A Branching Model for Crack Propagation

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Summary – A branching model for crack propagation is proposed, a 'branch' corresponding to an existing microfissure or flaw in the material, and the propagation of the crack to the coalescence of such branches. Increase in external stress increases the probability that a given branch will link into more than a specified number of further branches. Such increases can continue until a critical state is reached when the mean number of branches linking into a given branch is equal to unity; beyond this point, the system becomes unstable, and any slight movement is likely to lead to catastrophic rupture. The distribution of the sums of the lengths of the branches linked together in a cracking episode is investigated, and shown to lead, in the critical case, to a Gutenberg-Richter type relation with parameter $b = 0.75$. Departures from this value are attributed to the influence of the distribution of the lengths of preexisting fissures, this distribution varying with the strength of the material and its stress history. Some difficulties with the theoretical model of Scholz are raised, and it is suggested that a more complete analysis of Scholz's model should lead to results qualitatively similar to those obtained for the branching model.

1. Introduction

Most theoretical studies of crack propagation refer to the growth of a single crack in a homogeneous medium. However, experimental studies of rock fracture, particularly those of HOEK and BIENIAWSKI (1965), have suggested that, at least in the final rupture stage, the dominating mechanism may be that of coalescence of existing cracks. More recently, studies of 'dilatancy' by Brace, Scholz and other writers confirm the impression that microfracturing plays a central role in determining the sequence of physical changes leading up to rupture. SCHOLZ (1968a,b) has also studied the relationship of these changes to changes in the energy distribution of microfractures. This work led him to postulate a certain theoretical model for crack propagation; starting from the statistical distribution of microscopic variations in strength and stress, he obtained a qualitatively attractive formula for the dependence of the b-value (the parameter in the Gutenberg-Richter relation) on the applied stress. The main purpose of the present note is to propose an alternative model for crack propagation, based on the notion of crack coalescence referred to above. This model also leads to a version of the Gutenberg-Richter law, and to some qualitative inferences concerning the variation of b-value with applied stress, but not to a direct dependence of b-value on stress. In

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section 6 of the paper I shall put forward some grounds for supposing that Scholz's formula is not soundly based, and that a more rigorous formulation of his model would lead to conclusions closer to those in the present paper.

As a starting point for the discussion, consider the stress distribution round the tip of an existing crack in a material under uniaxial compressive stress. BRACE and BOMBOLAKIS (1963) showed that even in a homogeneous material such as glass, such cracks will not in general propagate catastrophically, but only to a limited extent, turning away from the direction where they are subject to maximum stress concentration at the tip, towards a closer alignment with the direction of the applied stress. If it is accepted that this is likely to be a general property of crack propagation under compressive stress, two further features need to be incorporated before discussing a highly fractured, heterogeneous material such as rock. Firstly, we should take into account that not only the orientation of the crack, but also the strength of the material in the vicinity of the tip, and the extent to which the given crack is shielded from the applied stress by other cracks, will vary from crack to crack. Thus it is to be expected that some fractures will extend further and at lower applied stresses than others, so that a statistical treatment is needed (indeed a similar argument forms the basis of Scholz's statistical treatment). Secondly, one should incorporate the possibility that the initial extension so produced may encounter further cracks or flaws in the material. It is this latter feature which will play the key role in our discussion. We may idealize it by treating it as a 'Branching Process' of the same general type used to describe the progress of a nuclear reaction in fissile material, and in many other contexts (a comprehensive account of the range of applications and variants of the model is contained in the monograph by HARRIS (1963)).

To specify the evolution of such a process – to be interpreted as an episode of crack coalescence – two component distributions are needed. The first of these, the 'offspring distribution,' to be denoted by p_n; $n = 0, 1, \ldots$, will be interpreted as the number of further cracks which may be encountered during the initial period of extension when a given crack is subject to an increase in the applied stress. We shall assume that the tail probabilities $t_n = p_n + p_{n+1} + \cdots$ of linking into more than a given number n of further cracks are increasing functions of the applied stress, other properties being held constant. An important corollary of this assumption is that the mean number of such 'offspring' cracks, which is a key parameter for the discussion, also increases with the applied stress. This follows from the fact that the mean can also be written in the form

$$\nu = \Sigma n p_n = \Sigma t_n.$$

The second distribution of importance concerns the lengths of the individual branches. We envisage the process of extension and coalescence as continuing through a series of movements, each of which consists in principle of two components, an initial extension from the tip of one crack, and the activation of the further crack reached by this initial extension. Of the two lengths, we presume it is the latter which will normally predominate. If this is so, the distribution of branch lengths should reflect the distribution of lengths of existing cracks when the process of coalescence is initiated. In any case we shall denote by $F(x)$ the cumulative distribution function for the branch lengths, so that $F(x) = \text{Prob (branch length} \leq x)$.

