Pageoph, Vol. 115 (1977), Birkhäuser Verlag, Basel

Non-linear Laminar Flow into Well Partially Penetrating a Porous Aquifer of Finite Thickness

By K. K. UPADHYAY[1])

Abstract – In steady state condition, non-linear laminar flow of fluid into a well partially penetrating a porous aquifer of finite thickness is considered. The influence of such a flow on discharge and its dependence on related physical quantities are investigated. It is observed that the discharge into the well decreases as the depth of the well is decreased and the region of non-linear laminar flow is widened, which is quite obvious from physical considerations. As a particular case, result for a fully penetrating well has been deduced.

Key words: Hydrology; Non-linear flow; Aquifer.

1. Introduction

Although laminar flow exists in the producing formation of wells under normal flow rates, non-linear laminar and turbulent flow takes place in the vicinity of the well. As the flow rate increases the region which permits the flow to be non-linear laminar and turbulent becomes comparatively wider. Also, under open flow conditions in relatively deep wells a considerable portion of the flow through the sand is found to be different from laminar. Besides, the intricacy of the nature of porous media does not always justify the natural flow of fluid into porous medium to be purely laminar and it appears most justifiable to consider the flow to be either non-linear laminar or turbulent [1]. Consequently, JAIN and UPADHYAY [2], ELENBAAS and KATZ [3], ENGELUND [4] obtained specific solutions of certain non-linear laminar and turbulent flow problems.

In the present paper, we consider non-linear laminar flow into a well partially penetrating a porous aquifer of finite thickness. It is observed that the flow pattern is characterised by two different zones in which discharge is greater or less than that which would have been in case of purely laminar flow. Specifically, it is found that the discharge into the well decreases as the depth of the well is decreased and the region of non-linear laminar flow is widened. From physical considerations the findings appear to be quite obvious.

In particular, results for a fully penetrating well have been deduced and compared with those obtained by UPADHYAY [5].

---

[1]) 79 Bhagat Singh Marg, Dewas (M.P.), India.

2. *Basic equations of fluid flow in porous medium*

The laws governing laminar and non-linear laminar flow of fluid in porous media are [1]

$$v = -K\frac{dh}{ds} \tag{1}$$

and

$$\frac{dh}{ds} = av + bv^2 \tag{2}$$

respectively, where v, K, dh/ds denote seepage velocity, seepage coefficient, hydraulic gradient and the constants a and b, according to ENGELUND [4] are

$$a = \frac{2000\,\mu}{\rho g d^2}, \qquad b = \frac{35}{gd} \tag{3a, b}$$

In case of laminar flow, the expression for total head H at a radial distance r from the axis of well, penetrating a depth t in the porous aquifer of thickness T is obtainable in the form [6]

$$H = C + \frac{Q \log(r/R)}{2\pi Kt[1 + 7\sqrt{r/2t}\cos \pi\bar{T}/2]}, \quad \bar{T} = t/T \tag{4}$$

where Q and R denote the flow rate and radius of contour of intake respectively, C being constant to be determined with the help of boundary conditions.

3. *Statement of the problem*

In steady state condition, we consider radially symmetrical flow into a circular cylindrical well of radius r_w and depth t, penetrating a porous aquifer of finite thickness T. The aquifer is assumed to be homogeneous. The pressure at the contour of well and at the contour of intake are prescribed as p_w and p_c respectively.

If r be the radial distance measured from the axis of well, we assume that flow is (i) non-linear laminar within a cylindrical zone $r_w \leq r < r_t$, and (ii) laminar within $r_t < r \leq R$. Let p_t be the pressure at the transition boundary $r = r_t$ (Fig. 1).

The problem is to examine the influence of non-linear laminar flow on discharge and its dependence on related physical quantities.

