Usual Assumptions in the Treatment of Wave Propagation in Heterogeneous Elastic Media[1]

By Ravi Ravindra[2]

Summary – Several assumptions are usually made in seismology in the treatment of wave propagational problems in heterogeneous elastic media. These assumptions are pointed out and some of them are critically examined.

1. Introduction

Any realistic model of the earth must necessarily consist of some regions in which the elastic properties of the materials change continuously from place to place. The equations of motion for such heterogeneous media are very complicated mathematically and do not lend themselves to any general solutions. One is forced to make several simplifying assumptions in order to render these equations mathematically manageable. There is a danger that one might take mathematical tractability as the overriding consideration and forget the physics of the problem. It is desirable to look at the assumptions which are usually made in the treatment of wave propagation in heterogeneous media. Unfortunately, however, any proper evaluation of the validity of an assumption must involve a solution of the problem without that particular assumption. Such a solution is likely to be mathematically quite difficult if not impossible. However, a few of the assumptions can still be checked for their reasonableness or otherwise. Relevance of the problem to seismology is obvious.

2. Theoretical considerations

One usually starts from the well-known equation of motion:

$$\varrho \frac{\partial^2 U_i}{\partial t^2} = \sum_{k=1}^{3} \frac{\partial}{\partial x_k} (\lambda \, \theta \, \delta_{ik} + 2 \, \mu \, e_{ik})$$

[1]) This paper was presented in a modified form at the meetings of the Canadian Association of Physicists in Toronto in June, 1967.

[2]) Department of Physics, Dalhousie University, Halifax, Nova Scotia, Canada. The author was formerly known as Ravindra N. Gupta.

where

$$\theta = \nabla \cdot \vec{U},$$

$$e_{ik} = \frac{1}{2}\left(\frac{\partial U_k}{\partial x_i} + \frac{\partial U_i}{\partial x_k}\right), \tag{1}$$

$\vec{U}$ = Displacement vector,

λ and μ are Lamé's parameters,

and ϱ is density.

This equation of motion is valid only for *isotropic, elastic* media as long as the strains are infinitesimal. In a vectorial form, for a heterogeneous medium, this equation can be written as:

$$\left.\begin{aligned}\varrho \frac{\partial^2 \vec{U}}{\partial t^2} = \nabla[(\lambda + 2\,\mu)\,\nabla\cdot\vec{U}] - \nabla\times[\mu\,\nabla\times\vec{U}] + \\ + 2[(\nabla\,\mu\cdot\nabla)\,\vec{U} - (\nabla\,\mu)\,(\nabla\cdot\vec{U}) + (\nabla\,\mu)\times(\nabla\times\vec{U})]\,.\end{aligned}\right\} \tag{2}$$

There is hardly anything one can do with this equation in this form; so some simplifying assumptions are made: (a) The properties of the medium change only in one direction (say z). (b) The waves are two-dimensional (with the field taken independent of y). (c) The Poisson's ratio of the medium is constant. (This assumption is not necessary if only λ varies with depth but μ and ϱ remain constant. For most earth materials, however, the assumption of a constant Poisson's ratio is perhaps more justified.) If these assumptions are allowed, then equation (2) reduces to (see Hook [3][3]))

$$\left.\begin{aligned}\varrho \frac{\partial^2 \vec{U}}{\partial t^2} = \gamma\,\nabla(\mu\,\nabla\cdot\vec{U}) - \nabla\times(\mu\,\nabla\times\vec{U}) + \\ + 2\left(\frac{\partial\mu}{\partial z}\right)\left(\frac{\partial\vec{U}}{\partial z} - i_z\,\nabla\cdot\vec{U} + i_z\times(\nabla\times\vec{U})\right)\end{aligned}\right\} \tag{3}$$

where

$$\nabla = \left[i_x \frac{\partial}{\partial x} + i_z \frac{\partial}{\partial z}\right],$$

$$\nabla\times\vec{U} = \left[-\,i_x \frac{\partial U_y}{\partial z} + i_z \frac{\partial U_y}{\partial x}\right],$$

$$\gamma = \frac{\lambda + 2\,\mu}{\mu} = 2\,\frac{(1-\sigma)}{(1-2\,\sigma)},$$

where σ is Poisson's ratio and i_x and i_z are unit vectors in the x- and z-direction, respectively.

Even in this simplified form, however, the equation is unmanageable. Physically,

---

[3]) Numbers in brackets refer to References, page 17.

one of the sources of the difficulty can be appreciated by imagining the medium to consist of a large number of thin homogeneous layers. At every interface there will be an interconversion between P and S waves; as the boundaries of these layers come closer, it will become more and more difficult to separate the P and the S wave motion.