For simplicity, we shall also assume that the numbers and lengths of the branches are mutually independent, and that the distributions $\{p_n\}$, $F(x)$ are constant over the whole region considered. Both assumptions are clearly approximations; the former is unlikely to critically influence the results unless the dependence is a very strong feature; the latter (in effect the assumption of statistical homogeneity) limits the range of validity of the discussion – we should not expect our conclusions to hold on a scale larger than that for which homogeneity was a reasonable assumption.

The quantity with which we shall be concerned particularly is the sum of the lengths of all the branches which coalesce to form the new crack. We shall assume that this total length is proportional to the total energy emitted during the coalescence. The appropriateness of this assumption may well be queried in view of the variety of relationships which, depending on the shape of the crack and the applied field, may hold between the dimensions of a crack and the energy emitted during its formation even in a homogeneous medium. The assumption of proportionality asserts, in effect, that on average, taken over all possible configurations and orientations, a unit increase in the total length activated will produce a given increase in average energy emitted. In any case some simple assumption governing the relationship between crack-length and energy will be needed. Even if direct proportionality were replaced by proportionality between energy and the αth power of the length, the net effect would not be worse than a multiplication of all b-values obtained by the constant α. However it is precisely the choice $\alpha = 1$ which seems to lead to conclusions in best agreement with the observational evidence.

2. *Some properties of the branching process*

For the sake of completeness, and to establish results and notation for the following section, we sketch here the derivation of the basic formulae for the distribution of the total length.

We assume that the process of crack coalescence starts from an initial crack of length L_1, and denote by $L_2, L_3, \ldots$ the sums of the lengths of the cracks up to $2, 3, \ldots$ links away from the initial crack. Thus L_2 denotes the sum of L_1 and the lengths of any cracks directly linking into the initial crack; generally L_{k+1} can be represented as L_k augmented by the sum of the lengths of any cracks linking into those of the preceding (kth) stage. Clearly $L_1 \leq L_2 \leq L_3 \leq \cdots$. If the crack is finite the L_k reach a terminating value and then cease to increase; but mathematically it is conceivable that they increase indefinitely. The quantities $L_1, L_2, \ldots$, are random variables in general, and we shall denote by $F_1(x), F_2(x), \ldots$ the corresponding distribution functions. The monotonicity of the sequence of random variables L_k is reflected in the corresponding property of their distribution functions; for every fixed x we have $F_1(x) \geq F_2(x) \geq F_3(x) \cdots$. Consequently, as $k \to \infty$, the values $F_k(x)$ converge to a certain limit function $G(x)$. If the total length is finite with probability one, the limit function will be a proper distribution function ($\lim_{x \to \infty} G(x) = 1$) and will represent the distribution of the total length. If there is a non-zero probability that the total length is infinite, the limit distribution function will be improper, the value $\eta = \lim_{x \to \infty} G(x)$ will represent the probability that the total crack length is finite, and the deficiency $1 - \eta$ will represent the probability that the crack propagates indefinitely.

Further analysis is most easily carried through in terms of the Laplace (Stieltjes) transforms

$$f_n^*(s) = \int e^{-sx} dF_n(x) = E(e^{-sL_n}).$$

Suppose for simplicity that the initial crack has the basic crack length distribution $F(x)$ (so that $F_1(x) = F(x), f_1^*(s) = f^*(s)$). We can write

$$L_{k+1} = L_1 + \sum_{i=1}^{N_1} L_k^{(i)}$$

where N_1 is the number of cracks directly linked to the initial crack and the $L_k^{(i)}$ are the k-step lengths of the crack-systems emanating from those cracks directly linked to the first crack. Using the independence and homogeneity assumptions, each of these latter has the same distribution as L_k, viz., $F_k(x)$. Then from the independence again, and the multiplication property for the Laplace transform of a sum of independent variables, we obtain

$$f_{k+1}^*(s) = f^*(s)[f_k^*(s)]^{N_1}.$$

The RHS has, however, to be averaged over the different possible values of N_1 (which also is to be treated as a random variable). If we introduce the generating function

$$P(z) = \sum p_n z^n = E(z^N)$$

for the number of cracks linking into a given crack, the averaging process yields the recursive equation

$$f_{k+1}^*(s) = f^*(s) P(f_k^*(s)). \tag{1}$$

Furthermore, as $k \to \infty$, (note again there is a monotonic convergence) we find for the Laplace transform $g^*(s)$ of the limit function $G(x) = \lim F_k(x)$,

$$g^*(s) = f^*(s) P(g^*(s)). \tag{2}$$

This equation for $g^*(s)$ is the basis of the further analysis.