4. *Solution*

Using (4), the expression for pressure distribution in laminar zone is obtained in the form

$$p = C' + \frac{Q\rho g \log(r/R)}{2\pi tK(1 + \beta\sqrt{r})}, \quad r_t < r \leq R \tag{5}$$

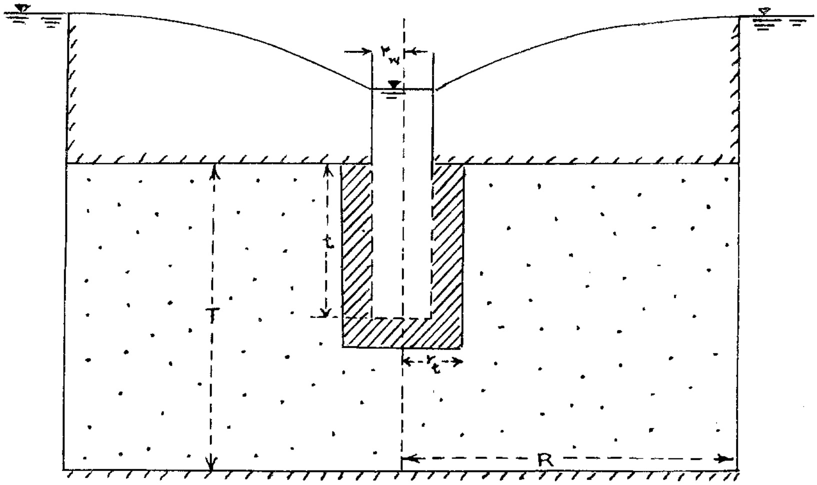

Figure 1
Geometry of the problem.

where $\beta = 7/\sqrt{2t}\cos(\pi\overline{T}/2)$ and constant C' is determined by the boundary conditions

$$\begin{aligned} p &= p_c \quad \text{at} \quad r = R, \\ &= p_t \quad \text{at} \quad r = r_t. \end{aligned} \tag{6}$$

Hence

$$p_t = p_c + \frac{Q\rho g \log(r_t/R)}{2\pi Kt(1 + \beta\sqrt{r_t})} \tag{7}$$

From (2) the expression for pressure distribution in non-linear laminar zone can be written as

$$\frac{1}{\rho g}\frac{dp}{dr} = av + bv^2, \quad r_w \leq r < r_t. \tag{8}$$

The seepage velocity v may be deduced from relation (4) in the form

$$v = \frac{Q}{2\pi t}\left[\frac{1 + \beta\sqrt{r}\,(1 - \log\sqrt{r/R})}{r(1 + \beta\sqrt{r})^2}\right], \tag{9}$$

and therefore equation (8) becomes

$$\frac{dp}{dr} = \frac{aQ}{2\pi t}\left\{\frac{1 + \beta\sqrt{r}\,(1 - \log\sqrt{r/R})}{r(1 + \beta\sqrt{r})^2}\right\} + \frac{bQ^2}{4\pi^2 t^2}\left\{\frac{1 + \beta\sqrt{r}\,(1 - \log\sqrt{r/R})}{r(1 + \beta\sqrt{r})^2}\right\}^2 \tag{10}$$

Integrating (10) and then evaluating the constant of integration with the help of boundary conditions

$$\begin{aligned} p &= p_w \quad \text{at} \quad r = r_w, \\ &= p_t \quad \text{at} \quad r = r_t, \end{aligned} \tag{11}$$

we obtain

$$p_t = \frac{aQ\rho g \log (r_t/R)}{2\pi t(1 + \beta\sqrt{r_t})} + \frac{bQ^2}{4\pi^2 t^2}\left[\int_{r_w}^{r_t}\left\{\frac{1 + \beta\sqrt{r}\,(1 - \log\sqrt{r/R})}{r(1 + \beta\sqrt{r})^2}\right\}^2 dr\right] \tag{12}$$

At the boundary of transition from laminar to non-linear laminar flow, the relation between critical Reynold's number $\xi_c = 0.07$ and critical velocity v_c is [4]

$$v_c = \frac{Q}{2\pi r}\left\{\frac{1 + \beta\sqrt{r_t}\,(1 - \log\sqrt{r_t/R})}{r_t(1 + \beta\sqrt{r_t})^2}\right\} = 0.07(a/b) \tag{13}$$

