we have the relation

$$W_\rho(\mathbf{K}) = \iiint_{-\infty}^{\infty} \left\langle \frac{\delta\rho(0)}{\rho_0} \frac{\delta\rho(\xi)}{\rho_0} \right\rangle e^{i\mathbf{K}\cdot\xi}\, d^3\xi = \left\langle \left(\frac{\delta\rho}{\rho_0}\right)^2 \right\rangle P(\mathbf{K})$$
$$P(\mathbf{K}) = \iiint_{-\infty}^{\infty} N(\xi) e^{i\mathbf{K}\cdot\xi}\, d^3\xi \tag{56}$$

where $N(\xi)$ is the normalized correlation function, ξ is the spatial lag vector and $P(\mathbf{K})$ is the normalized power spectrum of the density perturbations. For the cross-spectra we have

$$W_{\rho\lambda}(\mathbf{K}) = \iiint_{-\infty}^{\infty} \left\langle \frac{\delta\rho(0)}{\rho_0} \frac{\delta\lambda(\xi)}{\lambda_0 + 2\mu_0} \right\rangle e^{i\mathbf{K}\cdot\xi}\, d^3\xi. \tag{57}$$

If the random medium is also statistically isotropic, the 1-D power spectrum $P_1(K)$ is related with its 3-D power spectrum $P(K)$ by the relation (47).

In the following we list several most widely used correlation functions and their power spectra (all for statistically isotropic and uniform random media)

(1) Gaussian Correlations function,

$$N(r) = \exp(-r^2/a^2)$$
$$P_1(K) = \sqrt{\pi}a \exp(-K^2a^2/4) \tag{58}$$
$$P(K) = (\sqrt{\pi}a)^3 \exp(-K^2a^2/4).$$

(2) Exponential Correlation function,

$$N(r) = \exp(-|r|/a)$$

$$P_1(K) = \frac{2a}{1 + K^2a^2} \tag{59}$$

$$P(K) = 8\pi \frac{a^3}{(1 + K^2a^2)^2}.$$

(3) Von Kármán Correlation function,

$$N(r) = \frac{1}{2^{\nu-1}\Gamma(\nu)} \left[\frac{r}{a}\right]^\nu K_\nu(r/a),$$

$$P_1(K) = 2\sqrt{\pi} \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu)} \frac{a}{(1 + K^2a^2)^{\nu+1/2}}, \tag{60}$$

$$P(K) = 8\pi\sqrt{\pi} \frac{\Gamma(\nu + \frac{3}{2})}{\Gamma(\nu)} \frac{a^3}{(1 + K^2a^2)^{\nu+3/2}},$$

where $\Gamma(\ )$ is the gamma function, and $K_\nu(\ )$ is the modified Bessel function of order ν. When $\nu = 1/2$, the correlation function is equivalent to the exponential correlation function, and $\nu = 1/3$, the Kolmogorov's turbulence.

If $\delta\rho$, $\delta\lambda$ and $\delta\mu$ are from heterogeneities with the same composition, i.e., these perturbations are totally correlated, we can use (45) to calculate the scattered fields for each realization. In this case,

$$\langle|U|^2\rangle = \langle \mathbf{U}^* \cdot \mathbf{U} \rangle = \langle (\mathbf{C}^* \cdot \mathbf{C})(D^*D) \rangle = \langle |C|^2 \rangle \langle D^*D \rangle = V\langle |C|^2 \rangle P(\tilde{\mathbf{k}}), \tag{61}$$

where $P(\tilde{\mathbf{k}})$ is the "shape spectrum", i.e., the 3-D power spectrum of the parameter distributions function

$$P(\tilde{\mathbf{k}}) = \frac{1}{V} \langle D^*(\tilde{\mathbf{k}}) D(\tilde{\mathbf{k}}) \rangle. \tag{62}$$

In deriving (61), we use the independence of C and D. From (58) and (42), we can write (61) as

$$\langle |U|^2 \rangle = \frac{A^2V}{(4\pi r)^2} \langle |E|^2 \rangle k^4 P(\tilde{\mathbf{k}}) \tag{63}$$

where E is the elastic-wave-Rayleigh-scattering pattern function and

$$\langle |^PE^P| \rangle = \left\langle \left\{ (\hat{\mathbf{o}} \cdot \hat{\mathbf{i}}) \frac{\delta\rho}{\rho_0} - \frac{\delta\lambda}{\lambda_0 + 2\mu_0} - (\hat{\mathbf{o}} \cdot \hat{\mathbf{i}})^2 \frac{2\delta\mu}{\lambda_0 + 2\mu_0} \right\}^2 \right\rangle$$