Now equation (1) shows that the $f_k^*(s)$ are the successive approximations which are obtained in finding an iterative solution of the functional equation

$$x = zP(x) \tag{3}$$

in which $z = f^*(s)$ and the approximations are started from the trial value $x_0 = 1$. As before, the successive iterates are monotonic decreasing. The resulting solution, say $x = \Gamma(z)$, of (3) has an interpretation in its own right as the generating function of the total *number* of cracks linked together in the process of coalescence; this will be finite if and only if the total length is finite. Then the solution to (1) can be written in the form

$$g^*(s) = \Gamma[f^*(s)], \tag{4}$$

the interpretation of this equation being that the total length is the sum of a random number of independent lengths each with Laplace transform $f^*(s)$, the number in the sum having probability generating function $\Gamma(z)$.

The nature of the solutions to (2) and (3) can be determined by examining the graph of the function $y = P(x)$. The critical parameter is the value of the derivative $P'(1) = \Sigma np_n = \nu$, the mean number of cracks linking into a given initial crack. If $\nu < 1$ (subcritical case) or $\nu = 1$ (critical case), the situation is as shown in Fig. 1. The curve $y = P(x)$ lies everywhere above the line $y = x$ for $0 \leq x < 1$, and intersects the line $y = x/z$ $(0 < z < 1)$ at a unique point $x = \Gamma(z)$ in the range $0 < x < 1$. Furthermore as $z \to 1$ it is clear that $\Gamma(z) \to 1$, so that the corresponding distribution is proper. Thus in the subcritical and critical cases the total length is finite with probability 1.

In the supercritical case, (Fig. 2), the curve $y = P(x)$, which is convex downwards, continuous, and satisfies $P(0) = p_0 \geq 0$, must cut the line $y = x$ not only at $x = 1$ but also at a smaller value $x = \eta < 1$. $\Gamma(z)$ now lies in the range $0 \leq x < \eta$, and as $z \to 1$ we shall have $\Gamma(z) \to \eta$. We deduce that in the supercritical case, there is a positive probability $1 - \eta$ that the crack will propagate indefinitely.

We have assumed that the effect of increasing the applied stress will be to increase the parameter ν. As the stress is increased, therefore, we expect the system to pass through the subcritical stage and into the critical state. Once it passed beyond the critical stage however, any movement would have a non-zero probability of propagating indefinitely. Such an outcome would correspond in the physical system to the formation of a fracture extending beyond the region for which the initial assumptions

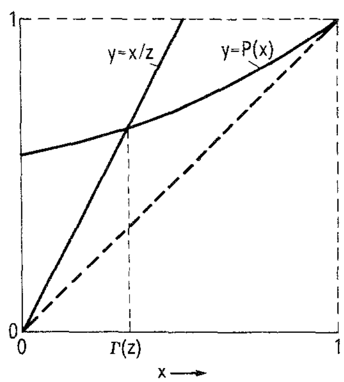

Figure 1
Solution to $x = zP(x)$ in subcritical case.

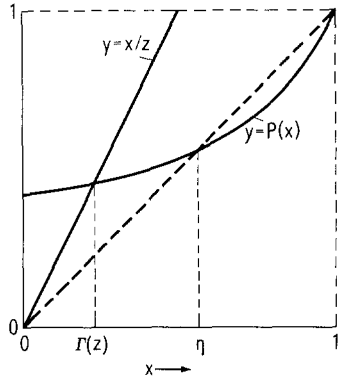

Figure 2
Solution to $x = zP(x)$ in supercritical case.

could be considered valid. In the laboratory context it seems reasonable to identify such an event with the final rupture of the specimen. On the tectonic scale the distinction between major and minor events may not be so clear-cut. It is tempting to suggest that the geological features or stress inhomogeneities terminating a major episode of crack propagation on one scale of magnitude could serve themselves as the microscopic features governing crack propagation on a larger scale, the ultimate limit of this process being imposed by the finiteness of the earth's surface. It also seems reasonable to suppose that the majority of tectonic earthquakes occur in critical or near-critical conditions, on the simple ground that significant cracking episodes will be infrequent until the critical stage is approached, while no system is likely to survive long in the supercritical state.

In the two sections which follow we shall try to augment this qualitative picture with quantitative estimates of the length (and hence energy) distributions. Section 3 is concerned with the subcritical case and section 4 is concerned with the critical case.

