The Theoretical Model of the Drop Spectrum Formation Process in Clouds

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Summary – In this paper the theoretical model of cloud spectrum formation process is built in which spectrum formation and its development are followed by and these observations find no contradictions, beginning with air ascent containing condensation of nuclei and up to large drop formation falling out off the cloud. Both the obtained results of velocity spectrum formation and the form of distribution function corresponds to the direct observations. The model does not use any empirical relations and empirical regularities; it is free from arbitrary suppositions and assumptions which are not examined by experiments, and the number of parameters in it is limited by ones (vertical velocity U_z, turbulent diffusion coefficient K_z and two parameter distribution function of nucleus condensation with super-saturation). First in the developed theory the principal contradictions not allowing till now to connect together condensational and coagulation stages of cloud spectrum development are overcome.

Processes taking place in clouds are within wide range of scales beginning from transition of condensation nuclei about 10^{-5} cm in size into drops and to cloud system formation with characteristic scales of about 10^7 cm. In connection with this physics of clouds in its approach to investigation of a phenomenon operates with various methods and is naturally subdivided into a number of sections the most significant of which is physics of macroprocesses and microprocesses in clouds. As microprocesses characterize changes occurring with separately taken drops, investigation of drop aggregate behaviour is determined to a certain extent by microprocesses and, on the other hand, it apparently can give interesting additional information on macrophenomena connected with precipitation, change of visibility, icing and etc. Behaviour of drop aggregates to a certain extent is related with intermediate scales and serves as a bridge which connects micro- and macroprocesses. For theoretical description of drop aggregate behaviour regularities the so-called kinetic equation is usually used which defines behaviour of the particle size distribution function.

However, it should be noted that at present there is no satisfactorily developed scheme for calculation of precipitation formation. It is caused by a number of fundamental contradictions which are not yet eliminated and not by the difficulty of solving this problem from the mathematical standpoint.

In this paper the theoretical model of kinetics of cloud spectrum formation is

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offered. The development of this model was presented in a series of our papers. Analysis of equations describing drop growth by condensation and by coagulation showed that improvement of the theory did not lead to considerable increase of velocity of drop radius change [1][2]), and turbulent coagulation [2, 3, 4] and electrostatic coagulation [5] even together with gravitational one [6, 7] did not introduce any substantial corrections in our motion of drop growth by coagulation and of existence of critical size below which impacts do not occur.

Since cloud spectrum formation is determined in the first place by the amount of drops taking part in the process, let us consider first of all kinetics of condensation nucleus transition into drops.

1. The theoretical model of condensation nuclei

Following JUNGE let us consider that the function of dry nucleus size distribution (differential one) can be given in the form

$$n(r_0) = a\, r_0^{-\nu}. \tag{1}$$

According to notions developed in [9, 10], nucleus activity can be expressed as a function of r_0;

$$C = b\, r_0^{2(1+\alpha)} \tag{2}$$

where b and α are some parameters whose value is determined by the composition and physical chemical properties of the nucleus. Then the function of nucleus size distribution can be obtained for any value of supersaturation δ_0:

$$\begin{aligned} n(r/\delta_0) = {} & \frac{a}{(1+\alpha)} \left(\frac{B}{b}\right)^{(1-\nu)/2(1+\alpha)} r^{-(\nu+\alpha)/(1+\alpha)} \\ & \times \left[1 - \frac{\delta_0}{(1-\delta_0)\, B}\, r\right]^{-(\nu+1+2\alpha)/2(1+\alpha)} \left(1 - \frac{3}{2}\frac{\delta_0}{(1-\delta_0)\, B}\, r\right). \end{aligned} \tag{3}$$

Knowledge of the distribution function (1) and the relation between C and r_0 (2) allows us to obtain the function of limiting supersaturation distribution which is very important for investigation of kinetics of drop formation. It should be noted that by limiting supersaturation we understand maximum supersaturation at which a nucleus of a given size r_0 or a given activity C has possibility of unlimited growth. According to [9]

$$n(\delta_0) = \frac{a}{1+\alpha}\left(\frac{4\, B^3}{27\, b}\right)^{(1-\nu)/2(1+\alpha)} (1-\delta_0)^{(\nu+2+3\alpha)/(1+\alpha)}\, \delta_0^{(\nu-2-\alpha)/(1+\alpha)} = P\, \delta_0^{l}. \tag{4}$$

In (3) and (4) $B = 2\,\sigma\, M/\varrho_w\, R\, T$ where σ is the coefficient of surface tension, ϱ_w is density, M is the molecular weight of water, R is the gas constant, T is absolute temperature.

[2]) Numbers in brackets refer to References, pages 334/335.

