The Propagation of Surface Waves in Elastic Mediums with Slightly Curved Boundaries of Sinusoidal Type

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Summary – The effect of slightly curved boundaries (free surfaces and interfaces) of the elastic mediums on the components of displacement of a particle in a medium due to the propagation of the surface waves has been investigated in this paper. It has been found that along with the usual displacement components for stratified boundaries (designated here as primary components), there exist secondary displacement components arising from the presence of curvature in the boundaries. They are constituted of different harmonic components with their amplitudes proportional to the parameters which measure the extent of curvature of the boundaries. The wave numbers of these harmonic constituents are related to those of the primary components in a definite way decided by the shapes of the boundaries.

Because of its closeness to the most of the natural situations, the study of the effect of the non-planer boundaries on the propagation of surface waves in elastic mediums has gained much of its importance.

As the analytical treatment of the irregularities of the surface in general encounters formidable mathematical difficulties, the most of the investigators concentrated their efforts with considerable successes in considering the cases of slightly curved surfaces of different types. While SATO [6][2]) studied the propagation of Love waves in a layer with an abrupt change in thickness, DE NOYER [2] considered the same in a layer over a half-space with a sinusoidal interface and KUO and NAFE [4] investigated the propagation of Rayleigh waves in a similar model. On the other hand MAL [5] and ABUBAKAR [1] also considered the effect of the curved boundaries in the presence of buried line sources. All these investigations were led to the development of the frequency equations of the corresponding wave motions. We, however, propose to examine the problem from a different stand point, viz., to consider the effect of the curved boundaries on the displacement components. We anticipate that the displacement vector at any point in the medium shall have two parts. The first one is the same as that in the case of plane boundaries while the other corresponds to the additional effect due to the presence of non-zero (but small) curvature of the boundaries. Naturally the standard frequency equation of the stratified situation will be associated with the first type of displacement. Thus, what remain to be determined, are the amplitudes and the wave numbers of the additional displacement terms. It will be found from the analysis that follows that they depend on the aforesaid frequency equation and the geometrical shape of the curved boundaries apart from the physical parameters

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[2]) Numbers in brackets refer to References, page 117.

(densities and elastic constants) of the mediums concerned. This idea has been forwarded by HANDELMAN [3], although the working out of the problems in the particular cases of our consideration differs in some respect from the same suggested by him.

The proposed analysis is as follows:

Referred to a rectangular Cartesian system of co-ordinates (x, y, z), let

$$z = \varepsilon f(x) \tag{1}$$

be the equation of a slightly curved surface (of cylindrical type) where ε is a small quantity of a first order and $f(x)$ is finite in $-\infty \leq x \leq \infty$.

Then θ, the angle that the tangent line to the surface lying in the x-z plane makes with the x axis, is given by

$$\theta \simeq \tan\theta = \varepsilon f'(x) \tag{2}$$

assuming, of course, that $f'(x)$ remains finite in $-\infty \leq x \leq \infty$.

Now we assume x' axis in the direction of the tangent and z' axis at right angles to this in the x-z plane.

Then the stress-components of our interest, referred to this new set of axes, are related to those referred to the old set of axes as

$$\left.\begin{aligned} Z'_{z'} &= \sin^2\theta\, X_x + \cos^2\theta\, Z_z - 2\sin\theta\cos\theta\, Z_x\,,\\ Z'_{x'} &= \sin\theta\cos\theta\,(Z_z - X_x) + (\cos^2\theta - \sin^2\theta)\, Z_x\,,\\ Z'_{y} &= \cos\theta\, Y_z - \sin\theta\, X_y\,, \end{aligned}\right\} \tag{3}$$

where the notations convey their usual meanings.

Now as is evident from (2) that θ is a small quantity of the first order, the relations (3) reduce to

$$\left.\begin{aligned} Z'_{z'} &= Z_z - 2\,\theta\, Z_x\,,\\ Z'_{x'} &= \theta(Z_z - X_x) + Z_x\,,\\ Z'_{y} &= Y_z - \theta\, X_y\,. \end{aligned}\right\} \tag{4}$$

For considering Rayleigh waves, we have

$$\frac{\partial u}{\partial y} = v = \frac{\partial w}{\partial y} = 0\,, \tag{5}$$

(u, v, w) being the displacement components of a particle in the medium.

Then we have

$$\left.\begin{aligned} X_x &= (\lambda + 2\,\mu)\,\frac{\partial u}{\partial x} + \lambda\,\frac{\partial w}{\partial z}\,,\\ Y_y &= \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right),\\ Z_z &= \lambda\,\frac{\partial u}{\partial x} + (\lambda + 2\,\mu)\,\frac{\partial w}{\partial z}\,,\\ X_y &= Y_z = 0\,,\\ Z_x &= \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \end{aligned}\right\} \tag{6}$$

and hence

$$\left.\begin{aligned} Z'_{z'} &= \lambda \frac{\partial u}{\partial x} + (\lambda + 2\,\mu) \frac{\partial w}{\partial z} - 2\,\theta\,\mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \\ Z'_{x'} &= 2\,\theta\,\mu \left(\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x}\right) + \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \\ Z'_{y} &= 0\,. \end{aligned}\right\} \tag{7}$$

Similarly in the case of Love waves we have

$$u = \frac{\partial v}{\partial y} = w = 0 \tag{8}$$

and hence

$$\left.\begin{aligned} X_x &= Y_y = Z_z = Z_x = 0\,, \\ X_y &= \mu \frac{\partial v}{\partial x}\,, \\ Y_z &= \mu \frac{\partial v}{\partial z} \end{aligned}\right\} \tag{9}$$

and

$$\left.\begin{aligned} Z'_{z'} &= Z'_{x'} = 0\,, \\ Z'_{y} &= \mu \left(\frac{\partial v}{\partial z} - \theta \frac{\partial v}{\partial x}\right). \end{aligned}\right\} \tag{10}$$

We shall, herefrom, consider separately the following two cases in some details:

1. Rayleigh waves in a half-space with a sinusoidal free-surface.
2. Love waves in a layer with a sinusoidal free-surface and a similar interface between the layer and the underlain half-space.