for P-P scattering, and

$$\langle|^P E^S|^2\rangle = [1-(\hat{\mathbf{o}}\cdot\hat{\mathbf{a}})^2]\left\langle\left\{\frac{\delta\rho}{\rho_0} - (\hat{\mathbf{o}}\cdot\hat{\mathbf{i}})\left[\frac{\beta_0}{\alpha_0}\right]\frac{2\delta\mu}{\lambda_0+2\mu_0}\right\}^2\right\rangle,$$

$$\langle|^S E^P|^2\rangle = (\hat{\mathbf{o}}\cdot\hat{\mathbf{a}})^2\left\langle\left\{\frac{\delta\rho}{\rho_0} - (\hat{\mathbf{o}}\cdot\hat{\mathbf{i}})\left[\frac{\beta_0}{\alpha_0}\right]\frac{2\delta\mu}{\mu_0}\right\}^2\right\rangle, \tag{64}$$

$$\langle|^S E^S|^2\rangle = [1-(\hat{\mathbf{o}}\cdot\hat{\mathbf{a}})^2]\left\langle\left\{\frac{\delta\rho}{\rho_0} - (\hat{\mathbf{o}}\cdot\hat{\mathbf{i}})\left[1-\frac{(\hat{\mathbf{o}}\cdot\hat{\mathbf{a}})^2}{1-(\hat{\mathbf{o}}\cdot\hat{\mathbf{a}})^2}\right]\frac{\delta\mu}{\mu_0}\right\}^2\right\rangle$$
$$+\left\langle\left\{(\hat{\mathbf{o}}\cdot\hat{\mathbf{a}})(\hat{\mathbf{I}}\cdot\hat{\mathbf{i}})\frac{\delta\mu}{\mu_0}\right\}^2\right\rangle,$$

in the case of P-S, S-P, and S-S scattering. Here we drop the subscript "0" for $\delta\rho$, $\delta\lambda$ and $\delta\mu$ because of the statistical uniformity of these random variables.

Directional Scattering Coefficient

We define the directional scattering coefficient for P-P scattering as 4π times the mean scattered power in $\hat{\mathbf{o}}$ direction per unit solid angle by a unit volume of random medium for a unit incident field (i.e., unit power flux density) in $\hat{\mathbf{i}}$ direction. From (63) we obtain

$$g^{PP}(\hat{\mathbf{o}},\hat{\mathbf{i}}) = \frac{4\pi r^2}{A^2 V}\langle|^P U^P|^2\rangle = \frac{1}{4\pi}\langle|^P E^P|^2\rangle k_\alpha^4 P(\tilde{\mathbf{k}}) \tag{65}$$

where $P(\tilde{\mathbf{k}})$ is the power spectrum of the random parameter distributions, we have dropped the subscript.

Note that g^{pp} defined in (65) is 4π times the total differential scattering cross-section in the case of discrete scatterers

$$g^{pp} = 4\pi n_s \sigma_s, \tag{66}$$

where n_s is the number of scatterers per unit volume, and σ_s is the average differential scattering cross-section for individual scatterers. By defining the scattering coefficient, we can treat both the continuous and the discrete random media in a unified approach within the weak scattering regime.

In the same manner as in (65) we can define the other scattering coefficients g^{ps}, g^{sp}, and g^{ss}.

In the case of P wave incidence, $(\hat{\mathbf{o}}\cdot\hat{\mathbf{a}}) = (\hat{\mathbf{o}}\cdot\hat{\mathbf{i}}) = \cos\theta$, where θ is the scattering angle. If we assume

$$\frac{\delta\rho}{\rho_0} = \epsilon, \quad \frac{\delta\lambda}{\lambda_0+2\mu_0} = \frac{m}{3}\epsilon, \quad \frac{\delta\mu}{\lambda_0+2\mu_0} = \frac{n}{3}\epsilon, \quad \frac{\delta\mu}{\mu_0} = l\epsilon,$$

we obtain

$$
\begin{aligned}
g^{pp}(\theta) &= \frac{1}{4\pi}\langle\epsilon^2\rangle\left(\cos\theta - \frac{m}{3} - \frac{2n}{3}\cos^2\theta\right)^2 k_\alpha^4 P(\tilde{\mathbf{k}}),\\
g^{ps}(\theta) &= \frac{1}{4\pi}\langle\epsilon^2\rangle\left(\sin\theta - l\frac{\beta_0}{\alpha_0}\sin 2\theta\right)^2 k_\beta^4 P(\tilde{\mathbf{k}}).
\end{aligned}
\tag{67}
$$