Knowledge of distribution of (3), (4) type is of great importance because using the r distribution function it is possible to study fluctuations of visibility in the atmosphere and some other processes and on the base of (4) type distribution the process of drop formation can be investigated. However it should be said that (3) and (4) have been obtained under the assumption that there is equilibrium between nuclei and their environment. But if environmental conditions change with time, some lag from equilibrium may take place. Let us estimate significance of this effect by considering the characteristic time when drops come to in equilibrium. The equation of drop growth can be given in the following form:

$$\frac{dr}{dt} = D\frac{\varrho}{\varrho_w}\frac{\chi(r)}{r^2}\delta(r, \delta_0) \tag{5}$$

$\chi(r)$ characterizes corrections for growth velocity, ϱ is density, D is the coefficient of vapour diffusion and δ is supersaturation above the drop surface which can be presented as

$$\delta(r, \delta_0) = \delta_0 - \left(\frac{B}{r} - \frac{C}{r^3}\right). \tag{6}$$

Under equilibrium conditions $\delta=0$ and $dr/dt=0$. When supersaturation, δ_0, changes, the nucleus equilibrium radius r_e changes as well, δ becomes different from 0 and the particle changes its size according to changed conditions. Let us assume that a particle is near the equilibrium point where $\delta(r_e, \delta_0)=0$. Then in the first approximation

$$\delta(r, \delta_0) \approx \left(\delta_0 - \frac{2\,C}{r_e^3}\right)\left(\frac{r - r_e}{r}\right) = 2\,B\left(\frac{1}{r_b} - \frac{1}{r_e}\right)\frac{\Delta r}{r} \tag{7}$$

where

$$\Delta r = r - r_e, \quad \frac{1}{r_b} \approx \frac{3\,\delta_0}{2\,B}.$$

It should be noted that that linear approximation over Δr can be used as the upper value of the characteristic time since linear dependence (7) results in underestimated values of supersaturation in comparison to (6). Substituting (7) into (5) and $\chi(r)/r^2$ for $\chi(r_e)/r_e^2$ it is possible to obtain the following solution:

$$r = r_e + (r_0 - r_e)\,e^{-t/\tau} \tag{8}$$

where

$$\tau = \frac{\varrho_w\, r_e^4}{2\,B\,D\,\varrho\,\chi(r_e)}\frac{r_b}{r_b - r_e} = T\frac{r_b}{r_b - r_e} \tag{9}$$

T highly depends on the nucleus radius. Estimations show that for $r_e \approx 10^{-5}$ cm, $T \approx 4 \cdot 10^{-2}$ sec and for $r_e \approx 10^{-4}$ cm, $T \approx 5$ sec. Nevertheless, since the variable part of the nucleus spectrum is, as a rule, within this size range, T is sufficiently small and it is reasonable to suppose that changes of the spectrum, in general, follow changes of environmental conditions which have time scales of the order of tens of seconds and higher. As τ is also det erminedby the factor $r_b/r_b - r_e$, notice that for $\delta_0 < 0 (r_b < 0)$ this factor is less than 1 and $\tau < T$, and for $\delta_0 > 0$ it is higher than 1 and $\tau > T$. At positive δ_0 the value of $r = r_b$ separates the area of nuclei from that of drops [9] and with the approach to this boundary characteristic time τ tends to infinity. When considering transition of nuclei into drops it is very important to estimate the width of size range for which characteristic time is higher than that of the process. Using the supersaturation distribution function the zone width characterizes the aggregate of particles which lag behind in growth and have no chance to follow δ change. As drop formation takes place mainly on nuclei with the radius of the order of 10^{-5} cm and during the time of about several seconds, equating $\tau \approx 10$ sec it can be majorized the total amount of particles which lag behind.

From (9) we have

$$\frac{r_e - r_b}{r_b} = \frac{4 \cdot 10^{-2}}{\tau} \approx 4 \cdot 10^{-3}.$$

For a spectrum of JUNGE's type ($n = a\, r^{-\nu}$) the ratio of the amount of particles lagging behind to the amount of particles which have already transited into drops, N, is determined by:

$$\frac{\Delta N}{N} \approx \nu \frac{\Delta r}{r} \approx \nu\, 4 \cdot 10^{-3} \ll 1.$$

Thus, as estimations show, use of distribution function in accordance with the assumption of equilibrium existence should not lead to significant errors.

2. *Kinetics of the initial stage of drop spectrum formation*

Let us consider the process of drop formation from condensation nuclei. According to [11] this process is described by the following system of equations:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial r}(\dot{r} f) = n(\delta_0) \frac{d\delta_0}{dt} \delta_1 (r - r_b) \tag{10,1}$$

$$\dot{r} = D \frac{\varrho_0}{\varrho_w} \frac{\chi(r)}{r^2} \delta_0 \tag{10,2}$$

$$\frac{d\delta_0}{dt} = (1 - \delta_0) \left[\left(1 - \frac{L M}{C_p M_1 T}\right) \frac{1}{p} \frac{dp}{dt} - 4 \pi D \varphi_1 \left(1 + \frac{L M}{R T} \frac{L}{C_p T} \frac{\varrho_0}{\varrho_1}\right) \delta_0 \right] \tag{10,3}$$

$$\frac{dp}{dt} \approx - g \varrho_1 U_z \tag{10,4}$$

$$\varphi_1 = \int_{rb}^{\infty} \chi(r) f(r, t)\, dr \tag{10,5}$$

Here $n(\delta_0)$ = the function of nucleus distribution over limiting supersaturation which is determined by (4), $\varrho_0, \varrho_1, \varrho_w$ = density of saturating vapour, air and water, respectively; L = condensation heat; C_p = specific heat of air; U_z = vertical air velocity; g = gravity acceleration; M_1 = the molecular weight of air; $\chi(r)$ = the function which for the transitional area between kinetic and diffusion regime of condensation can be presented in the form:

$$\chi(r) = \frac{r^2}{\zeta + r} \tag{11}$$

where

$$\zeta \approx \frac{4}{3} \frac{\lambda}{\alpha}$$

λ = free path of molecules; α = the condensation coefficient ($\alpha \approx 0.036$). In (10,1) $\delta_1(r - r_b)$ is the delta function. Substitution of (10,4) and (10,5) into (10,3), integration of (10,2) and writing of formal solution for (10,1) lead the problem to the following form:

$$\left.\begin{aligned} y''(t) = K_1 - K_2\, y'(t) \int_0^{\bar{t}} [\zeta_1^2 + y(t) - y(t_0)]^{1/2}\, y'^{l}(t_0)\, y''(t_0)\, dt_0 \\ + 2 K_2 \zeta_1\, y'(t) \int_0^{\bar{t}} y'^{l}(t_0)\, y''(t_0)\, dt_0 - K_2 \zeta_1^2\, y'(t) \int_0^{\bar{t}} \frac{y'^{l}(t_0)\, y''(t_0)\, dt_0}{\sqrt{\zeta_1^2 + y(t) - y(t_0)}} \end{aligned}\right\} \tag{12}$$

$$f(r, t) = \frac{P\, \delta_0^{l-1}(t_0)}{K_3} (r + \zeta) \frac{d\delta_0(t_0)}{dt} \tag{13}$$

where

$t_0 = t_0(r, t)$ is the integral of (10,2)

$$y(t) = \int_0^t \delta_0(t)\, dt$$

and

$$K_1 = \left(\frac{L M}{R T} - \frac{C_p M_1}{R}\right) \frac{\gamma_a}{T} U_z$$

$$K_2 = \left(2 D \frac{\varrho_0}{\varrho_w}\right)^{1/2} \cdot 4 \pi D P \left(1 + \frac{L M}{R T} \frac{L}{C_p T} \frac{\varrho_0}{\varrho_1}\right)$$

$$K_3 = D \frac{\varrho_0}{\varrho_w}$$

$$\bar{t} = \begin{Bmatrix} t & t < t_M \\ t_M & t > t_M \end{Bmatrix} \quad y''(t_M) = 0$$

$$\zeta_1 = \frac{\zeta}{\sqrt{2 K_3}}$$

γ_a is the adiabatic gradient.

The equation (12) which characterizes change of supersaturation values with time can be considered independently in the form it was obtained. The similar type of an equation was obtained in papers [11, [12, [13, [14]. It should be noticed that (12) was obtained under the assumption that the effect of the initial radius of particles can be neglected.

It is not difficult to take r_0 into account and it can be done in the way analogous to [11]. But since estimations and some calculations [11] carried out by us show that the effect of r_0 is insignificant, we considered it possible to neglect this effect and further to investigate the equation of the (12) type which in the form it was written down allows decreasing the equation order. Integration of (12) and transformation of the variables, $y=\beta z$, $t=\gamma \tau$, lead to the form:

$$\left.\begin{aligned} z' &= \tau - \frac{2}{3}\int_0^{\bar{\tau}} \{[a^2 + z(\tau) - z(\tau_0)]^{3/2} - a^2\} (z(\tau_0)')^l z''(\tau_0)\, d\tau_0 \\ &+ \frac{2\,a}{l+1}\int_0^{\bar{\tau}} (z'(\tau_0))^{l+2}\, d\tau_0 + \frac{2\,a}{l+1} (z'(\tau_M))^{l+1} [z(\tau) - z(\tau_M)]\, \theta(\tau - \tau_M) \\ &- 2\,a^2 \int_0^{\bar{\tau}} \{[a^2 + z(\tau) - z(\tau_0)]^{1/2} - a\} (z'(\tau_0))^l z''(\tau_0)\, d\tau_0 \end{aligned}\right\} \tag{14}$$

where

$$a = \frac{\zeta_1}{\sqrt{\beta}} = \frac{\zeta (K_1^{3/2+l} K_2)^{1/(3+l)}}{(2\,K_1\,K_3)^{1/2}}$$

$$\beta = K_1 (K_1^{3/2+l} K_2)^{-2/(3+l)}$$

$$\gamma = (K_1^{3/2+l} K_2)^{-1/(3+l)}$$

$$\theta = \begin{cases} 0 & \tau < \tau_M \\ 1 & \tau > \tau_M \,. \end{cases}$$

The equation (14) was solved numerically on the electronic computer Minsk-2. The function z' for various values of a and $l=1$ is given in Fig. 1. Comparison of obtained results with those of [11] shows that regard to correction for ζ does not break the character of the process though increases the time of reaching the maximum and the value of maximum supersaturation. As all these changes are of no particular significance, the results obtained in (11) remain valid and must be corrected for a certain factor the value of which is of the order of 1. The dependence of supersaturation on main parameters remains in the same form because additionally introduced parameter, a, changes very little. On the base of obtained results it is easy to calculate the number of formed particles, the time of reaching the supersaturation maximum and using (13) to obtain the distribution function $f(r, t)$. These problems are considered in detail in [11], therefore here we shall give only the main results.

