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Cauchy Completions of Nearness Frames

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Abstract. A nearness frame is Cauchy complete if every regular Cauchy filter on the nearness frame is convergent and we show that the category **CCNFrm** of Cauchy complete nearness frames is coreflective in the category **NFrmC** of nearness frames and Cauchy homomorphisms and that the coreflection of a nearness frame is given by the strict extension associated with regular Cauchy filters on the nearness frame. Using the same completion, we show that the category **CCSNFrm** of Cauchy complete strong nearness frames is coreflective in the category **SNFrm** of strong nearness frames and uniform homomorphisms.

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1. Introduction

The concept of nearness frames has been introduced by Banaschewski and Pultr to generalize uniform frames and they introduced two concepts of complete nearness frames, namely complete and Cauchy complete nearness frames and they construct completions of nearness frames and show that for the category of almost uniform nearness frames and uniform homomorphisms, the completion gives rise to a functor (see [2, 4]).

The purpose of this paper is to show that the full subcategory **CCNFrm** of Cauchy complete nearness frames is coreflective in the category **NFrmC** of nearness frames and Cauchy homomorphisms and that the coreflection is given by the strict extension associated with regular Cauchy filters.

Furthermore, observing that every Cauchy filter on a strong nearness frame contains a unique regular Cauchy filter, we show that the category **CCSNFrm** of Cauchy complete strong nearness frames is coreflective in the category **SNFrm** of strong nearness frames and uniform homomorphisms.

For the general background of frames and frame homomorphisms, we refer to [7] and for the category theory, we refer to [1]. In a frame L, e (0, resp.) denotes the top (bottom, resp.) of L and $Cov(L)$ the set of covers C of L, i.e., $\bigvee C = e$. For $C, D \in Cov(L)$, C is said to *refine* D if for any $c \in C$, there is $d \in D$ with $c \leqslant d$ and in the case, we write $C \leqslant D$. It is clear that $\leqslant$ is a quasi order on $Cov(L)$. For C in $Cov(L)$ and $x \in L$, $\bigvee\{c \in C : c \wedge x \neq 0\}$ will be denoted by Cx. The following is due to Banaschewski and Pultr ([4]).

DEFINITION 1. Let L be a frame and $\mathcal{N} \subseteq Cov(L)$. Then $\mathcal{N}$ is said to be a *nearness* on L if it satisfies the following:

(1) $\mathcal{N}$ is a filter on $(Cov(L), \leqslant)$, that is, $\mathcal{N}$ is a nonempty upset in $(Cov(L), \leqslant)$ such that for any $C, D \in \mathcal{N}$, $C \wedge D = \{c \wedge d : c \in C \text{ and } d \in D\} \in \mathcal{N}$;

(2) For any $a \in L$, $a = \bigvee\{x \in L : x \lhd a\}$, where $x \lhd a$ means that there is $C \in \mathcal{N}$ such that $Cx \leqslant a$.

In this case, the pair $(L, \mathcal{N})$ is called a *nearness frame*.

Remark 2. Since $\lhd$ implies the well inside relation $\prec$ in a nearness frame, every nearness frame is a regular frame.

DEFINITION 3. Let $(M, \mathcal{M}), (L, \mathcal{N})$ be nearness frames and $h : M \to L$ a frame homomorphism.

(1) h is said to be *uniform* if for any $C \in \mathcal{M}$, $h(C) \in \mathcal{N}$.

(2) h is said to be a *surjection* if h is onto and for any $D \in \mathcal{N}$, $h_*(D)$ is a cover of M and $\{h_*(D) : D \in \mathcal{N}\}$ generates the filter $\mathcal{M}$, where h_* denotes the right adjoint of h.

Now we recall strict extensions of frames. For a set X of filters on a frame L, let $P(X)$ denote the power set lattice of X and $L \times P(X)$ the product frame of L and $P(X)$. Then $s_X L = \{(x, \Sigma) \in L \times P(X) : \text{for any } F \in \Sigma,\ x \in F\}$ is a subframe of $L \times P(X)$ and the restriction $s : s_X L \to L$ of the first projection is an onto, dense and open homomorphism (see [5, 8]). For any $x \in L$, let $\Sigma_x = \{F \in X : x \in F\}$, then the right adjoint s_* of s is given by $s_*(x) = (x, \Sigma_x)$ $(x \in L)$. Since s_* preserves meets, $s_*(L)$ is closed under finite meets and hence the subframe $t_X L$ of $s_X L$ generated by $s_*(L)$ is given by $t_X L = \{\bigvee\{(x, \Sigma_x) : x \in A\} : A \subseteq L\}$.