Case 1: Rayleigh waves in a half-space with a sinusoidal free-surface.

Let

$$z = \varepsilon \sin m\,x \tag{11}$$

be the free-surface of a homogeneous elastic half-space with ϱ and μ as its density and modulus of rigidity.

Comparing (11) with (1) and making use of (2) we find that in this case

$$\theta = \varepsilon\, m \cos m\,x \tag{12}$$

and hence

$$\left.\begin{aligned} Z'_{z'} &= \lambda \frac{\partial u}{\partial x} + (\lambda + 2\,\mu) \frac{\partial w}{\partial z} - 2\,\varepsilon\, m \cos m\,x \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \\ Z'_{x'} &= 2\,\varepsilon\, m\,\mu \cos m\,x \left(\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x}\right) + \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \\ Z'_{y} &= 0\,. \end{aligned}\right\} \tag{13}$$

Now the equations of motion of a particle for the propagation of Rayleigh waves are given by

$$\left.\begin{aligned} \varrho \frac{\partial^2 u}{\partial t^2} &= (\lambda + 2\,\mu) \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial z^2} + (\lambda + \mu) \frac{\partial^2 w}{\partial x\, \partial z}\,, \\ \varrho \frac{\partial^2 w}{\partial t^2} &= (\lambda + \mu) \frac{\partial^2 u}{\partial x\, \partial z} + \mu \frac{\partial^2 w}{\partial x^2} + (\lambda + 2\,\mu) \frac{\partial^2 w}{\partial z^2}\,. \end{aligned}\right\} \tag{14}$$

Let us assume

$$(u, w) \sim \operatorname{Exp}\{\nu\, z + i(\omega\, t - k\, x)\}\,. \tag{15}$$

Then substituting in (14) we get

$$\left.\begin{aligned} \{\varrho\, \omega^2 - (\lambda + 2\,\mu)\, k^2 + \mu\, \nu^2\}\, u - i\, k\, \nu(\lambda + \mu)\, w = 0\,, \\ i\, k\, \nu(\lambda + \mu)\, u - \{\varrho\, \omega^2 - \mu\, k^2 + (\lambda + 2\,\mu)\, \nu^2\}\, w = 0\,, \end{aligned}\right\} \tag{16}$$

whence eliminating u/w we get

$$\{\varrho\, \omega^2 - (\lambda + 2\,\mu)\, k^2 + \mu\, \nu^2\}\, \{\varrho\, \omega^2 - \mu\, k^2 + (\lambda + 2\,\mu)\, \nu^2\} + k^2\, \nu^2 (\lambda + \mu)^2 = 0\,, \tag{17}$$

which yields four roots for ν, viz., $\pm\, \nu_1$, $\pm\, \nu_2$, where

$$\left.\begin{aligned} \nu_1^2 &= k^2 - \frac{\varrho\, \omega^2}{\lambda + 2\,\mu}\,, \\ \nu_2^2 &= k^2 - \frac{\varrho\, \omega^2}{\mu}\,. \end{aligned}\right\} \tag{18}$$

Thus the general solutions for u and w become

$$\left.\begin{aligned} u &= i[k(A\, e^{\nu_1 z} + B\, e^{-\nu_1 z}) - \nu_2 (C\, e^{\nu_2 z} - D\, e^{-\nu_2 z})]\, e^{i(\omega t - k x)}\,, \\ w &= [-\, \nu_1 (A\, e^{\nu_1 z} - B\, e^{-\nu_1 z}) + k(C\, e^{\nu_2 z} + D\, e^{-\nu_2 z})]\, e^{i(\omega t - k x)}\,. \end{aligned}\right\} \tag{19}$$

Now if we measure z positive in the downward direction (i.e., within the half-space), then from the condition of surface waves (i.e., the vanishing of the displacements at an infinite depth) we shall have

$$\left.\begin{aligned} u &= i[k\, B\, e^{-\nu_1 z} + \nu_2\, D\, e^{-\nu_2 z}]\, e^{i(\omega t - k x)}\,, \\ w &= [\nu_1\, B\, e^{-\nu_1 z} + k\, D\, e^{-\nu_2 z}]\, e^{i(\omega t - k x)}\,, \end{aligned}\right\} \tag{20}$$

where ν_1 and ν_2 are the positive roots of the corresponding equations given by (18).

The boundary conditions which are to be satisfied by the displacement components are the vanishing of the stress components given by (13) at the free-surface. These yield the following equations

$$\left.\begin{aligned} \lambda \frac{\partial u}{\partial x} + (\lambda + 2\,\mu) \frac{\partial w}{\partial z} - 2\, \varepsilon\, m\, \mu \cos m\, x \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = 0\,, \\ \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) - 2\, \varepsilon\, m \cos m\, x \left(\frac{\partial u}{\partial x} - \frac{\partial w}{\partial z}\right) = 0\,, \\ \text{at} \quad z = \varepsilon \sin m\, x\,, \end{aligned}\right\} \tag{21}$$

which, correct upto first order of ε, are the same as

$$\left.\begin{aligned} &\lambda \frac{\partial u}{\partial x} + (\lambda + 2\,\mu)\,\frac{\partial w}{\partial z} = 0\,, \\ &\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} - 2\,\varepsilon\, m \cos m\, x \left(\frac{\partial u}{\partial x} - \frac{\partial w}{\partial z}\right) = 0\,, \end{aligned}\right\} \quad \text{at} \quad z = \varepsilon \sin m\, x\,. \tag{22}$$