3. Asymptotic behaviour in the subcritical case

We investigate the behaviour of $\Gamma(z)$, supposing first that $P(z)$ is analytic at $z = 1$, a condition equivalent to the exponential decay of the probabilities $t_n = \sum_{k \geq n} p_k$, and and implying the existence of all moments of the distribution $\{p_n\}$. In the subcritical case, the solution pair (1, 1) is a simple solution pair of the equation (3) and it follows from standard theorems on the reversion of power series that $x = \Gamma(z)$ is also analytic at $z = 1$ and can be expanded as a power series in $(1 - z)$:

$$1 - \Gamma(z) = m_1(1 - z) - \tfrac{1}{2}m_2(1 - z)^2 + \cdots \qquad (5)$$

where m_1 is the expected number $E[Z]$ of cracks linked together, and the further coefficients on the right can be interpreted as the successive factorial moments $m_k = E[Z(Z - 1)\cdots(Z - k + 1)]$ of this total number of cracks. Expanding $P(z)$ similarly as a power series in $(1 - z)$, substituting both expansions in the functional equation (3), and equating coefficients, we find after some manipulation that the mean and variance for the total number of cracks are given respectively by the expressions

$$m_1 = 1/(1 - \nu)$$
$$v = \sigma^2/(1 - \nu)^3,$$

where σ^2 is the variance of the underlying distribution $\{p_n\}$. These expressions show clearly how the mean and variance of the total number of cracks approach infinity as ν, the mean number of cracks per branching node, approaches unity. The assumptions we have made on the distribution $\{p_n\}$ in deriving the above results can be weakened to the existence of the first three moments. In general it seems reasonable to suppose that there will be no more than a small number of new branches for each branch of the crack, and hence that the distribution p_n will be sufficiently well-behaved, for the above to be valid.

The situation is rather different when we come to examine the crack length distribution, for here the available evidence suggests that the distribution of lengths of existing cracks will frequently have a very long tail, i.e. with a small but non-negligible proportion of very long cracks. The results we shall need are, that if the Laplace transform $f^*(s)$ of a distribution $F(x)$ satisfies the condition

$$1 - f^*(s) \sim L(s) \cdot s^{\alpha} \qquad (s \to 0, 0 < \alpha < 1)$$

then the tail of the distribution function satisfies

$$1 - F(x) \sim L(x)x^{-\alpha}/\Gamma(1 - \alpha)$$

and conversely. Here $L(x)$ denotes a slowly varying function, and $\Gamma(\alpha)$ is the gamma function, not to be confused with the solution of (3). Such Tauberian theorems are discussed, for example, in FELLER (1971) chapter XIII. Since we are not concerned here with a rigorous exposition we shall suppose simply that if $1 - f^*(s) \sim cs^{\alpha}$ $(0 < \alpha < 1)$, then $1 - F(x) \sim cx^{-\alpha}/\Gamma(1 - \alpha)$ and conversely.

From equation (4) we have for the Laplace transform $g^*(s)$ of total length

$$1 - g^*(s) = 1 - \Gamma[f^*(s)] = m_1[1 - f^*(s)] + 0[1 - f^*(s)]^2 \qquad (6)$$

This shows that in the subcritical case, the tail behaviour of the distribution of total crack length reflects the tail behaviour of the distribution of the lengths of preexisting cracks. In particular, it follows from (6) that if $1 - F(x) \sim c \cdot x^{-\alpha}$ as $x \to \infty$, then also $1 - G(x) \sim d \cdot x^{-\alpha}$ where c, d are constants. Intuitively speaking, if the existing crack lengths have a very irregular distribution, then this effect swamps the effect of branching. Even where a crack is formed from the coalescence of several component cracks, the total length is likely to be dominated by the length of the largest individual component.

The parallel between the two distributions carries over to a degree even in the regular case when both $P(z)$ and $f^*(s)$ are analytic (at $z = 1$, $s = 0$, respectively) for then the solution functions $\Gamma(z)$ and $g^*(s)$ will also be analytic at $z = 1$ and $s = 0$. Both distributions in this case would have exponential type decay at infinity, though the decay parameters in general would be different, and for moderate values of x the actual forms of the distribution might be very different.

4. *Asymptotic behaviour in the critical case*

As the system approaches criticality, the character of the asymptotic distribution changes, for then branching becomes a dominant feature of the process which cannot be ignored. Mathematically this is reflected in the fact that in the critical case, even when $P(z)$ is analytic at $z = 1$, the solution pair (1, 1) is no longer an analytic solution pair for the equation (3), but corresponds to a branch point of order two. The solution $\Gamma(z)$ then has an expansion in terms of $(1 - z)^{1/2}$ rather than in integral powers of $(1 - z)$, the first terms being given by

$$1 - \Gamma(z) = (2/\beta)(1 - z)^{1/2} + o(1 - z)^{1/2}$$

where $\beta = \Sigma n(n - 1)p_n$ is the second factorial moment of the distribution $\{p_n\}$. This asymptotic form remains valid even under the weaker assumption that the $\{p_n\}$ distribution has the first three moments finite.

