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Love Waves in Slowly Varying Layered Media

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Summary – The problem of Love waves propagating in slowly varying layered media with a general geometry is solved by using the method of multiple scales. A first-order uniformly valid solution is obtained for the modulation of the amplitude as a function of the scale $x_1 = \varepsilon x$, where ε is a measure of the amplitude of the geometrical variation of the layer. This solution is particularly suited for computational procedures.

Introduction

The problem of surface waves propagating in layered media has been studied extensively [1]. Since the analysis of the wave propagation in the general case of layers with an arbitrary surface shape is very complex, the problem has been attacked under various assumptions by several authors.

The propagation of high-frequency guided elastic waves near curved surfaces and in layers of non-constant thickness has been investigated by RULF, ROBINSON and ROSENAU [2] who adapted an asymptotic method first introduced by KELLER [3]; high-frequency Love waves have also been treated by SMITH [4]. Some investigators have obtained solutions for Love waves in the case of special variations in the layer thickness: SATO [5] and KNOPOFF and HUDSON [6] analyzed the layer with a step change, while TAKAHASHI [7] considered a layer varying hyperbolically, HOMMA [8] one varying linearly and DE NOYER [9] one varying sinusoidally. WOLF [10] studied the scattering of Love waves in a layer with a slightly irregular free-surface lying on a plane elastic half-space and obtained the first-order solution in terms of an infinite series by means of contour integration and perturbation expansion. SLAVIN and WOLF [11] presented a method using a least squares procedure to approximate the scattering of Love waves in a surface layer with an irregular boundary for the case of a rigid underlying half-space.

In this study an asymptotic solution is presented to the problem of Love waves propagating in layers slowly varying both at the free surface and the interface with the underlying elastic half-space, which appears to be a more realistic model. A first-

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order solution is obtained by the method of multiple scales [12]; a main feature of the method is that in order to find the field quantities of one order it is necessary to proceed to the next order by making use of the 'integrability' condition which provides a condition of consistency on the lower order field. The dependence of the amplitude on the irregularities is derived from this condition. An advantage of the method presented here is that it yields the solution in a form suitable for numerical applications.

Analysis

Consider the problem of Love waves, that is SH-waves guided by an elastic layer of thickness h (positive number) lying on an elastic half-space (Fig. 1). The layer thickness is assumed slowly varying, that is the free surface is given by:

$$F_1(x, z) = z + h - \varepsilon f_1(x) = 0 \tag{1}$$

and the interface by:

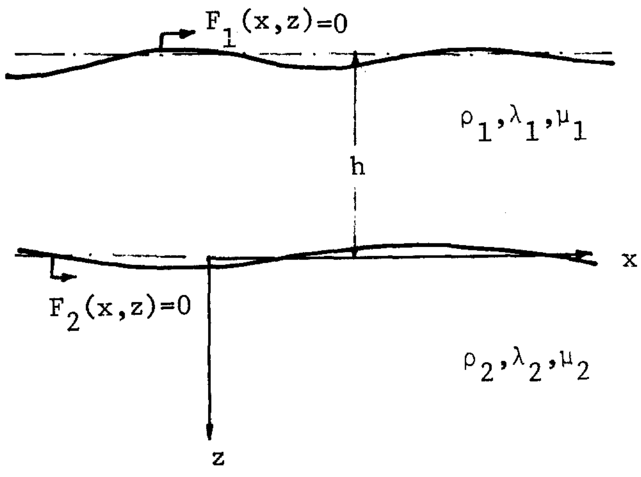
$$F_2(x, z) = z - \varepsilon f_2(x) = 0 \tag{2}$$

where ε is a small parameter and it is assumed that the functions $f_1(x)$ and $f_2(x) \varepsilon C^1$; that is, the assumptions made here are that the irregularities are smooth and their amplitude is small compared to the layer thickness.

The equations of motion of linear elasticity for SH-waves, that is when $\mathbf{u} = (0, u_y, 0)$, $\mathbf{u}$ being the displacement vector, reduce to:

$$\frac{\partial^2 u_i}{\partial x^2} + \frac{\partial^2 u_i}{\partial z^2} = \frac{1}{c_i^2}\frac{\partial^2 u_i}{\partial t^2} \qquad (i = 1, 2) \tag{3}$$

where $u_i = u_{y_i}$, the index i referring to the region R_i. The region R_i is defined by $-h + \varepsilon f_1(x) < z < \varepsilon f_2(x)$ and is occupied by an elastic material of density ρ_1, Lamé



constants μ_1, λ_1 and shear-wave velocity c_1. The region R_2 defined by $\varepsilon f_2(x) < z < \infty$ is occupied by an elastic material of density ρ_2, Lamé constants μ_2, λ_2 and shear-wave velocity c_2.