Since at this boundary dh/ds as given by (1) and (2) yield the same value, it follows from (13) that

$$1.07aK = 1. \tag{14}$$

Using (14) in (7) and then comparing with (12), we get

$$\begin{aligned} \frac{(p_c - p_w)}{\rho g} = \frac{aQ}{2\pi t}\left[\frac{\log (R/r_w)}{(1 + \beta\sqrt{r_w})} + \frac{0.07 \log (R/r_t)}{(1 + \beta\sqrt{r_t})}\right] \\ + \frac{bQ^2}{4\pi^2 t^2}\left[\int_{r_w}^{r_t}\left\{\frac{1 + \beta\sqrt{r}\,(1 - \log\sqrt{r/R})}{r(1 + \beta\sqrt{r})^2}\right\}^2 dr \end{aligned} \tag{15}$$

Combining equations (3a, b) and (13) with (15), we obtain

$$\begin{aligned} \frac{(p_c - p_w)\rho d^3}{\mu^2 r_w} = \frac{8000}{r_w}\cdot\frac{r_t(1 + \beta\sqrt{r_t})^2}{1 + \beta\sqrt{r_t}\,(1 - \log\sqrt{r_{t/R}})}\left[\frac{\log (R/r_w)}{(1 + \beta\sqrt{r_w})} + \frac{0.07 \log (R/r_t)}{(1 + \beta\sqrt{r_t})}\right. \\ \left. + \frac{0.07 r_t(1 + \beta\sqrt{r_t})^2}{1 + \beta\sqrt{r_t}\,(1 - \log\sqrt{r_{t/R}})}\int_{r_w}^{r_t}\left(\frac{1 + \beta\sqrt{r}\,(1 - \log\sqrt{r/R})}{r(1 + \beta\sqrt{r})^2}\right)^2 dr\right] \end{aligned} \tag{16}$$

If we assume purely laminar flow in the entire flow region $r_w \leq r \leq R$, then the flow rate Q_{lam} may be obtained from (4) by using the corresponding boundary conditions at the well and at the contour of intake. Hence

$$Q_{\text{lam}} = \frac{2\pi Kt(p_c - p_w)}{\rho g \log (R/r_w)}\cdot(1 + \beta\sqrt{r_w}) \tag{17}$$

Therefore, from (13) and (17), we get

$$\frac{Q}{Q_{\text{lam}}} = \frac{8560 \log (R/r_w)}{r_w}\left\{\frac{\mu^2 r_w}{(p_c - p_w)\rho d^3}\right\}\left\{\frac{r_t(1 + \beta\sqrt{r_t})^2}{1 + \beta\sqrt{r_t}\,(1 - \log\sqrt{r_{t/R}})}\right\} \tag{18}$$

Introducing dimensionless quantity X and ratio Y such that

$$X = \frac{\rho d^3(p_c - p_w)}{\mu^2 r_w}, \qquad Y = \frac{Q}{Q_{\text{lam}}} \tag{19a, b}$$

and combining (18) with (16), we obtain

$$1.07X = \frac{XY}{\log(R/r_w)}\left[\frac{\log(R/r_w)}{(1 + \beta\sqrt{r_w})} + \frac{0.07\log(R/r_t)}{(1 + \beta\sqrt{r_t})} + \frac{0.07XYr_w}{8560\log(R/r_w)}\int_{r_w}^{r_t}\left\{\frac{1 + \beta\sqrt{r}\,(1 - \log\sqrt{r/R})}{r(1 + \beta\sqrt{r})^2}\right\}^2 dr\right] \tag{20}$$

It may be inferred from (19a) that the value of X which is possible from physical considerations is $X > 0$, hence equation (20) becomes

$$1.07\log(R/r_w) = Y\left[\frac{\log(R(r_w)}{(1 + \beta\sqrt{r_w})} + \frac{0.07\log(R/r_t)}{(1 + \beta\sqrt{r_t})} + \frac{0.07XYr_w}{8560\log(R/r_w)}\int_{r_w}^{r_t}\left\{\frac{1 + \beta\sqrt{r}\,(1 - \log\sqrt{r/R})}{r(1 + \beta\sqrt{r})^2}\right\}^2 dr\right] \tag{21}$$

This is an integral equation corresponding to non-linear laminar flow of fluid into a well partially penetrating a homogeneous porous aquifer of finite depth.