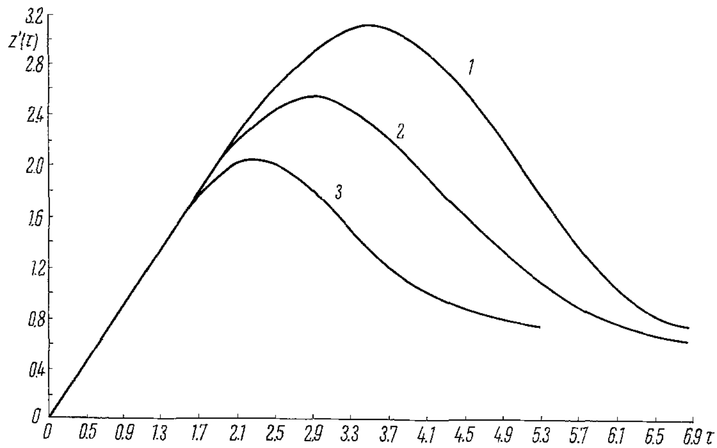

Figure 1
1. $a=3$; 2. $a=2$; 3. $a=1$

The number of formed drops N is described by

$$N = \frac{P}{l+1}\left(\frac{K_1^{3/2}}{K_2}\right)^{(l+1)/(l+3)} [z(\tau_M)]^{l+1}. \tag{15}$$

The maximum supersaturation reached in the process

$$\delta_{0\,max} = \left(\frac{K_1^{3/2}}{K_2}\right)^{1/(3+l)} z'(\tau_M). \tag{16}$$

The time of reaching the maximum supersaturation is defined by the formula

$$t_M = (K_1^{(3+2l)/2}\, K_2)^{-1/(3+l)}\, \tau_M. \tag{17}$$

The difference between maximum and minimum drop sizes

$$r_{max} - r_{min} \approx \left[\frac{K_1\, K_3}{3(l+1)}\right]^{1/2} \frac{\delta_{0\,max}^{(l+1)/2}\, t_M^2}{2}\, t^{-1/3}. \tag{18}$$

The calculation results of N, δ_0, t_M and $r_{max}-r_{min}$ for typical parameter values of the process are given in Table 1.
The following conclusions can be drawn basing on the results of calculations. The initial stage of condensation takes a short period of time (in comparison to the cloud

Table 1

U_z [cm/sec]	N [cm^{-3}]	$\delta_{0\,max}$ [%]	t_M [sec]
1	$6.1\cdot10$	$1.4\cdot10^{-2}$	35.9
5	$2.2\cdot10^{2}$	$2.6\cdot10^{-2}$	12.8
10	$3.2\cdot10^{2}$	$3.2\cdot10^{-2}$	8.5

$p=1000$ mb; $T=283$°K; $P=0.65\cdot10^{10}$ [cm^{-4}]; $l=1$

life duration) and small space scale $U_z t_M$ (in comparison to cloud sizes). Dispersion of the drop size distribution function is small and decreases with time. These facts as well as stability of the kinetic equation solution relative to variations of the initial distribution function (in accordance with the numerical calculations carried out) allow us to take use of the following simplifications when considering the problems on cloud spectrum formation. It is possible to assume that initial stage determines only the number of formed drops, and distribution of the delta-function type (or rather narrow distribution) with the average radius of about 1 micron can be used as an initial distribution function. Regular rate of drop formation within the cloud may not be taken into account. At the cloud boundary where entrainment of fresh portions of air takes place it is necessary to take into consideration the action of the surface source of drop formation the intensity of which is determined by relations obtained from calculations for the initial stage. Drop formation within the cloud occurs owing to the fluctuation mechanism, and disappearance of drops takes place due to zero boundary conditions at $r = r_b$, r_b can be regarded equal to zero.

3. On fluctuations of meteorological parameters in clouds. Stochastic condensation

Before considering the kinetic equation of cloud drop growth let us discuss the important problem of meteorological parameter fluctuations in the cloud taking place as a result of turbulence. This question in application to the atmosphere was intensively studied in the recent years in connection with the problem of electromagnetic wave propagation [15]. The nature of such pulsations considered for the first time relative to temperature [15] is conditioned by turbulent transport of air elements to a certain level from other levels at which the values of this or that parameter are different. Thus, turbulence mixes elements of the air and smoothes mean values of parameters, sharpens locally gradients, and increases fluctuation intensity of these parameters at a certain level. For the clear atmosphere these questions were investigated well, but when applied to clouds they were considered not long ago [16], [17]. From the standpoint of processes taking place in clouds it is more interesting to know pulsations of a certain value not at some point of the space but appropriate to a certain element of the air volume moving in the space. Under such consideration which can be called Lagrangian it is possible to study pulsations of not only such values as temperature, velocity, humidity but also pulsations of the values characteristic of clouds (liquid water content, vapour density, supersaturation). Since investigation results of all these problems are presented in [17], we shall consider here only pulsations of supersaturation. As supersaturation, δ_0, is determined by the relation $\delta_0 = 1 - \varrho_0/\varrho$, then taking into account that