Similarly, we can derive g^{sp} and g^{ss},

$$
\begin{aligned}
g^{sp}(\theta, \theta_A) &= \frac{1}{4\pi}\langle\epsilon^2\rangle\left(\cos\theta_A - 2l\frac{\beta_0}{\alpha_0}\cos\theta_A\cos\theta\right)^2 k_\alpha^4 P(\tilde{\mathbf{k}})\\
g^{ss}(\theta, \theta_A, \psi) &= \frac{1}{4\pi}\langle\epsilon^2\rangle\{[\sin\theta_A - l\sin\theta_A\cos\theta(1 - \mathrm{ctg}^2\theta_A)]^2\\
&\quad + (l\cos\theta_A\cos\psi)^2\}k_\beta^4 P(\tilde{\mathbf{k}})
\end{aligned}
\tag{68}
$$

where

$$
\cos\theta = (\hat{\mathbf{o}}\cdot\hat{\mathbf{i}}), \quad \cos\theta_A = (\hat{\mathbf{o}}\cdot\hat{\mathbf{a}}), \quad \cos\psi = (\hat{\mathbf{l}}\cdot\hat{\mathbf{i}}).
$$

Total Scattering Coefficient

Total scattering coefficient (or simply "scattering coefficient") g^p or g^s is the mean total scattered power by a unit volume of a random medium for a unit incident P or S wave. For P wave incidence

$$
g^p = g^{pp} + g^{ps} = \frac{1}{4\pi}\int_{4\pi} g^{pp}(\theta) + g^{ps}(\theta)\, d\Omega. \tag{69}
$$

Substituting (67) into (69), we can derive g^{pp} and g^{ps}. WU and AKI (1985b) gave the results for the case of exponential correlation function. For the low frequency approximation ($k_\alpha a \ll 1$)

$$
\begin{aligned}
g^{pp} &\approx 2\left[\frac{1}{3} + \frac{1}{9}m^2 + \frac{4}{45}n^2 + \frac{4}{27}mn\right]\langle\epsilon^2\rangle k_\alpha^4 a^3,\\
g^{ps} &\approx 4\left[\frac{1}{3} + \frac{4}{15}n^2\frac{\beta_0^2}{\alpha_0^2}\right]\langle\epsilon^2\rangle k_\beta^4 a^3.
\end{aligned}
\tag{70}
$$

In the high frequency range ($k_\alpha a \gg 1$)

$$
\begin{aligned}
g^{pp} &\approx 2\left\langle\left[\frac{\delta\alpha}{\alpha_0}\right]^2\right\rangle k_\alpha^2 a,\\
g^{ps} &\approx \frac{D}{2a}\left[\frac{\alpha_0}{\beta_0}\right]^2\langle\epsilon^2\rangle,
\end{aligned}
\tag{71}
$$

where D is a constant depending on the medium parameters.

Compare (70) and (71) with the scattering coefficients from the scalar wave theory (CHERNOV, 1960, equations (56), (58), and (59) for exponential correlation functions)

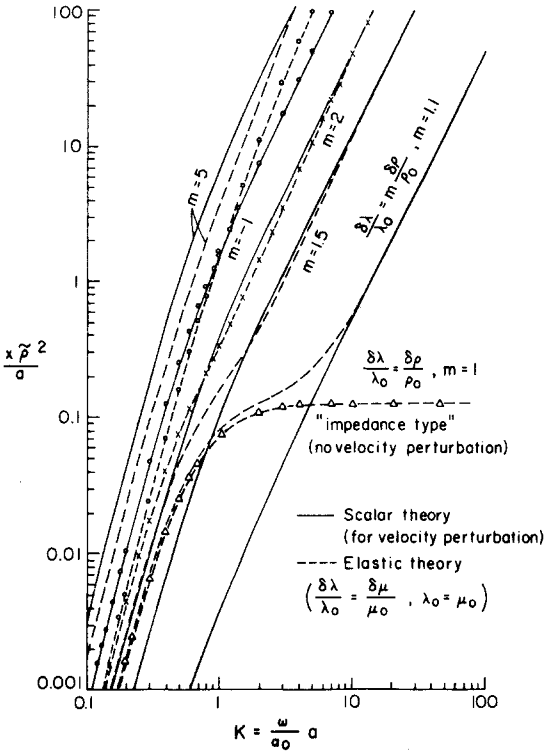
$$g_{\text{scalar}} \approx 8\left\langle \left[\frac{\delta\alpha}{\alpha_0}\right]^2 \right\rangle k_\alpha^4 a^3, \quad \text{when} \quad k_\alpha a \ll 1,$$
$$g_{\text{scalar}} \approx 2\left\langle \left[\frac{\delta\alpha}{\alpha_0}\right]^2 \right\rangle k_\alpha^2 a, \quad \text{when} \quad k_\alpha a \gg 1. \tag{72}$$