The boundary conditions are traction-free on the free surface and continuous traction and displacement at the interface; moreover the field must decay as $z \to \infty$. Thus, on $F_1(x, z) = 0$:

$$\mathbf{t}^n = 0 \qquad \text{or} \qquad n_{k_1}\tau_{kl_1} = 0 \tag{4}$$

where τ_{kl_i} $(k, l = 1, 3; i = 1, 2)$ is the stress tensor in the region R_i and n_{k_i} are the components of the vector normal to the surface $F_i(x, z) = 0$, which is given by:

$$\mathbf{n}_i = \frac{\nabla F_i}{|\nabla F_i|} = -\varepsilon \frac{df_i(x)}{dx}\mathbf{i} + \mathbf{k} \tag{5}$$

$\mathbf{i}$, $\mathbf{k}$ being the unit vectors along the axes x and z respectively, and on

$$F_2(x, z) = 0: \begin{cases} u_1 = u_2 \\ n_{k_2}\tau_{kl_1} = n_{k_2}\tau_{kl_2}. \end{cases} \tag{6}$$

By making use of (5) and of the stress-strain and strain-displacement relations the boundary conditions (4) and (6) are expressed respectively by:

$$-\varepsilon \frac{df_1}{dx}\frac{\partial u_1}{\partial x} + \frac{\partial u_1}{\partial z} = 0 \tag{7}$$

and

$$\begin{cases} u_1 = u_2 \\ -\varepsilon \dfrac{df_2}{dx}\dfrac{\partial u_1}{dx} + \dfrac{\partial u_1}{\partial z} = \left[-\varepsilon \dfrac{df_2(x)}{dx}\dfrac{\partial u_z}{\partial x} + \dfrac{\partial u_2}{\partial z}\right]\dfrac{\mu_2}{\mu_1}. \end{cases} \tag{8}$$

The new variables

$$x_1 = \varepsilon x, \qquad \theta = k(x_1)x - \omega t, \qquad z = z \tag{9}$$

are introduced, where ω is the circular frequency and k the wave number, and the solution is sought in the form:

$$u_i(x, z, t; \varepsilon) = [u_{0_i}(x_1, z) + \varepsilon u_{1_i}(x_1, z)]e^{i\theta} + c.c + 0(\varepsilon^2) \tag{10}$$

c.c. standing for the complex conjugate of the preceding terms and 0( ) being the Landau symbol [12]. With the relations for the derivatives:

$$\frac{\partial(\ )}{\partial t} = -\omega\frac{\partial(\ )}{\partial\theta}, \qquad \frac{\partial^2(\ )}{\partial t^2} = \omega^2\frac{\partial^2(\ )}{\partial\theta^2}$$

$$\frac{\partial(\ )}{\partial x} = k\frac{\partial(\ )}{\partial\theta} + \varepsilon\frac{\partial(\ )}{\partial x_1}, \qquad \frac{\partial^2}{\partial x^2} = k^2\frac{\partial^2(\ )}{\partial\theta^2} + \varepsilon\left\{2k\frac{\partial^2(\ )}{\partial\theta\partial x_1} + \frac{dk}{dx_1}\frac{\partial(\ )}{\partial\theta}\right\} \tag{11}$$

equations (3) and boundary conditions (7) and (8) respectively yield:
in R_i:

$$0(\varepsilon^0): u''_{0_i} + \left[\frac{\omega^2}{c_i^2} - k^2\right]u_{0_i} = 0 \tag{12}$$

$$0(\varepsilon^1):\ u''_{1_i} + \left[\frac{\omega^2}{c_i^2} - k^2\right] u_{1_i} = i\left\{2k\frac{\partial u_{0_i}}{\partial x_1} + \frac{dk}{dx_1} u_{0_i}\right\}. \tag{13}$$

On $F_1(x, z) = 0$:

$$0(\varepsilon^0):\ u'_{0_1} = 0 \tag{14}$$

$$0(\varepsilon^1):\ u'_{1_1} - ik\frac{df_1}{dx} u_{0_1} = 0. \tag{15}$$

On $F_2(x, z) = 0$:

$$0(\varepsilon^0):\ u_{0_1} - u_{0_2} = 0 \tag{16}$$

$$u'_{0_1} - \frac{\mu_2}{\mu_1} u'_{0_2} = 0 \tag{17}$$

$$0(\varepsilon^1):\ u_{1_1} - u_{1_2} = 0 \tag{18}$$

$$u'_{1_1} - \frac{\mu_2}{\mu_1} u'_{1_2} = ik\left\{\frac{df_1}{dx} u_{0_1} - \frac{\mu_2}{\mu_1}\frac{df_2}{dx} u_{0_2}\right\} \tag{19}$$

where $(\ )' = \dfrac{\partial(\ )}{\partial z}$.

It may be noticed that the validity of equation (13) lies on the assumption that $k \sim 0(1)$ and that of equation (19) on df_1/dx, $df_2/dx \sim 0(1)$.