We notice in Eqn. (3) that for a homogeneous medium some of the terms drop out and then we can use the well-known Helmholtz representation for a vector field and obtain the standard scalar wave equations, which correspond to the longitudinal and the transverse waves. If we follow a very similar method of attack on the equation of motion for a heterogeneous medium, as has been done by GRANT and WEST [1] (pp. 43–7), we find that P and S wave potentials are separable only under the following restrictions: (a) Density, ϱ, is constant throughout the medium. (b) Second and higher derivatives of λ and μ can be ignored. (c) The waves are incident more or less normally, i.e., along the direction of variation of the elastic parameters. (The last assumption reduces the model essentially to a one-dimensional one.) If we make these assumptions, the separated equations of motion can be written as

$$\varrho \frac{\partial^2 \theta}{\partial t^2} = (\lambda + 2\,\mu)\,\nabla^2\,\theta + 2(\lambda' + 2\,\mu')\frac{\partial \theta}{\partial z}$$

or

$$\frac{\partial^2 \theta}{\partial t^2} = \nabla^2[\alpha^2\,\theta]\,.$$

Similarly

$$\frac{\partial^2 \xi}{\partial t^2} = \nabla^2[\beta^2\,\xi]\,, \tag{4}$$

where

$$\theta = \nabla \cdot \vec{U}\,,\quad \xi = \nabla \times \vec{U}\,,$$

$$\alpha = \left(\frac{\lambda + 2\,\mu}{\varrho}\right)^{1/2},\quad \beta = \left(\frac{\mu}{\varrho}\right)^{1/2},$$

and prime indicates a derivative with respect to z.

We can add here the equation of motion for S–H wave which separates out without any of the last three assumptions.

$$\varrho \frac{\partial^2 U_y}{\partial t^2} = \mu\,\nabla^2\,U_y + \mu'\,\frac{\partial U_y}{\partial z}. \tag{5}$$

It should be noted that almost all the usual expressions used in seismology, even for oblique incidence, for heterogeneous elastic media are variations of these equations – including those in seismic ray theory.

We can estimate the errors introduced by the last two assumptions – ignoring the second and higher derivatives of λ and μ and assuming near vertical incidence – by redefining the displacement potentials and assuming a particular model in which α and

β increase linearly. If we define

$$\vec{U} = \mu \nabla(\mu^{-1} \phi) + \mu^{-1} \nabla \times (\mu \vec{\psi}), \tag{6}$$

and have

$$\alpha = \alpha_1(1 + b z),$$
$$\beta = \beta_1(1 + b z),$$
$$\lambda = \mu$$
$$\varrho = \text{constant},$$

then the separated potentials satisfy the following scalar equations of motion (see GUPTA [2]):

$$\left.\begin{aligned} \frac{\partial^2 \phi}{\partial t^2} &= \alpha^2 \nabla^2 \phi \\ \text{and} \quad \frac{\partial^2 \psi}{\partial t^2} &= \beta^2 \nabla^2 \psi \end{aligned}\right\} \tag{7}$$

where $\psi = \vec{\psi}_y$.

Neither of the above separated waves is pure; i.e., purely rotational or longitudinal. If we concentrate our attention only on the 'pseudo P' wave, we can write (7) as

$$\frac{\partial^2}{\partial t^2} (\nabla^2 \phi) = \nabla^2[\alpha^2 \nabla^2 \phi]. \tag{8}$$

Compare this with the equation obtained by GRANT and WEST [1], viz (4).

It should be remembered that (8) is derived without assuming normal incidence and without ignoring second and higher derivative of λ and μ. Now, (4) will be exactly the same as (8) only if

$$\theta \equiv \nabla \cdot \vec{U} = \nabla^2 \phi \tag{9}$$

which, it turns out from (6), is true only when the medium is homogeneous and we can follow the ordinary method of separation of P and S wave potentials. In the present case, however, we get from (6)

$$\theta = \nabla^2 \phi - \frac{\partial}{\partial z}\left(\frac{\mu'}{\mu} \phi\right) + \frac{\partial}{\partial x}\left(\frac{\mu'}{\mu} \psi\right). \tag{10}$$

Out of the two correction terms in (10) the first one arises because of the terms which were not ignored as second and higher derivatives of μ and λ; the second one gives the error introduced because of the assumption of normal incidence. For the exact estimates of these errors, we will have to calculate ϕ and ψ which obviously depend on the particular boundary conditions and therefore vary from problem to problem. The intention here is merely to point out the existence of these correction terms. Their magnitude will be discussed elsewhere in connection with a specific problem.