$$\frac{d\varrho_0}{dt} = \frac{d}{dT}\left(\frac{p_0\,M}{R\,T}\right)\frac{dT}{dt} = \left(\frac{M}{R\,T}\,\frac{dp_0}{dT} - \frac{p_0\,M}{R\,T^2}\right)\frac{dt}{dT} = \frac{\varrho_0}{T}\left(\frac{L\,M}{R\,T} - 1\right)\frac{dT}{dt},$$

and keeping in mind that

$$\frac{d\varrho}{dt} = 4\,\pi\,D\,\varphi_1(\varrho_0 - \varrho) + \frac{\varrho\,M_1}{\varrho_1\,R\,T}\frac{dp}{dt} - \frac{\varrho}{T}\frac{dT}{dt},$$

we shall obtain

$$\frac{1}{1-\delta_0}\frac{d\delta_0}{dt} = \frac{R}{C_p\,M_1}\left(\frac{C_p\,M_1}{R} - \frac{L\,M}{R\,T}\right)\frac{1}{p}\frac{dp}{dt} - \beta\left(1 + \frac{L\,M}{R\,T}\frac{L}{C_p\,T}\frac{\varrho}{\varrho_1}\right)\delta_0 \tag{19}$$

where $\beta = 4\,\pi\,D\,\varphi_1$. (For the cases, when cloud formation occurs owing not to movement but to radiation processes δ_0 change can be easily related to the temperature changes). Further simplifications are connected with the following assumptions. Estimates show that the value of δ_0 is always much lower than 1 and therefore it can be taken $1-\delta_0 \approx 1$. For the same reason ϱ in the second term of the right part of (19) can be substituted for ϱ_0.

Further:

$$\frac{dp}{dt} \approx \frac{\partial p}{\partial t} - U_z\,g\,\varrho_1 \tag{20}$$

(z-axis is directed from the Earth's surface). The equation (19) has a stochastic sense because in the right part of it there are random variables $\partial p/\partial t$ and U_z. The process of δ_0 change is of relaxational character with relaxation time $1/\beta$ of about several seconds. The order of magnitude of $\partial p/\partial t$ can be evaluated if to use measurement results of pressure micropulsations in the atmosphere [18], [19]. It can be assumed (taking into account that the regular part of $\partial p/\partial t$ is small) that

$$\frac{\partial p}{\partial t} \sim \left[\overline{\left(\frac{\partial p}{\partial t}\right)^2}\right]^{1/2} = \int_0^\infty \omega^2\,F(\omega)\,d\omega \tag{21}$$

where $F(\omega)$ is spectral density of pressure fluctuations. Using the data given in [18] it is possible to calculate the integral (21) which is equal to $\sim 0.2\,\sigma_p$ where $\sigma_p = (7.5 \pm 1.4)$ dyne/cm^2. Thus, $\partial p/\partial t \sim (1.5 \pm 0.3)$ dyne/cm^2 sec. Taking into consideration that for the atmosphere U_z (here U_z is mean and pulsational component of velocity) is of the order of several tens of centimeters, it can be concluded that contribution of $\partial p/\partial t$ to the total derivative dp/dt does not exceed 7–10%. Thus, in the first approximation it is possible to consider that pulsations of δ_0 occur due to velocity pulsations and to investigate the equation more simple than (19):

$$\frac{d\delta_0}{dt} = b\,U_z - c\,\delta_0 \tag{22}$$

where

$$b = \left(\frac{L\,M}{R\,T} - \frac{C_p\,M_1}{K}\right)\gamma_a; \quad c = 4\,\pi\,D\,\varphi_1\left(1 + \frac{L\,M}{R\,T}\frac{L}{C_p\,T}\frac{\varrho_0}{\varrho_1}\right).$$

In the quasi-stationary approximation

$$\delta_0 = \frac{b}{c} U_z . \tag{23}$$

This relation is widely used for the case when U_z is the mean value of vertical velocity, and in this sense the equality (23) is beyond any doubt. When considering the pulsational component of δ_0, validity of (23) is determined to a considerable extent by the frequency characteristic of U_z. Let us present U_z in the form of Fourier's integral:

$$U_z = \int_0^\infty (\alpha \cos \omega t + \sigma \sin \omega t)\, d\omega .$$

Then considering b and c constant we shall find the corresponding expansion of δ_0. We have

$$\delta_0 = \int_0^\infty (\gamma \cos \omega t + \lambda \sin \omega t)\, d\omega .$$

Using (22) it can be easily obtained

$$\gamma = b \frac{c\,\alpha - \omega\,\sigma}{\omega^2 + c^2}, \quad \lambda = b \frac{\omega\,\alpha + c\,\sigma}{\omega^2 + c^2} .$$

Let us introduce Lagrangian spectral function for vertical pulsations of velocity $F_U(\omega)$.