We see that for low frequencies the scattering coefficients for elastic waves are more complicated and depend not only on the velocity perturbations (as in the case of scalar waves) but on all the perturbations of density and Lamé constants. On the other hand, the h-f asymptotics for both the elastic and scalar cases are equal. This implies that in the case of weak perturbation of parameters, the travel time fluctuation in the forward direction will dominate the scattered field when the wavelength is very short compared to the sizes of heterogeneities. This explains why the scattering coefficient (by the Born approximation) should not be taken as an attenuation coefficient, especially in the h-f range. The k^2 frequency dependence of g^{pp} also demonstrates the inapplicability of the Born approximation to the h-f limit, because g^{pp} will go to infinity (according to the Born approximation) when the frequency goes to infinity.

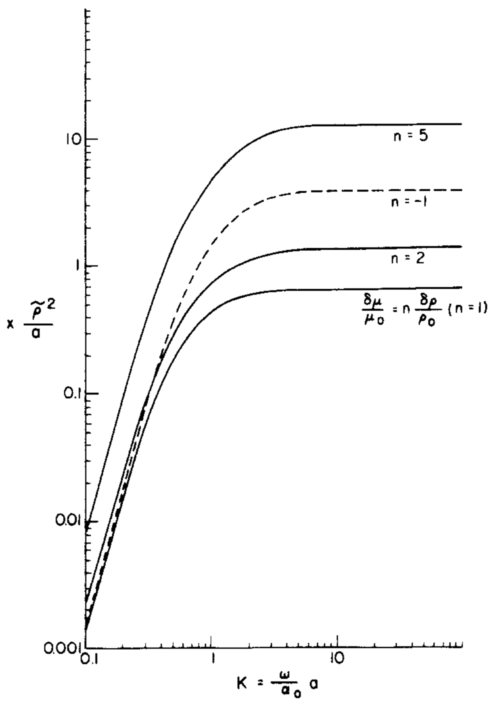
Frequency Dependence of the Scattering Coefficients

From (67) and (68), we see the frequency dependences of $g(\theta)$'s are determined by $k^4 P(\tilde{\mathbf{k}})$. From (58) to (60) we see that for low spatial frequency ($Ka \ll 1$), $P(K)$ is flat for all the cases, while for high $K(Ka \gg 1)$ the slope of the spectra depends on the form of $P(K)$. Gaussian medium has a Gaussian shape falloff of the spectrum in the high-**K** range, while exponential and von Kármán media have power-law falloffs with power index -4 and $-(2\nu + 3)$, respectively. For a fixed scattering angle, $\tilde{k} = |\mathbf{k}^{in} - \mathbf{k}^{sc}|$ is proportional to the frequency of the wave field. Therefore, the frequency dependence of $g(\theta)$ in the low frequency range is ω^4, which is typical for Rayleigh scattering and identical to scalar wave scattering. However, in the h-f range, the frequency dependence can deviate from the scalar wave case substantially. The frequency dependence also varies with the scattering angle. In the forward direction, $\theta = 0$, so the common-type scattering $g^{pp}(0)$ and $g^{ss}(0)$ have always the ω^4 dependence. For other angles, there are critical frequencies above which the ω^4 frequency dependences give way to the h-f asymptotes. The backscattering ($\theta = \pi$) has the smallest critical frequency.

The total scattering coefficients have similar frequency dependences: ω^4 for the low-f and different asymptotes for h-f. Figure 5 shows the frequency dependences of the total scattering coefficients g^{pp} and g^{ps} for the case of exponential media. In the figure we also plot the dependences of the scalar wave case for comparison. For the common-type scattering g^{pp}, the frequency dependences can behave quite differently from the scalar case in the intermediate frequency range, depending on the combination of parameter perturbations. For high frequencies, forward scattering


m = 5
m = -1
m = 2
m = 1.5
$\frac{\delta\lambda}{\lambda_0} = m\frac{\delta\rho}{\rho_0}$, m = 1.1
$\frac{\delta\lambda}{\lambda_0} = \frac{\delta\rho}{\rho_0}$, m = 1
"impedance type"
(no velocity perturbation)
Scalar theory
(for velocity perturbation)
Elastic theory
$\left(\frac{\delta\lambda}{\lambda_0} = \frac{\delta\mu}{\mu_0},\ \lambda_0 = \mu_0\right)$
$\frac{x\tilde{\rho}^2}{a}$
$K = \frac{\omega}{a_0}a$

(a)


n = 5
n = -1
n = 2
$\frac{\delta\mu}{\mu_0} = n\frac{\delta\rho}{\rho_0}$ (n = 1)
$x\,\frac{\tilde{\rho}^2}{a}$
$K = \frac{\omega}{a_0}a$