Then in place of (6) we have for the asymptotic form of $g^*(s)$

$$1 - g^*(s) = (2/\beta)[1 - f^*(s)]^{1/2} + o[(1 - f^*(s)]^{1/2}. \qquad (7)$$

Here there are effectively two situations to consider. If the distribution $F(x)$ of existing crack lengths has a finite mean a, then $1 - f^*(s) = as + o(s)$, and from (7) we obtain $1 - g^*(s) \sim cs^{1/2}$ as $s \to 0$, so that $1 - G(x) \sim dx^{-1/2}$, where c and d are constants. On the other hand, if $F(x)$ has an infinite mean, and in particular if $1 - F(x) \sim cx^{-\alpha}$ $(0 < \alpha < 1)$ as $x \to \infty$, then we obtain from (7) that $1 - G(x) \sim ds^{-\alpha/2}$. In both cases the distribution is of power law form asymptotically, with exponent $-\frac{1}{2}$ or $-\alpha/2$, whichever is the smaller.

5. *Interpretation in terms of energy distribution and b-value*

We now have to ask, to what extent are the observations from laboratory experiments and seismology capable of interpretation in terms of the above model? To provide the link between model and observation we make two assumptions. The first, to which we have already referred, is that the energy radiated during a given cracking episode is proportional to the total length of crack activated during the episode. The second is that the amplitude A of a signal observed at some distance r from the source is related to the radiated energy by an equation of the type

$$A = f(r)E^{2/3}. \qquad (8)$$

A relationship of the general form $A = f(r)g(E)$ is needed to account for the fact that the same form of distribution is found both for the Ishimoto-Iida relationship, where the observed distribution of amplitudes is not weighted by a distance factor, and the Gutenberg-Richter relationship, where it is. That $g(E)$ has the particular form $E^{2/3}$, at least as a first approximation, is the essential content of the energy-magnitude relationship

$$\log_{10} E = 11.8 + 1.5M;$$

if we bear in mind that M is an amplitude weighted by a distance factor. It is of course debatable whether the latter relationship, derived from seismological evidence concerning tectonic earthquakes, should be postulated for laboratory experiments on a totally different scale of magnitude and using quite different instrumentation. The main grounds for such extrapolation is perhaps that the relationship (8) must represent a basic physical property of the attenuation. In any case we shall adopt both assumptions *faute de mieux*.

Taken together, the two assumptions imply that if the distribution of total crack length has a tail of power-law type, $1 - G(x) \sim cx^{-\alpha}$, then the observed distribution of amplitudes should have a power-law tail of the form

$$1 - H(A) \sim dA^{-b}$$

where $b = 3\alpha/2$, and the constant d incorporates an integration over the volume through which cracking takes place. This is a relationship of Ishimoto-Iida type, and in the seismological context the parameter b can be identified further with the parameter in the Gutenberg-Richter magnitude/log frequency relation.

The main thesis of the present discussion is that cracking will take place according to a branching process operating in critical or near-critical conditions. The most persuasive evidence that this mathematical model reflects at least some part of the physical reality is the coincidence between the b-value predicted for this case, $b = (\frac{3}{2})(\frac{1}{2}) = 0.75$, and the b-values observed both in the field and in the laboratory. It should be noted that the b-value predicted in this way does not arise simply as a transformation of some plausible but equally hard to explain assumption concerning the distribution of some prior variable. For example, probably few would quarrel with the suggestion that if the lengths of existing cracks had a power-law form then the energies released when those cracks were activated would also have a power-law form; but this argument would simply shift the onus of explanation from the Gutenberg-Richter Law in its usual form to the law governing the distribution of existing crack lengths. It is at least one merit of the model we have described that it provides one mechanism by which a power law distribution of crack lengths can be initiated without requiring more than the most general assumptions concerning the component distributions. Indeed one may conjecture that it is only the simplest example of a large range of processes, all embodying some kind of aggregating or linking mechanism, which exhibit an approach to a critical state and a power-law distribution for the sizes of the aggregates produced as the critical state is reached. Thus it is probably not necessary to insist on the precise branching mechanism we have described to obtain power law behaviour with parameter b near 0.75 as a critical state is approached.