So far nothing has been said about the assumption of constant density. This assumption is hardly right. Let us mention another assumption and then we can look at both of these together. It seems to have become a habit to call α and β, as defined in (4), the P and the S *velocities* of the medium. What sort of velocity is α? Phase velocity or signal velocity? As long as we are dealing with homogeneous media this lack of precision does not matter because all the velocities are the same, and since the separated equation of motion for P and for S waves are exactly in the form of the standard wave equation, from the *physics* of the problem, we write the phase velocity (which is the same as group velocity, U) as

$$v = U = \left(\frac{\lambda + 2\,\mu}{\varrho}\right)^{1/2} = \alpha$$

(or β for S waves). But there is no a priori reason to believe that phase velocity for a heterogeneous medium does not involve frequency dependence and the gradients of elastic parameters. It turns out, in fact, that α is neither phase velocity nor group velocity nor any other velocity. It is only in the high-frequency limit that U and v approach α. An example will make it clear. Let us assume the following variation of ϱ, α, and β:

$$\left.\begin{aligned} \alpha &= \alpha_1(1 + b\,z) \\ \beta &= \beta_1(1 + b\,z) \\ \varrho &= \varrho_1(1 + b\,z)^p \end{aligned}\right\} \tag{11}$$

where the law of variation of density changes depending on the value of p. Notice that for $p=0$ we get constant density – the case we had earlier. Let us confine our attention only to normal incidence for which the following scalar wave equations hold:

$$\left.\begin{aligned} &\left(\frac{\partial^2}{\partial z^2} + \frac{\mu'}{\mu}\frac{\partial}{\partial z} - \alpha^{-2}\frac{\partial^2}{\partial t^2}\right) U_z = 0 \\ &\text{and} \\ &\left(\frac{\partial^2}{\partial z^2} + \frac{\mu'}{\mu}\frac{\partial}{\partial z} - \beta^{-2}\frac{\partial^2}{\partial t^2}\right)\begin{pmatrix} U_x \\ U_y \end{pmatrix} = 0. \end{aligned}\right\} \tag{12}$$

Confining our attention only to the P wave, the solution of (12) corresponding to the progressive wave travelling in the positive z direction is

$$U_z = A(1 + b\,z)^{-\left(\frac{p+1}{2}\right)} \exp\left\{- i\,\omega\,t + \frac{i\,\omega}{\alpha_1\,b}\,S\ln(1 + b\,z)\right\} \tag{13}$$

where

$$S = \left[1 - \left(\frac{p+1}{2}\right)^2 \left(\frac{b\,\alpha_1}{\omega}\right)^2\right]^{1/2}$$

and ω is the angular frequency.

The phase velocity, v, is given by

$$v = \frac{dz}{dt} = \alpha_1(1 + b\,z)\,S^{-1} = \alpha\,S^{-1}. \tag{14}$$

The group velocity, U, is given by the well-known relationship

$$U = v\left[1 - \frac{\omega}{v}\frac{dv}{d\omega}\right]^{-1} \tag{15}$$

Therefore,

$$U = \alpha\,S \tag{16}$$

It is obvious from (13), (14), and (15) that for $\omega \to \infty$, $S \to 1$, and therefore, $U \to v \to \alpha$. On the other hand, as $\omega \to \frac{(p+1)}{2} b\,\alpha_1$, $S \to 0$ and therefore $v \to \infty$ and $U \to 0$; i.e., there is no propagation of energy.

We see from above that the group velocity with which the signal travels depends on S which contains both p (which determines the law of variation of density) and ω, the frequency, in addition to b which determines the rapidity with which α changes with depth. The extent of this dependence is discussed elsewhere (RAVINDRA [4]). It is clear, however, that a heterogeneous medium is necessarily a dispersive medium, and that the extent of dispersion is dependent on the gradient of density in the medium.

In summary, various assumptions that are usually made in the treatment of wave propagation in heterogeneous media have been pointed out. Some of these assumptions are less unreasonable than the others; but as we deal with lower and lower frequencies in seismology, the need for a proper appreciation of these restrictions increases.

Acknowledgements

This research has been supported by a grant from the National Research Council of Canada.

REFERENCES

[1] F. S. GRANT and G. F. WEST, *Interpretation Theory in Applied Geophysics* (McGraw-Hill, New York 1965).

[2] RAVINDRA N. GUPTA, *Reflection of elastic waves from a linear transition layer*, Bull. Seism. Soc. Am. *56* (1966), 511.

[3] J. F. HOOK, *Separation of vector wave equation of elasticity for certain types of isotropic, inhomogeneous media*, Jour. Acoust. Soc. Am. *33* (1961), 302.

[4] R. RAVINDRA, *Dispersion of waves in a continuously stratified elastic medium*, Pure and Appl. Geophys. (1968) (in press).

(Received 14th August 1967)