(b)

caused by velocity perturbations becomes dominant, so that g^{pp} behaves as in the case of scalar wave scattering except for the case of

$$\frac{\delta\lambda}{\lambda_0} = \frac{\delta\mu}{\mu_0} = \frac{\delta\rho}{\rho_0}$$

("impedance type" scattering). g^{ps} has flat h-f asymptotes for exponential media. For a general von Kármán type medium, the h-f spectrum slope of type-converting scattering is $4 - (2\nu + 3) = 1 - 2\nu$. When $\nu = 1/3$ (Kolmogorov spectrum), the slope increases slightly with frequency. For a Gaussian medium, g^{ps} goes to zero exponentially.

Fore- and Backscattering, and the Scalar Wave Approximation

In the nearly forward direction, $\theta \approx 0$, which we call forescattering, the directional scattering coefficients (67) and (68) become

$$\begin{aligned} g^{pp}(\theta \approx 0) &\approx \frac{1}{\pi}\left\langle\left[\frac{\delta\alpha}{\alpha_0}\right]^2\right\rangle k_\alpha^4 P(0) \\ g^{ps}(\theta \approx 0) &\approx 0, \\ g^{sp}(\theta \approx 0) &\approx 0, \\ g^{ss}(\theta \approx 0) &\approx \frac{1}{\pi}\left\langle\left[\frac{\delta\beta}{\beta_0}\right]^2\right\rangle k_\beta^4 P(0). \end{aligned} \tag{73}$$

In deriving we used the relation

$$\frac{\delta\alpha}{\alpha_0} = \frac{1}{2}\left[-\frac{\delta\rho}{\rho_0} + \frac{\delta\lambda + 2\delta\mu}{\lambda_0 + 2\mu_0}\right].$$

We see that for forescattering the scattered waves are decoupled, and the mean scattered power is proportional to the mean square of P or S velocity perturbation only. In this case the scalar wave approximation can be applied.

Similarly, in the backward direction, i.e., in the case of backscattering ($\theta \approx \pi$)

$$\begin{aligned} g^{pp}(\theta \approx \pi) &\approx \frac{1}{\pi}\left\langle\left[\frac{\delta Zp}{Zp_0}\right]^2\right\rangle k_\alpha^4 P(0) \\ g^{ps}(\theta \approx \pi) &\approx 0, \\ g^{sp}(\theta \approx \pi) &\approx 0, \\ g^{ss}(\theta \approx \pi) &\approx \frac{1}{\pi}\left\langle\left[\frac{\delta Zs}{Zs_0}\right]^2\right\rangle k_\beta^4 P(0). \end{aligned} \tag{74}$$

Figure 5

(a) Frequency dependences of P-P scattering coefficient. The solid lines are from the scalar wave theory and the broken lines, elastic wave theory, (b) frequency dependences of P-S scattering coefficient from elastic wave scattering theory.

Here we use the relation

$$\frac{\delta Zp}{Zp_0} = \frac{1}{2}\left[\frac{\delta\rho}{\rho_0} + \frac{\delta\lambda + 2\delta\mu}{\lambda_0 + 2\mu_0}\right]$$

and

$$\frac{\delta Zs}{Zs_0} = \frac{1}{2}\left[\frac{\delta\rho}{\rho_0} + \frac{\delta\mu}{\mu_0}\right],$$

where $Zp = \rho\alpha$ and $Zs = \rho\beta$ respectively. We see that the backscattered waves are also decoupled (under the Born approximation) and the mean scattered power is proportional to the mean-square perturbation of P wave impedance or S wave impedance accordingly. The scalar wave approximation can be used in this case too.

In conclusion, both fore- and backscattering of elastic waves are similar to the scalar wave scattering, but with different scalar quantities as the perturbation parameter: velocity for the forescattering and impedance for the backscattering. We should of course bear in mind that the analysis is only valid for weak scattering (Born approximation). For strong heterogeneities, multiple scattering inside the random medium becomes important and waves are no longer decoupled.