To render the asymptotic expansion uniform to $0(\varepsilon^1)$ the 'integrability' condition is applied [13], which expresses the requirement that the inhomogeneous solution of the $0(\varepsilon^1)$ problem be orthogonal to every solution of the adjoint homogeneous $0(\varepsilon^0)$ problem, the two problems having the same self-adjoint operator $L = d^2/dz^2 + \omega^2/c_i^2 - k^2$. Thus equation (13) is multiplied by u_{0_i} and integrated by parts along the z axis in the region R_i to yield:

$$i\int_{-h+\varepsilon f_1(x)}^{\varepsilon f_2(x)} \left(2k\frac{\partial u_{0_1}}{\partial x_1} u_{0_1} + \frac{dk}{dx_1} u_{0_1}^2\right) dz = [u_{0_1} u'_{1_1} - u'_{0_1} u_{1_1}]_{-h+\varepsilon f_1(x)}^{\varepsilon f_2(x)} \tag{20}$$

$$i\int_{\varepsilon f_2(x)}^{\infty} \left(2k\frac{\partial u_{0_2}}{\partial x_1} u_{0_2} + \frac{dk}{dx_1} u_{0_2}^2\right) dz = -(u_{0_2} u'_{1_2} - u'_{0_2} u_{1_2})|_{\varepsilon f_2(x)} \tag{21}$$

where use has been made of (12) and of the requirement that all field quantities vanish at infinity.

To eliminate all the $0(\varepsilon^1)$ quantities from equations (20) and (21), equation (21) is multiplied by μ_2/μ_1 and added to (20) to yield the 'integrability' condition:

$$\int_{-h+\delta f_1(x)}^{\varepsilon f_1(x)} \left(2k\frac{\partial u_{0_1}}{\partial x_1} u_{0_1} + \frac{dk}{dx_1} u_{0_1}^2\right) dz + \frac{\mu_2}{\mu_1}\int_{\varepsilon f_2(x)}^{\infty} \left(2k\frac{\partial u_{0_2}}{\partial x_1} u_{0_2} + \frac{dk}{dx_1} u_{0_2}^2\right) dz$$
$$= -k\frac{df_1}{dx} u_{0_1\,-h+\varepsilon f_1(x)}^2 + k\left(\frac{df_1}{dx} u_{0_1} - \frac{\mu_2}{\mu_1}\frac{df_2}{dx} u_{0_2}\right) u_0|_{\varepsilon f_1(x)}. \tag{23}$$

If the solution of the $0(\varepsilon^0)$ problem is expressed in the form

$$u_{0_i} = A(x_1)\bar{u}_i(x_1, z) \tag{24}$$

then $\bar{u}_i$ satisfies the equation of the $0(\varepsilon^0)$ problem ((12), (14), (16), (17)). The solution of this eigenvalue problem is obtained depending parametrically on x_1, through the boundary conditions applied on F_1 and F_2. Introducing (24) into (23) yields:

$$\frac{dA}{dx_1} + \frac{H(x_1, z)}{G(x_1, z)} A = 0 \tag{25}$$

where

$$H(x, z) = 2k \left[\int_{-h+\varepsilon f_1(x)_1}^{\varepsilon f_1(x_1)} \bar{u}_1 \frac{\partial \bar{u}_1}{\partial x_1} dz + \frac{\mu_2}{\mu_1} \int_{\varepsilon f_2(x_1)}^{\infty} \bar{u}_2 \frac{\partial \bar{u}_2}{\partial x_2} dz \right]$$

$$+ \frac{dk}{dx_1} \left[\int_{-h+\varepsilon f_1(x)}^{\varepsilon f_2(x)} \bar{u}_1^2 dz + \frac{\mu_2}{\mu_1} \int_{\varepsilon f_2(x_1)}^{\infty} \bar{u}_2^2 dz \right]$$

$$+ \frac{df_1}{dx} k \, \bar{u}_1^2 \bigg|_{-h+\varepsilon f_1(x)} - k \, \bar{u}_1 \left(\bar{u}_1 \frac{df_1}{dx} - \frac{\mu_2}{\mu_1} \bar{u}_2 \frac{df_2}{dx} \right|_{\varepsilon f_2(x)}$$

and

$$G(x_1, z) = 2k \left[\int_{-h+\varepsilon f_1(x)}^{\varepsilon f_2(x)} \bar{u}_1^2 dz + \frac{\mu_2}{\mu_1} \int_{\varepsilon f_2(x)}^{\infty} \bar{u}_2^2 dz \right]. \tag{26}$$

The functions $H(x_1, z)$ and $G(x_1, z)$ are determined once $\bar{u}_i(x_1, z)$ has been obtained in the previous step. The solution of (25) is given by

$$A(x_1) = A_0 \exp \left[-\int_{x_{10}}^{x_1} \frac{H(x_1)}{G(x_1)} dx_1 \right]. \tag{27}$$

The amplitude modulation with x_1 has thus been determined up to a constant A_0 which will be specified given the wave amplitude at some x_{1_0} station.

Conclusions

A uniformly valid asymptotic solution has been presented for the modulation of SH-waves travelling in slowly varying wave-guides of any geometry. The restrictions imposed are that the amplitude of the irregularities should be small compared to the layer thickness, that they possess continuous first derivatives and that $k\varepsilon \ll 1$, that is, this method would not be appropriate for very short wave lengths. It has the advantage that it is particularly suited for practical applications since a computational procedure can be easily applied to a problem of specific geometry.

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