If we develope TAYLOR's expansion of (22) in the neighbourhood of $z = 0$ and then preserve terms correct upto first order of ε, we shall then have

$$\left.\begin{aligned} &\lambda \frac{\partial u}{\partial x} + (\lambda + 2\,\mu)\,\frac{\partial w}{\partial z} + \varepsilon \sin m\, x \left\{\lambda\, \frac{\partial^2 u}{\partial x\, \partial z} + (\lambda + 2\,\mu)\, \frac{\partial^2 w}{\partial z^2}\right\} = 0\,, \\ &\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \varepsilon \sin m\, x \left\{\frac{\partial^2 w}{\partial x\, \partial z} + \frac{\partial^2 u}{\partial z^2}\right\} - 2\,\varepsilon\, m \cos m\, x \left(\frac{\partial u}{\partial x} - \frac{\partial w}{\partial z}\right) = 0\,, \\ &\qquad\qquad \text{at} \quad z = 0\,. \end{aligned}\right\} \tag{23}$$

Let us now assume

$$\left.\begin{aligned} u &= u_0 + \varepsilon\, u_1\,, \\ w &= w_0 + \varepsilon\, w_1\,, \end{aligned}\right\} \tag{24}$$

where (u_0, w_0) are Rayleigh wave-displacement components in the case of a half-space with a plane free-surface (i.e. $\varepsilon = 0$) and (u_1, w_1) are the perturbation terms in the displacement components due to the presence of small curvature of the free surface (i.e. $\varepsilon \neq 0$, but small).

Evidently (u_0, w_0) satisfy the differential equations

$$\left.\begin{aligned} \varrho\, \frac{\partial^2 u_0}{\partial t^2} &= (\lambda + 2\,\mu)\, \frac{\partial^2 u_0}{\partial x^2} + \mu\, \frac{\partial^2 u_0}{\partial z^2} + (\lambda + \mu)\, \frac{\partial^2 w_0}{\partial x\, \partial z}\,, \\ \varrho\, \frac{\partial^2 w_0}{\partial t^2} &= (\lambda + \mu)\, \frac{\partial^2 u_0}{\partial x\, \partial z} + \mu\, \frac{\partial^2 w_0}{\partial x^2} + (\lambda + 2\,\mu)\, \frac{\partial^2 w_0}{\partial z^2} \end{aligned}\right\} \tag{25}$$

and also the boundary conditions

$$\left.\begin{aligned} &\lambda \frac{\partial u_0}{\partial x} + (\lambda + 2\,\mu)\, \frac{\partial w_0}{\partial z} = 0\,, \\ &\frac{\partial w_0}{\partial x} + \frac{\partial u_0}{\partial z} = 0\,. \end{aligned}\right\} \quad \text{at} \quad z = 0\,. \tag{26}$$

The expressions for u_0 and w_0 are readily obtainable. They are

$$u_0 = i\, k\, B_0 \left[e^{-\nu_1 z} - \frac{2\, \nu_1\, \nu_2}{k^2 + \nu_2^2}\, e^{-\nu_2 z}\right] e^{i(\omega t - k x)}\,, \tag{27}$$

$$w_0 = \nu_1\, B_0 \left[e^{-\nu_1 z} - \frac{2\, k^2}{k^2 + \nu_2^2}\, e^{-\nu_2 z}\right] e^{i(\omega t - k x)} \tag{28}$$

with

$$(k^2 + \nu_2^2)^2 - 4\,\nu_1\,\nu_2\,k^2 = 0 \tag{29}$$

as the frequency equation of the corresponding motion.

Now substituting (24) in (14) and making use of (25), we obtain the differential equations to be satisfied by u_1 and w_1 only, which are

$$\left.\begin{aligned}
\varrho\,\frac{\partial^2 u_1}{\partial t^2} &= (\lambda + 2\,\mu)\,\frac{\partial^2 u_1}{\partial x^2} + \mu\,\frac{\partial^2 u_1}{\partial z^2} + (\lambda + \mu)\,\frac{\partial^2 w_1}{\partial x\,\partial z}\,,\\
\varrho\,\frac{\partial^2 w_1}{\partial t^2} &= (\lambda + \mu)\,\frac{\partial^2 u_1}{\partial x\,\partial z} + \mu\,\frac{\partial^2 w_1}{\partial x^2} + (\lambda + 2\,\mu)\,\frac{\partial^2 w_1}{\partial z^2}\,.
\end{aligned}\right\} \tag{30}$$

Similarly the boundary conditions to be satisfied by u_1 and w_1 can be developed by substituting (24) in (23) and then making use of (26). They are

$$\left.\begin{aligned}
&\lambda\,\frac{\partial u_1}{\partial x} + (\lambda + 2\,\mu)\,\frac{\partial w_1}{\partial z} + \sin m\,x\left\{\lambda\,\frac{\partial^2 u_0}{\partial x\,\partial z} + (\lambda + 2\,\mu)\,\frac{\partial^2 w_0}{\partial z^2}\right\} = 0\,,\\
&\frac{\partial w_1}{\partial x} + \frac{\partial u_1}{\partial z} + \sin m\,x\left(\frac{\partial^2 w_0}{\partial x\,\partial z} + \frac{\partial^2 u_0}{\partial z^2}\right) - 2\,m\cos m\,x\left(\frac{\partial u_0}{\partial z} - \frac{\partial w_0}{\partial x}\right) = 0\,,\\
&\qquad\qquad\text{at}\quad z = 0
\end{aligned}\right\} \tag{31}$$

correct upto first order of ε.