$$F_U(\omega) = \frac{\alpha^2(\omega) + \sigma^2(\omega)}{T_1}$$

where T_1 is a rather long period of time determining the interval of time averaging of the process the sense of which will not be discussed here. Let us introduce the spectral function of supersaturation in an analogous way:

$$F_\delta(\omega) = \frac{b^2}{\omega^2 + c^2} F_U(\omega) = \chi(\omega)\, F_U(\omega) \tag{24}$$

where $\chi(\omega)$ is the transitional function characterizing change of δ_0 spectrum at a known spectral function $F_U(\omega)$. Thus, for example, if to take that Lagrangian correlation function of vertical pulsations of velocity $R_L(t) = e^{-t/\tau}$, then

$$F_U(\omega) = \frac{2}{\pi} \overline{(U_z')^2} \frac{\tau}{1 + \omega^2 \tau^2} \quad \text{and} \quad F_\delta(\omega)$$

can be easily found

$$F_\delta(\omega) = \frac{2\,b^2}{\pi} \overline{(U_z')^2} \frac{\tau}{(\omega^2 + c^2)(\omega^2 t^2 + 1)} .$$

Thus, pulsations of δ_0 in general form can be easily investigated. However, it should be noted that the value of β is of the order of 1 sec^{-1}, therefore the high-frequency end of the spectrum with characteristic time less than several seconds smooths (the corresponding estimations can be easily made with the help of relations given above). In principle high-frequency pulsations do not effect drop growth. Taking this into account it is possible for simplicity to use the equality (23) assuming that U_z characterizes both the mean value $\bar{U}_z$ and the pulsational part U'_z. A principally new idea of equality (23) will consist in this. Further discussion will be trivial. If δ_0 is a random variable, the equation of drop growth acquires a stochastic sense and describes so-called stochastic condensation

$$\frac{dr^2}{dt} = 2\,D\,\frac{\varrho}{\varrho_w}\,\delta_0 = 2\,D\,\frac{\varrho}{\varrho_w}\,\frac{b}{c}\,U_z = A\,U_z\,. \tag{25}$$

If to consider that φ_1 slightly changes with time, then it is very easy to integrate the obtained equation

$$r^2(t) - r_0^2 = A\left[\bar{U}_z\,t + z'\right]. \tag{26}$$

Supposing that change of the vertical co-ordinate z' (in conformity with ideas sometimes being assumed in the problem of turbulent diffusion of aerosols) satisfies the normal law

$$\psi(z') = \frac{1}{(4\,\pi\,K_z\,t)^{1/2}}\exp\left(-\frac{(z')^2}{4\,K_z\,t}\right) \tag{27}$$

the distribution function for r can be easily obtained

$$\varphi(r) = \frac{r}{(\pi\,A^2\,K_z\,t)^{1/2}}\exp\left\{-\frac{[r^2 - r_0^2 - \bar{U}_z\,A\,t]^2}{4\,A^2\,K_z\,t}\right\}. \tag{28}$$

Thus, regard for the fact that δ_0 is random results in the distribution function of (28) type with dispersion increasing with time. It is very important because this process leads to appearance in the volume of some amount of large drops, coagulation for them being permitted. It should be said that within the limits of (25) it is impossible to take into consideration other mechanisms of particle growth and transport. Therefore this equation has a limited sense by itself. Next part of our paper will be devoted to working up the mathematical apparatus which would allow to account all the factors determining drop behaviour.

4. The stochastic kinetic equation of cloud drop growth

Let us write down a kinetic equation for the distribution function $f(\sigma)$ for the elementary volume $[\sigma = r^2]$:

$$\frac{\partial f}{\partial t} + [U_i - v(\sigma)\,\delta_{iz}]\,\frac{\partial f}{\partial x_i} + \frac{\partial}{\partial \sigma}\left(\frac{d\sigma}{dt}\cdot f\right) = H(f,f) + \Phi(\sigma, \vec{U}) \tag{29}$$

where $H(f,f)$ = integral of collisions

$$H = \frac{1}{2}\int_0^\sigma \frac{\sigma^{1/2}}{(\sigma^{3/2} - \sigma_1^{3/2})^{1/3}}\,\beta(\sigma^{3/2} - \sigma_1^{3/2}, \sigma_1^{3/2})\,n\left[(\sigma^{3/2} - \sigma_1^{3/2})^{2/3}, \vec{r}, t\right]\,n(\sigma_1, t)\,d\sigma_1$$

$$- n(\sigma)\int_0^\infty \beta(\sigma^{3/2}, \sigma_1^{3/2})\,n(\sigma_1, t)\,d\sigma_1$$