In other directions there are no simple relations as in the case of fore- and backscattering. The full elastic wave formulation has to be applied. In the case of $\delta\lambda/\lambda_0 = \delta\mu/\mu_0$ and $\lambda_0 = \mu_0$, we can decompose the perturbations into velocity and impedance perturbations using the relations

$$\begin{aligned} &\frac{\delta\rho}{\rho_0} = \frac{\delta Zp}{Zp_0} - \frac{\delta\alpha}{\alpha_0} = \frac{\delta Zs}{Zs_0} - \frac{\delta\beta}{\beta_0}, \\ &\frac{\delta\lambda}{\lambda_0} = \frac{\delta\mu}{\mu_0} = \frac{\delta Zp}{Zp_0} + \frac{\delta\alpha}{\alpha_0} = \frac{\delta Zs}{Zs_0} + \frac{\delta\beta}{\beta_0}. \end{aligned} \tag{75}$$

Substituting (75) into (64), we obtain the scattering patterns

$$\begin{aligned} \langle|^P E^P|^2\rangle &= \left\langle\left\{\frac{\delta Zp}{Zp_0}(C_I - B_I) - \frac{\delta\alpha}{\alpha_0}(C_I + B_I)\right\}^2\right\rangle \\ \langle|^P E^S|^2\rangle &= \left\langle\left\{\frac{\delta Zs}{Zs_0}(S_I - T_I) - \frac{\delta\beta}{\beta_0}(S_I + T_I)\right\}^2\right\rangle \\ \langle|^S E^P|^2\rangle &= \left\langle\left\{\frac{\delta Zs}{Zs_0}(C_A - T_A) - \frac{\delta\beta}{\beta_0}(C_A + T_A)\right\}^2\right\rangle \\ \langle|^S E^S|^2\rangle &= \left\langle\left\{\frac{\delta Zs}{Zs_0}(S_A - B_A) - \frac{\delta\beta}{\beta_0}(S_A + B_A)\right\}^2\right\rangle + \left\langle\left\{\left[\frac{\delta Zs}{Zs_0} + \frac{\delta\beta}{\beta_0}\right]D_A\right\}^2\right\rangle \end{aligned} \tag{76}$$

where

$$
\begin{aligned}
&C_I = \cos\theta, \quad B_I = \frac{1}{3} + \frac{2}{3}\cos^2\theta, \\
&S_I = \sin\theta, \quad T_I = \frac{\beta_0}{\alpha_0}\sin 2\theta, \\
&C_A = \cos\theta_A, \quad T_A = \frac{2\beta_0}{\alpha_0}\cos\theta_A\cos\theta, \\
&S_A = \sin\theta_A, \quad B_A = \sin\theta_A\cos\theta(1 - ctg^2\theta_A), \quad D_A = \cos\theta_A\cos\psi,
\end{aligned}
\tag{77}
$$

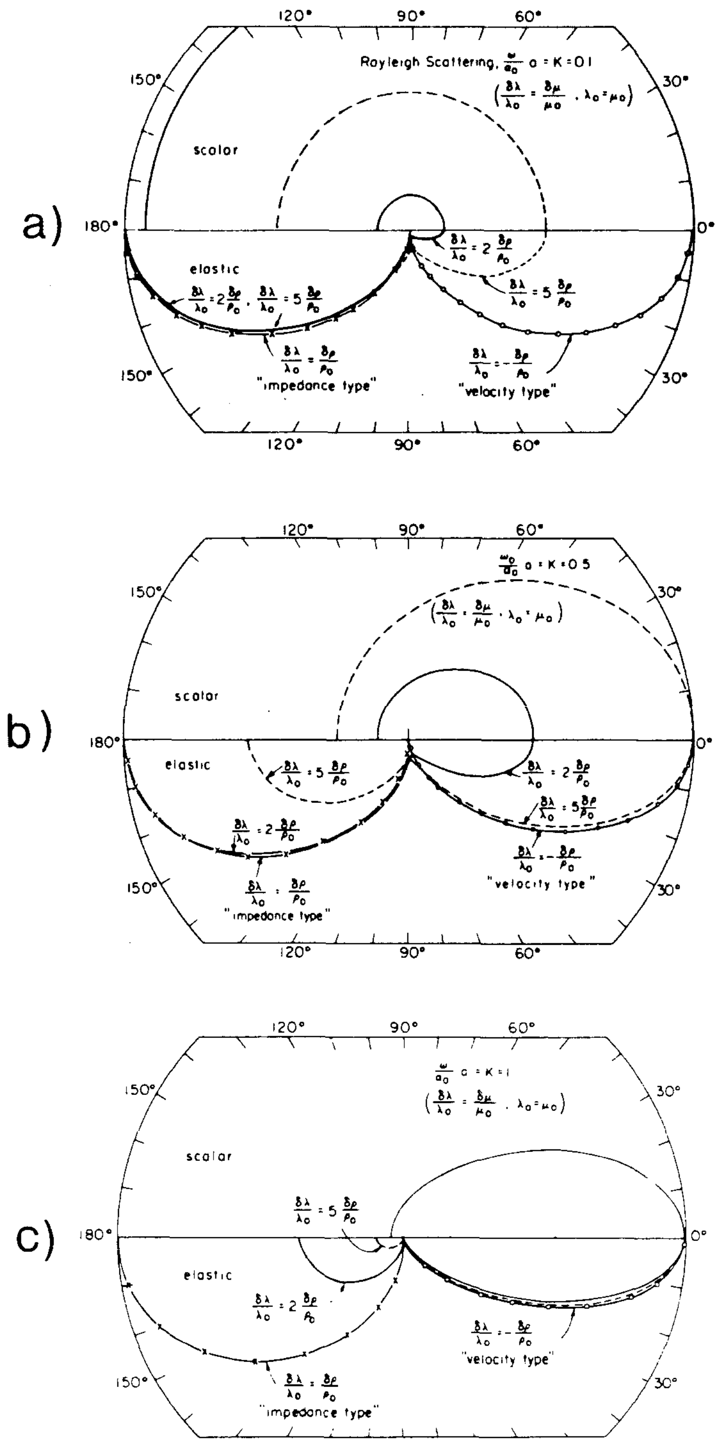
are constants depending on the scattering angle and the polarization directions.