Now corresponding to u_0 and w_0 given by (27) and (28), let us assume

$$\left.\begin{aligned}
u_1 = {}& i\left[(k + m)\,B_1\,e^{-\nu_{11} z} + \nu_{12}\,D_1\,e^{-\nu_{12} z}\right] e^{i(\omega t - \overline{k+m}\,x)}\\
&+ i\left[(k - m)\,B_1'\,e^{-\nu_{11}' z} + \nu_{12}'\,D_1'\,e^{-\nu_{12}' z}\right] e^{i(\omega t - \overline{k-m}\,x)}\,,
\end{aligned}\right\} \tag{32}$$

$$\left.\begin{aligned}
w_1 = {}& \left[\nu_{11}\,B_1\,e^{-\nu_{11} z} + (k + m)\,D_1\,e^{-\nu_{12} z}\right] e^{i(\omega t - \overline{k+m}\,x)}\,,\\
&+ \left[\nu_{11}'\,B_1'\,e^{-\nu_{11}' z} + (k - m)\,D_1'\,e^{-\nu_{12}' z}\right] e^{i(\omega t - \overline{k-m}\,x)}\,,
\end{aligned}\right\} \tag{33}$$

where

$$\left.\begin{aligned}
\nu_{11}^2 &= (k + m)^2 - \frac{\varrho\,\omega^2}{(\lambda + 2\,\mu)}\,,\\
\nu_{11}'^2 &= (k - m)^2 - \frac{\varrho\,\omega^2}{(\lambda + 2\,\mu)}\,,\\
\nu_{12}^2 &= (k + m)^2 - \frac{\varrho\,\omega^2}{\mu}\,,\\
\nu_{12}'^2 &= (k - m)^2 - \frac{\varrho\,\omega^2}{\mu}\,.
\end{aligned}\right\} \tag{34}$$

Evidently (32) and (33) satisfy the differential equations given by (30).

Then substituting (27), (28), (32) and (33) in (31), we obtain the equations

$$\left.\begin{aligned}&\{(\nu_{12}^2+\overline{k+m}^2)\,B_1+2\,(k+m)\,\nu_{12}\,D_1\}\,e^{-imx}\\&\qquad+\{(\nu_{12}'^2+\overline{k-m}^2)\,B_1'+2\,(k-m)\,\nu_{12}'\,D_1'\}\,e^{imx}\\&\qquad=B_0(\nu_1-\nu_2)\,(k^2+\nu_2^2)\sin m i\,x\,,\end{aligned}\right\}\tag{35}$$

$$\left.\begin{aligned}&\{2\,\nu_{11}(k+m)\,B_1+(\nu_{12}^2+\overline{k+m}^2)\,D_1\}\,e^{-imx}\\&\qquad+\{2\,\nu_{11}'(k-m)\,B_1'+(\nu_{12}'^2+\overline{k-m}^2)\,D_1'\}\,e^{imx}\\&\qquad=2\,k\,\nu_1\,B_0\left[(\nu_1-\nu_2)\sin m\,x-2\,m\left(\frac{\nu_2^2-k^2}{\nu_2^2+k^2}\right)\cos m\,x\right]\end{aligned}\right\}\tag{36}$$

to hold for all x's.

Therefore, when we equate seperately the coefficients of e^{-imx} and e^{imx} of the right hand sides and the left hand sides, we obtain from (35) and (36) the following four linear equations:

$$\{\nu_{12}^2+(k+m)^2\}\,B_1+2\,(k+m)\,\nu_{12}\,D_1=\frac{i\,B_0}{2}\,(\nu_1-\nu_2)\,(k^2+\nu_2^2)\,,\tag{37}$$

$$\{\nu_{12}'^2+(k-m)^2\}\,B_1'+2\,(k-m)\,\nu_{12}'\,D_1'=-\frac{i\,B_0}{2}\,(\nu_1-\nu_2)\,(k^2+\nu_2^2)\,,\tag{38}$$

$$2\,\nu_{11}(k+m)\,B_1+\{\nu_{12}^2+(k+m)^2\}\,D_1=k\,\nu_1\,B_0\left[i\,(\nu_1-\nu_2)-2\,m\left(\frac{\nu_2^2-k^2}{\nu_2^2+k^2}\right)\right],\tag{39}$$

$$2\,\nu_{11}'(k-m)\,B_1'+\{\nu_{12}'^2+(k-m)^2\}\,D_1'=-k\,\nu_1\,B_0\left[i\,(\nu_1-\nu_2)+2\,m\left(\frac{\nu_2^2-k^2}{\nu_2^2+k^2}\right)\right]\tag{40}$$

solving which we get,

$$B_1=B_0\frac{\left[2\,\nu_{12}(k+m)\left\{\dfrac{m(k^4-\nu_2^4)}{2\,\nu_2\,k}+i\,k\,\nu_1(\nu_1-\nu_2)\right\}-\dfrac{i}{2}\,\{\nu_{12}^2+(k+m)^2\}\,(\nu_1-\nu_2)\,(\nu_2^2+k^2)\right]}{4\,\nu_{11}\,\nu_{12}(k+m)^2-\{\nu_{12}^2+(k+m)^2\}^2}\,,\tag{41}$$

$$B_1'=B_0\frac{\left[2\,\nu_{12}'(k-m)\left\{\dfrac{m(k^4-\nu_2^4)}{2\,\nu_2\,k}+i\,k\,\nu_1(\nu_1-\nu_2)\right\}-\dfrac{i}{2}\,\{\nu_{12}'^2+(k-m)^2\}\,(\nu_1-\nu_2)\,(\nu_2^2+k^2)\right]}{4\,\nu_{11}'\,\nu_{12}'(k-m)^2-\{\nu_{12}'^2+(k-m)^2\}^2}\,,\tag{42}$$