Nevertheless there are some features which are less easily explained by a model of this kind. Notable among these is the sharp dependence of the b-value on applied stress obtained by Scholz in his laboratory experiments. We are not altogether convinced by Scholz's theoretical reasoning, but we have no reason to doubt the experimental work on which it is based. At first sight the present model would seem to suggest that at low stresses the energies would be subject to a distribution of exponential rather than power law type, and that as stress was increased the distribution would approach a power law form with $b = 0.75$.

However, this inference does not take into account, firstly, the possibility that even in the unstressed state, the distribution of existing crack lengths may have a power law form which would be reflected in the form of the distribution of energies; and, secondly, the effects of increasing stress on the mechanical properties of the medium.

In a very broken up or porous type of material, cracking is likely to be more frequent, but the lengths of the segments linked together in a cracking episode are likely to be small. This situation will tend to produce a distribution approximating more to exponential form, or, if a power law form, then with a high b-value. The opposite tendency is to be expected in a hard material containing a few long cracks. Here the frequency of cracking might be lower, but the distribution of energies should have a longer tail, corresponding to a lower b-value.

With increasing stress, cracks will tend to close up, and the strength of the material to increase, thus producing a move towards lower b-values. At still higher values of the stress, approaching the fracture stress, the phenomena associated with dilatancy will begin to appear. In terms of our model, this would correspond to the approach to the critical state. At this stage the linking of cracks becomes an important feature, and, irrespective of the initial distribution, the lengths of cracks produced by such linking episodes will tend to follow a power law form. But the cracks so produced do not disappear; they are available to serve as the individual segments of further cracking episodes. Thus the constancy of the segment length distribution, which was assumed in our previous discussion, is not really valid, but should be replaced by a progressive change towards longer cracks, and hence longer tails in the distribution, as the process of microfracturing continues.

These two features together seem to offer, in terms of our model, the most likely explanation of the laboratory observations that, at low stresses, some materials show b-values as high as 1.5 or 2, while with increasing stress the b-values progressively decrease until near the fracture stress, b-values as low as 0.3 or 0.2 may be observed.

The application of the model in the tectonic environment raises even greater uncertainties. New features need to be taken into account: the role played by pore pressure in determining the mechanical properties of the rock; the fact that the rock may be subjected to high confining pressures and perhaps high temperatures also; the possibility of large scale inhomogeneities in the medium; the effective unboundedness of the medium, which blurs the distinction one can draw in the laboratory situation between microfractures and catastrophic rupture of the specimen. At the time of writing it is by no means incontrovertibly established that the phenomenon of dilatancy, which offers the most convincing evidence in favour of crack formation at high stresses, is a necessary or even a possible precursor of earthquake activity.

As regards the role of pore pressure, the simplest point of view might be that this is an effect which enters towards the end of the process of stress accumulation and crack formation, and possibly only on a localized basis. An increase in pore pressure following a dilatant period during which the pore pressure was below normal would have the effect of suddenly lowering the strength of the material. This could be interpreted in terms of our model as a sudden increase in the criticality parameter ν, transferring the system from a subcritical to a supercritical state. Or it could be that the main effect was in terms of friction across existing cracks without necessarily affecting the formation of linkages. The net effect of the interaction with water might be to cause fractures to go off 'half-cock,' in conditions which were just below the critical conditions for rock fracture in dry laboratory specimens. Such a feature, together with the more obvious point that little fracturing will be observed until conditions are nearly critical, might underlie the fact that the range of b-values observed in the seismic environment seems to be narrower than that obtained from laboratory experiments, rarely dropping below 0.6 or rising above 1.5.

The observed convexity of log frequency-magnitude plots is also naturally explained in terms of a slightly sub-critical system. Perhaps the easiest way to illustrate this point, and also the analysis in the previous sections, is by way of a simple example. One of the few cases capable of explicit analytic solution occurs if the distribution $\{p_n\}$ has a geometric form, say $P(z) = (1 - \rho)/(1 - \rho z)$. Substitution in equation (3) leads to a quadratic equation for $\Gamma(z)$, to which the relevant solution is

$$\Gamma(z) = 1 - [1 - 4\rho(1 - \rho)z]^{1/2}/2.$$

The individual coefficients γ_n in $\Gamma(z)$ can be found by expansion and are given by