The directional scattering coefficients can be found from (76) by relations similar to (65). In order to see the differences between the scalar wave and elastic wave scattering, we show the comparison between the scalar case and the P-P scattering of the elastic case in Figure 6. Figures 6a,b and c are for $(\omega/\alpha_0)a = 0.1$, 1 and 10, respectively. For Rayleigh scattering (Figure 6a) the scattering pattern of scalar wave is isotropic (see the upper half plane), whereas the pattern for elastic waves is quite unsymmetric depending on the parameter combinations (the lower half plane). When the wavelength becomes increasingly shorter than the scale length a, the scattered energy becomes incrementally more concentrated in the forward lobe, except for the case of "impedance type" scattering which always has a big back lobe.

The Sensitivity of Backscattering and Forescattering to Heterogeneities with Different Scales

From (73) and (74) we see that the back- and forescattering sense not only the different elastic properties of the medium, but also the different spectral components of the medium power spectrum. The forescattering feels only the d.c. component of the random medium $P(0)$. Therefore it is most sensitive to the large-scale heterogeneities. This can be understood physically. When the wave passes through a random medium, the accumulated phase change is mainly determined by the large-scale variations, the effects from the small ones tend to cancel each other. On the other hand, the backscattering only senses $P(2k)$, the spectral component of the medium at $K = 2k$ where k is the wave number of the incident wave. It means that only the Fourier component of the medium with spatial period equal to half the wavelength of the incident field can be detected by backscattering. Therefore, it is most sensitive to the heterogeneities with scale length near the wavelength. Assuming an exponential medium with $P(k)$ of (59), we see from (74) that, for S-S scattering,

$$
g^{ss}(\pi) \propto \begin{cases} a^3, & \dfrac{\omega}{\beta_0}a \ll 1 \\[2ex] \dfrac{1}{a}, & \dfrac{\omega}{\beta_0}a \gg 1 \end{cases}. \tag{78}
$$


a)
Rayleigh Scattering, $\frac{\omega}{a_0} a = K = 0.1$
$\left(\frac{\delta\lambda}{\lambda_0} = \frac{\delta\mu}{\mu_0}, \lambda_0 = \mu_0\right)$
scalar
elastic
$\frac{\delta\lambda}{\lambda_0} = 2\frac{\delta\rho}{\rho_0}$
$\frac{\delta\lambda}{\lambda_0} = 5\frac{\delta\rho}{\rho_0}$
$\frac{\delta\lambda}{\lambda_0} = 2\frac{\delta\rho}{\rho_0}, \frac{\delta\lambda}{\lambda_0} = 5\frac{\delta\rho}{\rho_0}$
$\frac{\delta\lambda}{\lambda_0} = \frac{\delta\rho}{\rho_0}$
"impedance type"
$\frac{\delta\lambda}{\lambda_0} = -\frac{\delta\rho}{\rho_0}$
"velocity type"
b)
$\frac{\omega_0}{a_0} a = K = 0.5$
$\left(\frac{\delta\lambda}{\lambda_0} = \frac{\delta\mu}{\mu_0}, \lambda_0 = \mu_0\right)$
scalar
elastic
$\frac{\delta\lambda}{\lambda_0} = 5\frac{\delta\rho}{\rho_0}$
$\frac{\delta\lambda}{\lambda_0} = 2\frac{\delta\rho}{\rho_0}$
$\frac{\delta\lambda}{\lambda_0} = 5\frac{\delta\rho}{\rho_0}$
$\frac{\delta\lambda}{\lambda_0} = 2\frac{\delta\rho}{\rho_0}$
$\frac{\delta\lambda}{\lambda_0} = -\frac{\delta\rho}{\rho_0}$
"velocity type"
$\frac{\delta\lambda}{\lambda_0} = \frac{\delta\rho}{\rho_0}$
"impedance type"
c)
$\frac{\omega}{a_0} a = K = 1$
$\left(\frac{\delta\lambda}{\lambda_0} = \frac{\delta\mu}{\mu_0}, \lambda_0 = \mu_0\right)$
scalar
elastic
$\frac{\delta\lambda}{\lambda_0} = 5\frac{\delta\rho}{\rho_0}$
$\frac{\delta\lambda}{\lambda_0} = 2\frac{\delta\rho}{\rho_0}$
$\frac{\delta\lambda}{\lambda_0} = -\frac{\delta\rho}{\rho_0}$
"velocity type"
$\frac{\delta\lambda}{\lambda_0} = \frac{\delta\rho}{\rho_0}$
"impedance type"
180°
150°
120°
90°
60°
30°
0°