$$D_1=i\,B_0\frac{\left[\nu_{11}(k+m)\,(\nu_1-\nu_2)\,(k^2+\nu_2^2)+\left\{\dfrac{i\,m(k^4-\nu_2^4)}{2\,\nu_2\,k}-k\,\nu_1(\nu_1-\nu_2)\right\}\{\nu_{12}^2+(k+m)^2\}\right]}{4\,\nu_{11}\,\nu_{12}(k+m)^2-\{\nu_{12}^2+(k+m)^2\}^2}\,,\tag{43}$$

$$D_1'=i\,B_0\frac{\left[-\nu_{11}'(k-m)\,(\nu_1-\nu_2)\,(k^2+\nu_2^2)+\left\{\dfrac{i\,m(k^4-\nu_2^4)}{2\,\nu_2\,k}+k\,\nu_1(\nu_1-\nu_2)\right\}\{\nu_{12}'^2+(k-m)^2\}\right]}{4\,\nu_{11}'\,\nu_{12}'(k-m)^2-\{\nu_{12}'^2+(k-m)^2\}^2}\,.\tag{44}$$

These enable us to determine u_1 and w_1 completely.

The above analysis indicates that the propagation of Rayleigh waves is possible in an elastic half-space whose free-surface is of sinusoidal type of small amplitude. Corresponding to a given frequency ω, the primary wave propagates with the same wave number k as in the case of plane free-surface. But along with the primary waves, there travel two other secondary waves with their wave numbers equal to $k - m$ and $k + m$. Thus the frequency-wave number curve for the primary wave can also be used for the secondary wave by appropriate shifting of the origin along k axis. The phase velocities of the different secondary waves will, no doubt, be different from that of the primary waves, but the group velocities will remain unaltered.

Case 2: Love waves in a layer with a sinusoidal free-surface with a similar interface between the layer and the underlain half-space.

Let

$$z = \varepsilon_1 \sin m_1 x \tag{45}$$

be the free-surface of a homogeneous layer (medium 1) with (ϱ_1, μ_1) as its density and modulus of rigidity, which is underlain by a homogeneous elastic half-space (medium 2) with (ϱ_2, μ_2) as its density and modulus of rigidity, The interface between the mediums is given by

$$z = H + \varepsilon_2 \sin(m_2 x + \delta) . \tag{46}$$

Comparing (45) and (46) with (1) and making use of (2) we find that in this case

$$\left.\begin{aligned} \theta_1 &= \varepsilon\, m_1 \cos m_1 x , \\ \theta_2 &= \varepsilon_2\, m_2 \cos(m_2 x + \delta) , \end{aligned}\right\} \tag{47}$$

$$\left.\begin{aligned} (Z'_y)_1 &= \mu \left(\frac{\partial v}{\partial z} - \varepsilon_1 m_1 \cos m_1 x \frac{\partial v}{\partial x} \right) , \\ (Z'_y)_2 &= \mu \left\{ \frac{\partial v}{\partial z} - \varepsilon_2 m_2 \cos(m_2 x + \delta) \frac{\partial v}{\partial x} \right\} , \end{aligned}\right\} \tag{48}$$

where the suffixes 1 and 2 in θ and (Z'_y) refer to the free-surface and the interface respectively.

Let v_1 and v_2 be the Love-wave displacements in the media 1 and 2 respectively, satisfying the equations

$$\left.\begin{aligned} \varrho_1 \frac{\partial^2 v_1}{\partial t^2} &= \mu_1 \left(\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_2}{\partial z^2} \right) \quad \text{(medium 1)} , \\ \varrho_2 \frac{\partial^2 v_2}{\partial t^2} &= \mu_2 \left(\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial z^2} \right) \quad \text{(medium 2)} . \end{aligned}\right\} \tag{49}$$

Solutions of these equations in the case of plane boundaries (i.e., $\varepsilon_1 = \varepsilon_2 = 0$) are readily obtainable. They are

$$\left.\begin{aligned} v_{10} &= A_{10} \cos \nu_{10} z \cdot e^{i(\omega t - k x)} , \\ v_{20} &= A_{10} \cos \nu_{10} H \cdot e^{\nu_{20}(H-z) + i(\omega t - k x)} , \end{aligned}\right\} \tag{50}$$

with ν_{10} and ν_{20} being the real positive roots of the equations

$$\left.\begin{aligned} \nu_{10}^2 &= \frac{\varrho_1\,\omega^2}{\mu_1} - k^2\,,\\ \nu_{20}^2 &= k^2 - \frac{\varrho_2\,\omega^2}{\mu_2}\,. \end{aligned}\right\} \tag{51}$$

The frequency equation of the corresponding motion is given by

$$\tan \nu_{10} H = \frac{\mu_2\,\nu_{20}}{\mu_1\,\nu_{10}}\,. \tag{52}$$

Let us assume that correct upto first order of small quantities ε_1 and ε_2

$$\left.\begin{aligned} v_1 &= v_{10} + \varepsilon_1\,v_{11} + \varepsilon_2\,v_{12}\,,\\ v_2 &= v_{20} + \varepsilon_1\,v_{21} + \varepsilon_2\,v_{22}\,, \end{aligned}\right\} \tag{53}$$

where v_{11}, v_{12}, v_{21} and v_{22} satisfy the equations

$$\left.\begin{aligned} \varrho_1\,\frac{\partial^2 v_{11}}{\partial t^2} &= \mu_1\left(\frac{\partial^2 v_{11}}{\partial x^2} + \frac{\partial^2 v_{11}}{\partial z^2}\right),\\ \varrho_1\,\frac{\partial^2 v_{12}}{\partial t^2} &= \mu_1\left(\frac{\partial^2 v_{12}}{\partial x^2} + \frac{\partial^2 v_{12}}{\partial z^2}\right),\\ \varrho_2\,\frac{\partial^2 v_{21}}{\partial t^2} &= \mu_2\left(\frac{\partial^2 v_{21}}{\partial x^2} + \frac{\partial^2 v_{21}}{\partial z^2}\right),\\ \varrho_2\,\frac{\partial^2 v_{22}}{\partial t^2} &= \mu_2\left(\frac{\partial^2 v_{22}}{\partial x^2} + \frac{\partial^2 v_{22}}{\partial z^2}\right). \end{aligned}\right\} \tag{54}$$

Then the relations (53) satisfy the equations (49) respectively.