$$\gamma_n = \frac{(2n - 3)(2n - 1)\cdots 3.1}{n!}\frac{1}{2\rho}(\theta/2)^n$$

$$\sim \frac{1}{2\rho} n^{-3/2}\theta^n$$

where $\theta = 4\rho(1 - \rho) = 1 - 4(\frac{1}{2} - \rho)^2$. The criticality parameter $\nu = \Sigma n p_n$ is here equal to $\rho/(1 - \rho)$, so that the critical case corresponds to $\rho = \frac{1}{2}$. In the critical case, therefore, $\gamma_n \sim c\cdot n^{-3/2}$ and $\sum_{k\geq n}\gamma_k \sim d\cdot n^{-1/2}$, in accordance with the conclusion of our general analysis that in the critical case the tails decay according to an inverse power law with parameter $\frac{1}{2}$. The notable feature, however, is that even in the sub-critical case the distribution approximates closely to a power law form for a considerable range of values of n. Thus, for $\rho = 0.45$, we find $\nu = 0.82$, which is already quite some distance from criticality, but $\theta = 0.99$ so that the deviation from the power law form would not become appreciable until after the first hundred terms or so. In this case a graph of log frequency versus log n (corresponding roughly to magnitude) would show a considerable straight line portion, then a final section steepening downwards as the geometric term became dominant. A similar qualitative behaviour would be expected from the energies.

6. *Comparison with Scholz's analysis*

The analysis given in the preceding sections starts from a similar general picture – a medium with random flaws or inhomogeneities – to that of Scholz in his BSSA article (SCHOLZ, 1968a), but differs both qualitatively and quantitatively in the form of its conclusions. In this section we shall try to support our belief that the major reason for these differences can be traced to a faulty assumption in Scholz's analysis, and that if this fault is corrected, the two models lead to substantially similar qualitative conclusions.

The major difficulty we find is with the argumentation leading to equation (4) in his paper. Keeping to Scholz's notation this equation reads

$$g(A)dA = \frac{1 - F(S, \bar{\sigma})}{A} dA. \tag{9}$$

There is some difficulty even over the definition of the quantities $g(A)$ and $F(S, \bar{\sigma})$ appearing in this equation, but we understand that $g(A)$ refers to the *conditional* probability $g(A)dA$ = Prob (crack terminates when its area lies between A, $A + dA$, *given* that its area at least reaches A), while $F(S, \bar{\sigma})$ denotes the probability that the local stress will exceed the average local strength S when the overall applied field has a value $\bar{\sigma}$. If these interpretations are correct, then $g(A)$ is nothing other than the hazard function for the distribution of the area of the crack, and can be written in the form

$$g(A) = h(A)/[1 - H(A)]$$

where $H(A) = \int_0^A h(a)da$ is the cumulative distribution function of total crack area. Substituting this expression in (9) and solving the resulting differential equation for $H(A)$ yields

$$1 - H(A) = A^{-[1 - F(S,\sigma)]}$$

i.e. a power law form for area distribution. From this point assumptions similar to those used in our own analysis suggest that this result may be considered equivalent to a Gutenberg-Richter frequency-magnitude relation with parameter $b = [1 - F(S, \bar{\sigma})]$.

We do not quarrel with this part of the analysis, but with the form of equation (9) itself. It is only consistent with the whole tenor of the discussion that the net local stress (the difference between the local stress and the local strength) should be conceived of as a quantity $\mathscr{S}(\mathbf{x})$ which varies randomly, but according to some specified probability laws, from point to point of the medium. (More precisely, the family $\mathscr{S}(\mathbf{x})$ constitutes a *random field*.) Presumably also the crack will only cease to spread when the local stress exceeds the local strength at *all* points $\mathbf{x}$ on the boundary ∂A of the crack. But the calculation of this probability involves the consideration of the joint behaviour of the whole family $\mathscr{S}(\mathbf{x})$ for $\mathbf{x} \in \partial A$, and we do not see how it can be reduced to the right side of (9) where no joint distributions are involved. Indeed, the calculation would appear to be a formidable problem in general, involving not only the area of the crack, but also its shape, and not only the distribution of $\mathscr{S}(\mathbf{x})$ at a single point but also the continuity properties of $\mathscr{S}(\mathbf{x})$ as x varies round the boundary.

Some indication of a more likely form for the right-hand side of (9) can be drawn from the analogy with high-level crossings for a stationary stochastic process (see, for example, CRAMER and LEADBETTER (1967)). Under wide conditions it is known that such crossings can be approximated by a Poisson process. The probability of no such crossings would therefore be given by an exponential with parameter proportional to the length of interval considered. This suggests that, if anything, (9) might be replaced by an equation of the form

$$g(A) = \exp\,[-\lambda|\partial A|] \tag{10}$$

where $|\partial A|$ refers to the length of the boundary, and λ is a positive constant representing the average number of times the local stress exceeds the local strength along a line of unit length. Even this approach cannot be considered fully satisfactory, however, for it cannot be continued without introducing some further assumption concerning the shape of the crack, while in fact the shape will be determined by the same probabilistic mechanism governing the rest of the process, and so should not be treated as an independent aspect of the problem. Nevertheless, some conclusions can be drawn from (10), and in particular it follows from that equation that the crack cannot grow in a self-similar manner. For if it did we would have (assuming a plane crack) $|\partial A| \propto A^{1/2}$, and the right hand side of (10) would be an integrable function, corresponding to a distribution with a positive probability of taking an infinite value. Thus, no matter how small the applied stress, if cracks grew in a self-similar manner the medium would always be in a 'supercritical' state in which every crack, once initiated, had a positive probability of growing indefinitely.