Since the exact solution of equation (21) is not possible, therefore, we consider small values of β such that $t \to T$ and $(\beta\sqrt{r})_{\max} < 1$. In this situation, as a first approximation, equation (21) takes the form

$$1.07\log(R/r_w) = Y\left[\frac{\log(R/r_w)}{(1 + \beta\sqrt{r_w})} + \frac{0.07\log(R/r_t)}{(1 + \beta\sqrt{r_t})} + \frac{0.07XY}{8560\log(R/r_w)}\left\{\left(1 - \frac{r_w}{r_t}\right) + 8\beta\sqrt{r_w}\left(\sqrt{\frac{r_w}{r_t}} - 1\right) + 2\beta\sqrt{r_w}\left(\log\frac{R}{r_w} - \sqrt{\frac{r_w}{r_t}}\log\frac{R}{r_t}\right)\right\}\right], \tag{22}$$

which corresponds to the case of a partially penetrating well approaching to a fully penetrating well.

5. *Particular case*

For $\beta = 0$, equation (21) when combined with equation (18) gives

$$1.07\log(R/r_w) = Y\left[1.07\log(R/r_w) - 0.07\log\left\{\frac{XY}{8560\log(R/r_w)}\right\} + \frac{0.07XY}{8560\log(R/r_w)} - 0.07\right], \tag{23}$$

which represents the case of non-linear laminar flow of fluid into a fully penetrating well as obtained by Upadhyay [5]. It can also be derived from equation (22).

6. *Discussion*

From (19a), it is evident that X depends on the density of the fluid, grain size of the medium, pressure difference of the system, viscosity of the fluid and well radius. Since d and μ occur in higher powers in the expression for X, they highly affect the discharge. Moreover, from physical considerations it is obvious that X and Y are both positive.

Now, to get the definite idea of the result (22), we take $R/r_w = 3 \times 10^3$, that is the radius of the contour of intake is 3000 times the radius of contour of the well. Further, considering $r_t = 15$ cm, $r_w = 10$ cm and $\beta = 0.00247$ (i.e., $t = 0.99T$), numerical values of Y are obtained corresponding to different values of $X > 0$ and have been graphically represented in Fig. 2.

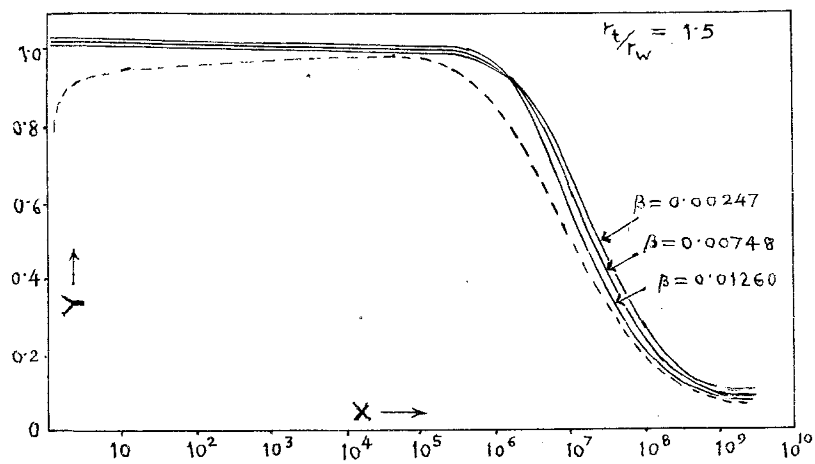

Figure 2
Discharge relative to laminar flow (Y) as a function of normalized pressure drop (X) for fixed β.