Therefore there exists an a_{opt} that has maximum response to the detecting field. By differentiating (74) with respect to the correlation length a and equation to zero, we can get a_{opt}

$$\frac{\omega}{\beta_0} a_{opt} \approx \sqrt{\tfrac{3}{4}} = 0.87 \tag{79}$$

i.e.,

$$a_{opt} \approx 0.14\lambda.$$

From this analysis we can expect that backscattering experiments using *S-S* waves will have a maximum response to heterogeneities with scale length comparable to the wavelength. For the 1 Hz short period *S* waves, the corresponding correlation length of maximum response is around 0.5 km.

Figure 7 shows the responses of heterogeneities with different correlation lengths to 1-Hz *S* waves ($\beta_0 = 3.5$ km/s). The maximum is at $a \approx 0.48$ km. For heterogeneities with correlation length $a \approx 10$ km the backscattering response drops to 1/11.8 of the maximum value. However, these large-scale heterogeneities have much stronger responses to the forescattering.

The above analysis helps us to understand those seemingly contradictory observations in seismology. Through the measurements of phase and amplitude fluctuations across a large array such as LASA or NORSAR, and the travel time inversion for 3D velocity distributions in different areas around the world, we recognized the existence of meso-scale velocity inhomogeneities with scale lengths of a few tens of km and velocity perturbations of a few percents. In the mean time the coda waves from a local earthquake are believed to be the backscattered waves from the heterogeneities in the lithosphere. However, if we calculate the backscattering coefficient from the meso-scale velocity heterogeneities derived from the transmission fluctuation measurements and the travel-time inversions, which are forescattering experiments, it became too small to explain the observed coda strengths. The analysis on the sensitivity of fore- and backscattering to the scale lengths shows that the two kinds of observation may originate from heterogeneities of different scale lengths. The fluctuation measurements sense the *meso-scale velocity inhomogeneities* (the scale length detectable is limited by the array sizes), while the coda strengths are mainly determined by the *small-scale impedance heterogeneities* in the lithosphere of the local region. Therefore, the lithosphere must have multi-scale heterogeneities, at least 2 scales to account for the above-mentioned two kinds of observation (this two-scale model of lithosphere was first proposed by WU, 1984;

Figure 6
(a) Comparison of scalar wave scattering (upper half plane) and elastic wave scattering (lower half plane) for *P* wave incidence when $(\omega/\alpha_0)a = 0.1$; (b) same as in (a), with $(\omega/\alpha_0)a = 0.5$; (c) same as in (a), with $(\omega/\alpha_0)a = 1$.

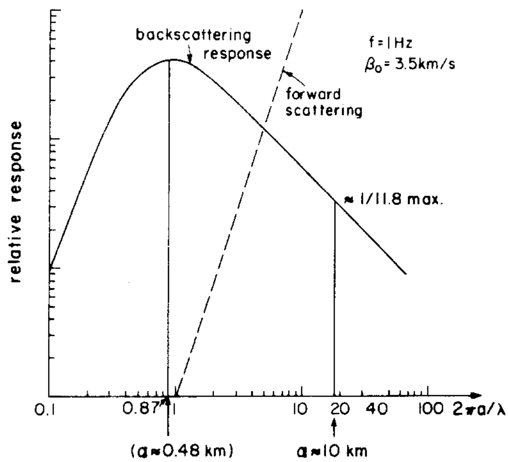


Figure 7
Backscattering responses of heterogeneities with different correlation lengths for 1-Hz wave.

WU and AKI, 1985b). Recent measurements on the angular coherence of transmitted waves using the NORSAR array (FLATTÉ and WU, 1988) suggest further that the two-scale heterogeneities may have different depth distributions. The small-scale ones are much shallower than the meso-scale ones; the meso-scale ones may begin from the middle of the crust and extend beyond the asthenosphere.

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