Conversely, since *only* the form $g(A) \propto A^{-1}$ leads to a power law distribution for the total area, any model leading to such a law must evolve in such a way that

$$g(A) = E(e^{-\lambda|\partial A|}) \sim c/A$$

where the expectation is taken over all shapes having total area A and weights them according to their relative probabilities in the evolution of the crack. It is this expression which seems to us to come closest to embodying in a more rigorous form the idea behind equation (9), but it is not easy to see how the asymptotic relation could be deduced from first principles, nor how the constant c could be related to the probability structure of the process. In a very approximate sense, it suggests that for a power law to emerge, the boundary should grow roughly according to the logarithm of the area.

Another method of gaining insight into the likely behaviour of the model is to formulate the analogous discrete model, following the general precept that probabilistic arguments are often easier to handle in a discrete setting. For example, we could consider the material as represented by a family of nodes lying on a square lattice. We consider a crack propagating from node to node, and suppose that it can link from a given node to a node adjacent to it with probability p, where p corresponds to the quantity $F(S, \bar{\sigma})$ in Scholz's analysis, and could be interpreted as the probability that the local stress along the link exceeded the strength of that link. As a first approximation, it would be natural to suppose that the individual links were independent. Then the probability that the crack will terminate after linking a total number N of nodes will be $p^{\partial N}$, where ∂N is the number of links leading from nodes within the crack to nodes outside it; clearly, this relation provides the discrete analogue to (11).

We may, however, proceed further in the discrete case, for the process we have described is nothing other than a 'percolation process' developed by HAMMERSLEY (1957) and other writers as a model for the percolation of fluid through a porous medium. The problem we have been concerned with, of finding the total area of the crack, corresponds in the percolation process context to the problem of finding the total 'wet area' when liquid is introduced at one point in the medium. Unfortunately the percolation processes are more intractable analytically than the branching processes we discussed earlier. Nevertheless, some facts are known. For example, it is known that there exists a certain critical value p_c of the probability p, determined by the type of lattice structure, such that for $p < p_c$ the total crack size is finite with probability one, and for $p > p_c$ there is a positive probability that the crack will propagate indefinitely. I have not been able to ascertain whether the asymptotic distribution of crack size follows a power law form in the critical case, but by analogy with the branching process model one might conjecture with reasonable confidence that it did, and even that the index would be $-\frac{1}{2}$, corresponding to a branch point of order 2 in the generating function.

7. *Conclusions and acknowledgements*

(i) From a qualitative point of view, perhaps the most important feature which emerges from the analysis is the likelihood of a critical stress state beyond which any small rupture may propagate catastrophically. The existence of such a critical state is not a feature confined to the branching process model, but can be expected in a wide variety of processes of agglomeration.

(ii) For the branching process at least, the asymptotic distribution of crack sizes when the critical state is approached follows a power law form which, under simple assumptions relating the size of the crack to the energy emitted and the amplitude of the observed signal, leads to a Gutenberg-Richter relation with parameter 0.75.

(iii) A possible limitation of the model is that it does not predict a direct dependence of b-value on stress, but rather a dependence of b-value on the stress *history*, insofar as fissures produced at earlier stages in the process become available as elements in the development of a larger fissure. This difference with SCHOLZ's (1968a) results is attributed to an unjustified step in Scholz's analysis rather than to a fundamental difference in the models.

(iv) The analysis does not go far enough to provide information about the dynamics of crack propagation. Indeed, the 1-dimensional cracks envisaged in the discussion should be regarded as a schematization of the physical picture rather than an attempt to model it realistically. The point is that any model of crack propagation in which coalescence and branching play dominant roles is likely to lead to similar qualitative results concerning the distribution of energies of individual cracking episodes.

(v) The possibility of relating the Gutenberg-Richter law to a branching process mechanism was raised in a tentative fashion in my review paper (VERE-JONES, 1975) presented at the 10th Symposium of Mathematical Geophysics. I am grateful to a number of colleagues, particularly to Dr. Colin Atkinson at Imperial College, London, for discussions which encouraged me to follow up this idea and relate it to the results of laboratory experiments on rock fracturing.

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