From Fig. 2 it is seen that as X increases, Y decreases asymptotically. The value $X = 2.5381 \times 10^5$ corresponds to the critical value $Y = 1$, at which flow pattern is characterised by two different zones $X \langle\text{or}\rangle\ 2.5381 \times 10^5$ in which the ratio Q/Q_{lam} assumes values greater or less than unity. It is inferred that discharge decreases as the density of the fluid, grain size of the medium and pressure difference of the system increase, viscosity of the fluid and the well radius decreases.

7. *Effect of increasing the depth of well*

To examine as to how the depth of the well affects the influence of non-linear laminar flow, we consider the cases $\beta = 0.00748, 0.01260$ corresponding to $t/T = 0.970, 0.950$. These cases have been graphically represented in Fig. 2. Therefrom, it is inferred that the influence of a non-linear laminar flow decreases and consequently the discharge increases as the depth of the well increases. The dotted curve represents the case $\beta = 0$ which corresponds to a fully penetrating well in which discharge from bottom ceases.

8. *Effect of widening the non-linear laminar zone*

The influence of widening the non-linear laminar region around the well is studied. The cases $r_t/r_w = 1.5, 2.5$ have been considered and represented graphically in Fig. 3. From these it is observed that widening of the non-linear laminar region reduces the discharge. From physical considerations the results appear to be quite obvious.

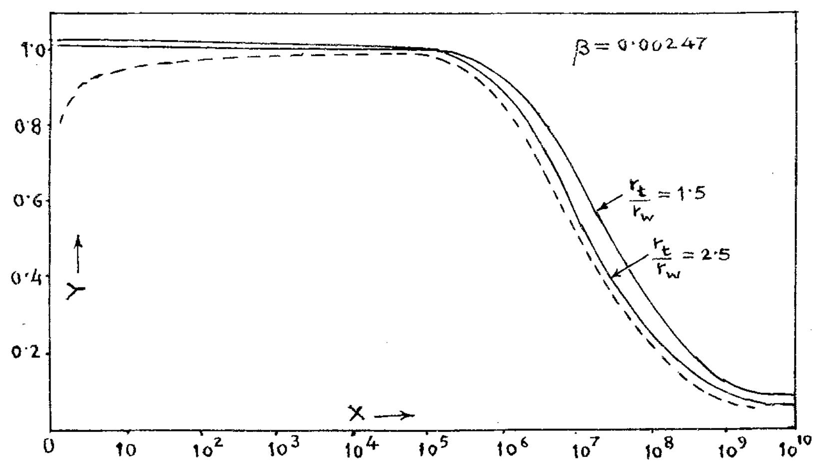

Figure 3
Discharge relative to laminar flow (Y) as a function of normalized pressure drop (X) for fixed r_t/r_w.

Acknowledgement

The author gratefully acknowledges the valuable guidance of Dr S. K. Jain, Department of Applied Mathematics, Shri G.S. Institute of Technology and Science, Indore in the preparation of this paper.

REFERENCES

[1] BEAR, J., *Dynamics of fluids in porous media* (American Elsevier Publishing Co. Inc., New York 1972).
[2] JAIN, S. K. and UPADHYAY, K. K. (1977)., *Non-linear laminar flow of fluid into a partially penetrating well,* Gerlands Beiträge Zur Geophysik, *85* (1976) 2, 151–156.
[3] ELENBAAS, J. R. and KATZ, D. L. (1948)., *A radial turbulent flow formula*, Trans. AIME, *174*, 25.
[4] ENGELUND, F. (1953), *On laminar and turbulent flow of ground water in homogeneous sand,* Trans. Dan. Acad. Tech. Sci. *3*, 1–105.
[5] UPADHYAY, K. K. (1977), *Non-linear laminar flow of fluid into a fully penetrating cylindrical well*, Ind. J. Theo. Phys. (To be published).
[6] HARR, M. E., *Ground Water and Seepage* (McGraw-Hill Book Co., New York 1962).

(Received 5th August 1976)