Now using the relations (48) we find that the boundary conditions to be satisfied by v_1 and v_2 are

$$\frac{\partial v_1}{\partial z} - \varepsilon_1\,m_1 \cos m_1\,x\,\frac{\partial v_1}{\partial x} = 0 \quad \text{at} \quad z = \varepsilon_1 \sin m_1\,x\,, \tag{55}$$

$$\left.\left.\begin{aligned} &v_1 = v_2\\ &\mu_1\left\{\frac{\partial v_1}{\partial z} - \varepsilon_2\,m_2 \cos(m_2\,x + \delta)\,\frac{\partial v_1}{\partial x}\right\}\\ &\quad = \mu_2\left\{\frac{\partial v_2}{\partial z} - \varepsilon_2\,m_2 \cos(m_2\,x + \delta)\,\frac{\partial v_2}{\partial x}\right\} \end{aligned}\right\} \text{at} \quad z = H + \varepsilon_2 \sin(m_2\,x + \delta)\,.\right\} \tag{56}$$

If we develope TAYLOR's expansion of (55) and (56) correct upto first order of ε_1 and ε_2 and then make use of (53), then we find that it will be sufficient if the solutions of (54) satisfy the following boundary conditions:

$$\frac{\partial v_{11}}{\partial z} + \sin m_1\,x\,\frac{\partial^2 v_{10}}{\partial z^2} - m_1 \cos m_1\,x\,\frac{\partial v_{10}}{\partial x} = 0 \quad \text{at} \quad z = 0\,, \tag{57}$$

$$\left.\begin{aligned} v_{11} &= v_{21}\\ \mu_1\,\frac{\partial v_{11}}{\partial z} &= \mu_2\,\frac{\partial v_{21}}{\partial z} \end{aligned}\right\} \quad \text{at} \quad z = H \tag{58}$$

and

$$\frac{\partial v_{12}}{\partial z} = 0 \quad \text{at} \quad z = 0\,, \tag{59}$$

$$\left.\begin{aligned} & v_{12} + \sin(m_2\,x + \delta)\,\frac{\partial v_{10}}{\partial z} = v_{22} + \sin(m_2\,x + \delta)\,\frac{\partial v_{20}}{\partial z}\,, \\ & \mu_1 \left\{\frac{\partial v_{12}}{\partial z} + \sin(m_2\,x + \delta)\,\frac{\partial^2 v_{10}}{\partial z^2} - m_2 \cos(m_2\,x + \delta)\,\frac{\partial v_{10}}{\partial x}\right\} \\ & \qquad = \mu_2 \left\{\frac{\partial v_{22}}{\partial z} + \sin(m_2\,x + \delta)\,\frac{\partial^2 v_{20}}{\partial z^2} - m_2 \cos(m_2\,x + \delta)\,\frac{\partial v_{20}}{\partial x}\right\} \\ & \qquad\qquad \text{at} \quad z = H\,. \end{aligned}\right\} \tag{60}$$

Now, corresponding to v_{10} and v_{20} given by (50) let us assume

$$\left.\begin{aligned} v_{11} = {} & (A_{11} \cos\nu_{11}\,z + B_{11} \sin\nu_{11}\,z)\,e^{i(\omega t - \overline{k + m_1} x)} \\ & + (A'_{11} \cos\nu'_{11}\,z + B'_{11} \sin\nu'_{11}\,z)\,e^{i(\omega t - \overline{k - m_1} x)} \\ \text{and} & \\ v_{21} = {} & A_{21}\,e^{\nu_{21}(H - z) + i(\omega t - \overline{k + m_1} x)} + A'_{21}\,e^{\nu'_{21}(H - z) + i(\omega t - \overline{k - m_1} x)}\,, \end{aligned}\right\} \tag{61}$$

where ν_{11}, ν'_{11}, ν_{21} and ν'_{21} are real positive roots of the equations

$$\left.\begin{aligned} \nu_{11}^2 &= \frac{\varrho_1\,\omega^2}{\mu_1} - (k + m_1)^2\,, \\ \nu_{11}'^2 &= \frac{\varrho_1\,\omega^2}{\mu_1} - (k - m_1)^2\,, \\ \nu_{21}^2 &= (k + m_1)^2 - \frac{\varrho_2\,\omega^2}{\mu_2}\,, \\ \nu_{21}'^2 &= (k - m_1)^2 - \frac{\varrho_2\,\omega^2}{\mu_2}\,. \end{aligned}\right\} \tag{62}$$