$\beta(\sigma, \sigma_1)$=the function characterizing probability of collisions of drops with sizes σ and σ_1, v=sedimentation velocity of a drop, $d\sigma/dt = A U_z$, and $\Phi(\sigma, \vec{u})$=the function characterizing formation of new drops. Since there are random variables in (29) let us make averaging analogous to that made by Reynolds with equations of movement. Taking into account that f is not conservative in approximation of the semi-empirical theory of turbulence in accordance with [20], [21] let us write down

$$\overline{f' u_j'} = -K_{ij}\left(\frac{\partial}{\partial x_i} + A\,\delta_{iz}\frac{\partial}{\partial \sigma}\right)\bar{f}, \tag{30}$$

where K_{ij}=tensor of turbulent diffusion coefficients. Keeping in mind that $\overline{\bar{f} f'} \approx \overline{\bar{f}\, U'} = \overline{\bar{u}\, u'} = \overline{f'\,\bar{u}} = 0$ and regarding that drop formation due to regular rise of air takes place only at the boundary and is taken into account with the help of initial distribution we shall obtain

$$\frac{\partial \bar{f}}{\partial t} + (\bar{U}_i - v\,\delta_{iz})\frac{\partial \bar{f}}{\partial x_i} + A\,\bar{u}_z\frac{\partial \bar{f}}{\partial \sigma} = \left(\frac{\partial}{\partial x_i} + A\,\delta_{iz}\frac{\partial}{\partial \sigma}\right) K_{ij}\left(\frac{\partial}{\partial x_j} + A\,\delta_{jz}\frac{\partial}{\partial \sigma}\right) f$$
$$+ H(\bar{f}, \bar{f}) + H(f', f') + \bar{\Phi}'. \tag{31}$$

Deriving (31) we neglected fluctuations of A. Let us also reject $H(f', f')$, because at the present state of knowledge it seems to us impossible to take into account fluctuations in integral of collisions. Having omitted the bar which denotes averaging let us finally write down

$$\frac{\partial f}{\partial t} + (U_i - v\,\delta_{iz})\frac{\partial f}{\partial x_i} + A\,U_z\frac{\partial f}{\partial \sigma} = \left(\frac{\partial}{\partial x_i} + A\,\delta_{iz}\frac{\partial}{\partial \sigma}\right) K_{ij}\left(\frac{\partial}{\partial x_j} + A\,\delta_{jz}\frac{\partial}{\partial \sigma}\right) f$$
$$+ H(f, f) + \Phi. \tag{32}$$

For a homogeneous infinite cloud at condensation stage of development the equation (32) will take the form

$$\frac{\partial f}{\partial t} + A\,U\frac{\partial f}{\partial \sigma} = A^2 K \frac{\partial^2 f}{\partial \sigma^2}. \tag{33}$$

Solution of this equation when initial distribution is the delta-function under the assumption that A is constant results in the distribution (28). This shows that in (32) the effect of stochastic condensation is taken into consideration. Refuse from so strong simplifying assumptions leads to very complicated solutions. Some results are given in [20], [21]. Analysis carried out by us showed that change of A (A is determined by (25) with the help of integral) could significantly effect the distribution function f. As in this case and also when H is taken into account the equation becomes nonlinear and integral-differential, the analytical methods of solution are rather limited. To investigate in principle possibilities of using a kinetic equation of (32) type for making concrete calculations we considered solution of the equation taking the model of infinite homogeneous cloud with consideration for variation of A

$$\left[A = \frac{2 D \varrho_0 \left(\frac{L M}{R T} - \frac{C_p M_1}{R} \right) \gamma_a}{4 \pi D \varrho_w \left(1 + \frac{L M}{R T} \frac{L}{C_p T} \frac{\varrho_0}{\varrho_1} \right) \int\limits_0^\infty \chi(\sigma^{1/2}) f(\sigma, t)\, d\sigma} \right]$$

and with regard to integral of collisions. The difference equation for the function $\varphi(r, t)$ of drop radius r distribution was written down in the form:

$$\left. \begin{aligned} \varphi_i^t = \Bigg[\varphi_i + \frac{K A^2 \Delta t}{4 r_i^2} & \frac{\left(\frac{r_i}{r_{i+1}} \varphi_{i+1} - \varphi_i \right) (r_i - r_{i-1}) - (r_{i+1} - r_i) \left(\varphi_i - \frac{r_i}{r_{i-1}} \varphi_{i-1} \right)}{(r_{i+1} - r)(r - r_{i-1})^2} \\ - \frac{K A^2 \Delta t}{4 r_i^2 (r_{i+1} - r_{i-1}) r_{i-1}} & \left(\frac{r_{i-1}}{r_{i+1}} \varphi_{i+1} - \varphi_{i-1} \right) - \frac{A U \Delta t}{2(r_i - r_{i-1}) r_i} \left(\varphi_i - \frac{r_i}{r_{i-1}} \varphi_{i-1} \right) \\ - \varphi_i \int\limits_0^\infty K(r, r') \varphi(r', t)\, dr' + & \int\limits_0^{r/2^{1/3}} \frac{r^2}{(r^3 - r'^3)^{2/3}} K\left((r^3 - r'^3)^{1/3}, r'\right) \\ \times \varphi\left((r^3 - r'^3)^{1/3}, t\right) \varphi(r', t)\, dr' \Bigg]^{t-1} . & \end{aligned} \right\} \tag{34}$$