Evidently, the equations (61) satisfy the first two equations of (54) respectively. Then substituting (50) and (61) in (57) and (58) we obtain the equations

$$\left.\begin{aligned} & \nu_{11}\,B_{11}\,e^{-i m_1 x} + \nu'_{11}\,B'_{11}\,e^{i m_1 x} = A_{10}(\nu_{10}^2 \sin m_1\,x + i\,k\,m_1 \cos m_1\,x) \\ & (A_{11} \cos\nu_{11}\,H + B_{11} \sin\nu_{11}\,H)\,e^{-i m_1 x} + (A'_{11} \cos\nu'_{11}\,H + B'_{11} \sin\nu'_{11}\,H)\,e^{i m_1 x} \\ & \qquad = A_{21}\,e^{-i m_1 x} + A'_{21}\,e^{i m_1 x}\,, \\ & \mu_1[\nu_{11}(- A_{11} \sin\nu_{11}\,H + B_{11} \cos\nu_{11}\,H)\,e^{-i m_1 x} \\ & \qquad + \nu'_{11}(- A'_{11} \sin\nu'_{11}\,H + B'_{11} \cos\nu'_{11}\,H)\,e^{i m_1 x}] \\ & \qquad = - \mu_2[A_{21}\,\nu_{21}\,e^{-i m_1 x} + A'_{21}\,\nu'_{21}\,e^{i m_1 x}] \end{aligned}\right\} \tag{63}$$

to hold for all x's.

Therefore, when we equate separately the coefficients of e^{-im_1x} and e^{im_1x} between the right hand sides and the left hand sides we obtain from (63) the following six equations

$$\left.\begin{aligned}
&\nu_{11} B_{11} = i A_{10} \frac{\nu_{10}^2 - k m_1}{2},\\
&\nu'_{11} B'_{11} = - i A_{10} \frac{\nu_{10}^2 + k m_1}{2},\\
&A_{11} \cos\nu_{11} H + B_{11} \sin\nu_{11} H = A_{21},\\
&A'_{11} \cos\nu'_{11} H + B'_{11} \sin\nu'_{11} H = A'_{21},\\
&\mu_1 \nu_{11}(A_{11} \sin\nu_{11} H - B_{11} \cos\nu_{11} H) = \mu_2 \nu_{21} A_{21},\\
&\mu_1 \nu'_{11}(A'_{11} \sin\nu'_{11} H - B'_{11} \cos\nu'_{11} H) = \mu_2 \nu'_{21} A'_{21},
\end{aligned}\right\} \tag{64}$$

which do not involve x.

The solutions of these equations in terms of A_{10} have been found out as

$$\left.\begin{aligned}
B_{11} &= \frac{i A_{10}}{2} \cdot \frac{(\nu_{10}^2 - k m_1)}{\nu_{11}},\\
B'_{11} &= \frac{- i A_{10}}{2} \cdot \frac{(\nu_{10}^2 + k m_1)}{\nu'_{11}},\\
A_{11} &= \frac{i A_{10}}{2} \cdot \frac{(\mu_1 \nu_{11} \cos\nu_{11} H + \mu_{21} \nu_{21} \sin\nu_{11} H)(\nu_{10}^2 - k m_1)}{(\mu_1 \nu_{11} \sin\nu_{11} H - \mu_2 \nu_{21} \cos\nu_{11} H)\nu_{11}},\\
A'_{11} &= \frac{- i A_{10}}{2} \cdot \frac{(\mu_1 \nu'_{11} \cos\nu'_{11} H + \mu_2 \nu'_{21} \sin\nu'_{11} H)(\nu_{10}^2 + k m_1)}{(\mu_1 \nu'_{11} \sin\nu'_{11} H - \mu_2 \nu'_{21} \cos\nu'_{21} H)\nu'_{11}},\\
A_{21} &= \frac{i A_{10}}{2} \cdot \frac{\mu_1(\nu_{10}^2 - k m_1)}{(\mu_1 \nu_{11} \sin\nu_{11} H - \mu_2 \nu_{21} \cos\nu_{11} H)},\\
A'_{21} &= \frac{- i A_{10}}{2} \cdot \frac{\mu_1(\nu_{10}^2 + k m_1)}{(\mu_1 \nu'_{11} \sin\nu'_{11} H - \mu_2 \nu'_{21} \cos\nu'_{11} H)}.
\end{aligned}\right\} \tag{65}$$

These enable us to determine v_{11} and v_{21} completely.

Similarly we assume

$$\left.\begin{aligned}
v_{12} &= A_1 \cos\nu_{12} z\, e^{i(\omega t - \overline{k + m_2 x})} + A'_{12} \cos\nu'_{12} z\, e^{i(\omega t - \overline{k - m_2 x})},\\
v_{22} &= A_{22} \cos\nu_{12} z\, e^{i(\omega t - \overline{k + m_1 x})} + A'_{22} \cos\nu'_{21} z\, e^{i(\omega t - \overline{k - m_2 x})},
\end{aligned}\right\} \tag{66}$$

where ν_{12}, ν'_{12}, ν_{22} and ν'_{22} are real positive roots of the equations

$$\left.\begin{aligned}
\nu_{12}^2 &= \frac{\varrho_1 \omega^2}{\mu_1} - (k + m_2)^2,\\
\nu_{12}'^2 &= \frac{\varrho_1 \omega^2}{\mu_1} - (k - m_2)^2,\\
\nu_{22}^2 &= (k + m_2)^2 - \frac{\varrho_2 \omega^2}{\mu_2},\\
\nu_{22}'^2 &= (k - m_2)^2 - \frac{\varrho_2 \omega^2}{\mu_2}.
\end{aligned}\right\} \tag{67}$$

Evidently, the equations (66) satisfy last two equations of (54) respectively.