The value of A and integrals in the right part of the equation were computated numerically by the trapezoid rule. Step of r change was chosen equal to $\Delta r = 3 \cdot 10^{-5}$ cm, step of change of time was chosen from conditions of computation stability. The results of capture coefficient computations obtained in [22] were used for K determination. An example of computations carried out is given in Fig. 2. Delta-shaped function with the mean drop radius of 10^{-4} cm and the initial number of drops $N \approx 10^3$ cm^{-3} was used an initial distribution. $K_z = 5 \cdot 10^4$ cm^2/sec, $U_z = 10$ cm/sec. To save the computation time integral of collisions was taken into account in computations two hours later the beginning of computation, therefore for $t < 2$ hr purely condensational development of spectrum is shown in the figure. Even with idealization which took place the obtained results qualitatively correspond to present ideas of spectrum development in stratus. In Fig. 2 we see, probably for the first time, the picture of transition of initially narrow highly disperse drop spectrum into a wide spectrum with availability of large drop fraction which captures small drop fraction (for $t > 2$ hr) and leads to a remarkable decrease of particle number in the system. Thus, in the closed form widening (and not narrowing) is obtained, possibility of rather large drop appearance during comparatively short period of time is shown, a satisfactory value of concentration of these drops is obtained and possibility of small drop existence in spite of pumping up vapour to the system is shown. Large drop distribution is shown

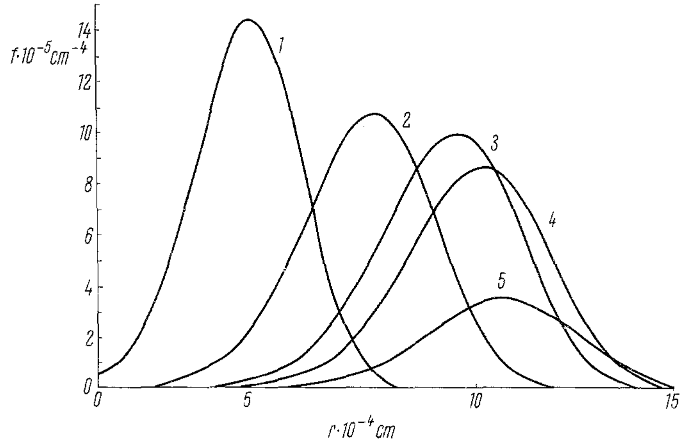

Figure 2
$K = 5 \cdot 10^4$ cm^2/sec; $U_z = 10$ cm/sec. 1. 10′; 2. 1^h; 3. 2^h; 4. 2^{h}30′; 5. 2^{h}50′

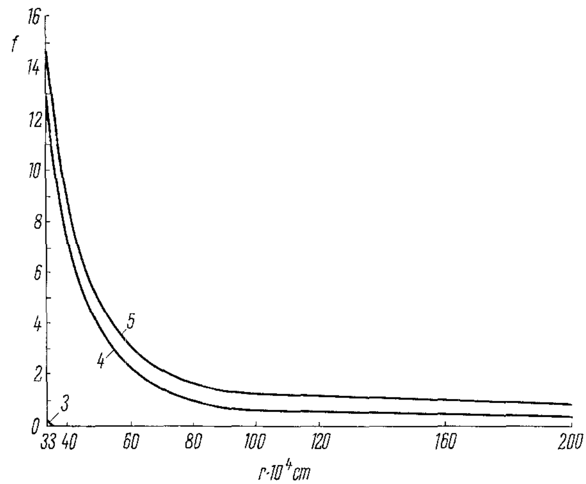

Figure 2a
K $= 5 \cdot 10^4$ cm^2/sec; $U_z = 10$ cm/sec. 3. 2^h; 4. 2^h 30′; 5. 2^{h}50′

in Fig. 2a. Unfortunately, possibilities of the chosen model are rather limited because appearance of new drops and also falling drops out of volume is not taken into account in it. Nevertheless, with the help of this model computations show once more that main contradictions in the theory of cloud spectrum formation which existed up to the present time are overcome.

The main contradiction of well-known schemes of cloud drop spectrum formation lay in existence of a size range which was impossible to get over because of coagulation prohibition in this size range. This did not allow to connect condensation and coagulation stage of drop growth. Considerations of stochastic condensation bore a qualitative character since during its description some random not tested by experiment assumptions were made, therefore these considerations could not be used for calculational schemes. It seems to us that in this paper for the first time the closed scheme of cloud drop spectrum formation is given which includes drop formation from nuclei; describes the initial stage of condensation, is based upon stochastic condensation, takes into account regular condensation and processes of coagulation and transport. The obtained kinetic equation does not contain random parameters and is determined by a small number of constants and functional relations, in general, allowing experimental test.

At last some words should be said about relation between the kinetic equation and equations of thermohydrodynamics of the cloud. Apparently, in many cases liquid water content of the cloud is not so high as to be necessary to take into account its influence on dynamics of movement. If we manage to describe radiation processes by some averaged values which do not require detailed knowledge of distribution function, then in this approximation the system of general equations allows splitting and then the kinetic equation and equations of thermohydrodynamics can be considered independently. We think that it is in the way that our further investigations should continue.

The obtained results permit to proceed to creation of the quantitative theory of precipitation formation.

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