Further the last equations of (66) satisfies the boundary condition (59).
Then substituting (50) and (66) in (60) we obtain

$$\left.\begin{aligned} &A_{12} \cos\nu_{12} H\, e^{-im_2 x} + A'_{12} \cos\nu'_{12} H\, e^{im_2 x} - \nu_{10} A_{10} \sin\nu_{10} H \sin(m_2 x + \delta) \\ &\qquad = A_{22}\, e^{-im_2 x} + A'_{22}\, e^{im_2 x} - \nu_{20} A_{10} \cos\nu_{10} H \sin(m_2 x + \delta)\,, \\ &\mu_1 [A_{12}\, \nu_{12} \sin\nu_{12} H\, e^{-im_2 x} + A'_{12}\, \nu'_{12} \sin\nu'_{12} H\, e^{im_2 x} \\ &\qquad + A_{10} \cos\nu_{10} H \{\nu_{10}^2 \sin(m_2 x + \delta) - i\,k\,m_2 \cos(m_2 x + \delta)\}] \\ &\qquad = \mu_2 [A_{22}\, \nu_{22}\, e^{-im_2 x} + A'_{22}\, \nu'_{22}\, e^{im_2 x} \\ &\qquad - A_{10} \cos\nu_{10} H \{\nu_{20}^2 \sin(m_2 x + \delta) + i\,k\,m_2 \cos(m_2 x + \delta)\}] \end{aligned}\right\} \quad (68)$$

to hold for all x's.

Then, on the basis of similar arguments as offered in the previous case, we obtain from (68) the following equations

$$\left.\begin{aligned} &A_{12} \cos\nu_{12} H - A_{22} = \frac{i\, e^{-i\delta}}{2} A_{10} (\nu_{10} \sin\nu_{10} H - \nu_{20} \cos\nu_{10} H)\,, \\ &A'_{12} \cos\nu'_{12} H - A'_{22} = \frac{-\,i\, e^{i\delta}}{2} A_{10} (\nu_{10} \sin\nu_{10} H - \nu_{20} \cos\nu_{10} H)\,, \\ &A_{12}\, \mu_1\, \nu_{12} \sin\nu_{12} H - A_{22}\, \mu_2\, \nu_{22} \\ &\qquad = -\,\frac{i\, e^{-i\delta}}{2} A_{10} \cos\nu_{10} H [(\mu_1 \nu_{10}^2 + \mu_2 \nu_{20}^2) - i\,k\,m_2 (\mu_1 - \mu_2)]\,, \\ &A'_{12}\, \mu_1\, \nu'_{12} \sin\nu'_{12} H - A_{22}\, \mu_2\, \nu_{22} \\ &\qquad = \frac{i\, e^{i\delta}}{2} A_{10} \cos\nu_{10} H [(\mu_1 \nu_{10}^2 + \mu_2 \nu_{20}^2) + i\,k\,m_2 (\mu_1 - \mu_2)]\,, \end{aligned}\right\} \quad (69)$$

which do not involve x.

The solutions of these equations in terms of A_{10} have been found out as

$$\left.\begin{aligned} A_{12} &= -\,\frac{i\, e^{-i\delta}}{2} A_{10} \cos\nu_{10} H \\ &\quad \times \frac{[\mu_1^2 (\nu_{10}^2 - k\, m_2) + \mu_1 \mu_2 (\nu_{20}^2 - \nu_{20} \nu_{22} + k\, m_2) - \mu_2^2\, \nu_{20} \nu_{22}]}{\mu_1 (\mu_1\, \nu_{12} \sin\nu_{12} H - \mu_2\, \nu_{22} \cos\nu_{12} H)}\,, \\ A'_{12} &= \frac{i\, e^{i\delta}}{2} A_{10} \cos\nu_{10} H \\ &\quad \times \frac{[\mu_1^2 (\nu_{10}^2 + k\, m_2) + \mu_1 \mu_2 (\nu_{20}^2 - \nu_{20} \nu'_{22} - k\, m_2) + \mu_2^2\, \nu_{20} \nu'_{22}]}{\mu_1 (\mu_1\, \nu'_{12} \sin\nu'_{12} H - \mu_2\, \nu'_{22} \cos\nu'_{12} H)}\,, \\ A_{22} &= \frac{i\, e^{-i\delta}}{2} A_{10} \cos\nu_{10} H \\ &\quad \times \frac{[\nu_{12} \nu_{20} (\mu_1 - \mu_2) \sin\nu_{12} H - \{(\mu_1 \nu_{10}^2 + \mu_2 \nu_{20}^2) - k\, m_2 (\mu_1 - \mu_2)\} \cos\nu_{12} H]}{(\mu_1\, \nu_{12} \sin\nu_{12} H - \mu_2\, \nu_{22} \cos\nu_{12} H)}\,, \\ A'_{22} &= \frac{-\,i\, e^{i\delta}}{2} A_{10} \cos\nu_{10} H \\ &\quad \times \frac{[\nu'_{12} \nu_{20} (\mu_1 - \mu_2) \sin\nu'_{12} H - \{(\mu_1 \nu_{10}^2 + \mu_2 \nu_{20}^2) + k\, m_2 (\mu_1 - \mu_2)\} \cos\nu'_{12} H]}{(\mu_1\, \nu'_{12} \sin\nu'_{12} H - \mu_2\, \nu'_{22} \cos\nu'_{12} H)}\,. \end{aligned}\right\} \quad (70)$$

These enable us to determine v_{12} and v_{22} completely.

The above analysis indicates that the propagation of Love waves is possible in a two layer medium when its free-surface and the interface are of sinusoidal types of small amplitudes. Corresponding to a given frequency ω, the primary wave propagates with the same wave number k as in the case of horizontal stratification. But along with the primary waves, there travel four different secondary waves with their wave numbers equal to $k + m_1$, $k - m_1$, $k + m_2$ and $k - m_2$ where m_1 and m_2 are parameters as involved in (45) and (46) respectively. Thus the same frequency curve for the primary wave can be used for the secondary waves by appropriate shifting of origin along k axis. Phase velocities of the different secondary waves will, no doubt, be different from that of the primary waves, but, the group velocities will remain unaltered.

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