Clearly the restriction $t : t_X L \to L$ of s is an onto dense homomorphism. The homomorphism $t : t_X L \to L$ or simply $t_X L$ is called the *strict extension* of L associated with X (see [3, 6] for the details). Clearly the right adjoint t_* of t is also given by $t_*(x) = (x, \Sigma_x)$ $(x \in L)$. Furthermore, for any $A \subseteq L$, let $\Sigma_A = \bigcup\{\Sigma_x : x \in A\} = \{F \in X : F \cap A \neq \phi\}$, then one has the following:

$$\bigvee\{(x, \Sigma_x) : x \in A\} = (\bigvee A, \Sigma_A) = (\bigvee \downarrow A, \Sigma_{\downarrow A}).$$

Thus $t_X L = \{(\bigvee A, \Sigma_A) : A = \downarrow A \subseteq L\}$. In the following, we may assume that every element of $t_X L$ has the form of $(\bigvee A, \Sigma_A)$ for some down set A in L.

2. Cauchy Completions of Nearness Frames

In this section, we construct the Cauchy completion of a nearness frame.

DEFINITION 4. A filter F on a nearness frame $(L, \mathcal{N})$ is said to be:

(1) a *Cauchy filter* if for any $C \in \mathcal{N}$, $C \cap F \neq \phi$,

(2) a *regular Cauchy filter* if it is a Cauchy filter and for any $x \in F$, there is $y \in F$ with $y \lhd x$.

We include some properties of Cauchy filters (see [4]).

PROPOSITION 5. *Let $(L, \mathcal{N})$ be a nearness frame and F a filter on L*

(1) *If F is a Cauchy filter, then for any $x \lhd y$, either $y \in F$ or $x^* \in F$, where x^* denotes the pseudocomplement of x.*

(2) *If F is a regular Cauchy filter, then it is a minimal Cauchy filter.*

(3) *A regular Cauchy filter is a regular filter.*

Proof. (1) Suppose that $x \lhd y$, then there is $C \in \mathcal{N}$ with $Cx \leqslant y$. Since F is a Cauchy filter, there is $c \in C \cap F$. If $c \wedge x = 0$, then $c \leqslant x^*$; hence $x^* \in F$. If not, then $c \leqslant y$ and therefore $y \in F$.

(2) Suppose that G is a Cauchy filter with $G \subseteq F$. For any $a \in F$, there is $b \in F$ with $b \lhd a$. Since G is a Cauchy filter, either $b^* \in G$ or $a \in G$. If b^* is an element of G, then $b \wedge b^* = 0 \in F$, which is a contradiction. Thus $a \in G$.

(3) It is immediate from the fact that $\lhd$ implies $\prec$. □

PROPOSITION 6. (1) *If $h : (M, \mathcal{M}) \to (L, \mathcal{N})$ is uniform, then for any Cauchy filter G on $(L, \mathcal{N})$, $h^{-1}(G)$ is a Cauchy filter on $(M, \mathcal{M})$.*

(2) *Suppose that $h : (M, \mathcal{M}) \to (L, \mathcal{N})$ is a dense surjection, then for any (regular, resp.) Cauchy filter F on $(M, \mathcal{M})$, $h(F)$ is also a (regular, resp.) Cauchy filter on $(L, \mathcal{N})$.*

Proof. The first part is immediate from the definition and for the second part, we first note that since h is onto, dense, $h(F)$ is a filter on L and we have the result from the fact that for any $C \in \mathcal{N}$, $h(h_*(C)) = C$ and that h preserves the relation $\lhd$. □

The following is again due to Banaschewski and Pultr ([4]).

DEFINITION 7. A nearness frame $(L, \mathcal{N})$ is said to be:

(1) *complete* if any dense surjection $h : (M, \mathcal{M}) \to (L, \mathcal{N})$ is an isomorphism;

(2) *Cauchy complete* if every regular Cauchy filter on $(L, \mathcal{N})$ is a completely prime filter.

Remark 8. (1) Clearly every completely prime filter on a nearness frame is a regular Cauchy filter; therefore a nearness frame $(L, \mathcal{N})$ is Cauchy complete if and only if regular Cauchy filters on $(L, \mathcal{N})$ are precisely completely prime filters.

(2) We recall that a filter F on a frame L is said to be *convergent* if for any cover S of L, $F \cap S \neq \phi$ (see [6] for the details) and that a regular filter on a frame L is convergent if and only if it is a completely prime filter on L ([3]). Thus a nearness frame $(L, \mathcal{N})$ is Cauchy complete if and only if every regular Cauchy filter on $(L, \mathcal{N})$ is convergent.

(3) Suppose that B generates a frame L, that is, for any $x \in L$, $x = \bigvee\{b \in B : b \leqslant x\}$. Then a filter F on L is convergent if and only if for any basic cover S of L, i.e., S is a cover of L and $S \subseteq B$, $F \cap S \neq \phi$.

In the remainder of the section, $(L, \mathcal{N})$ always denotes a nearness frame and X *the set of regular Cauchy filters on* L. The strict extension $t_X L$ of L associated with X will be denoted by cL and $t : cL \to L$ by c_L or c.

Now we introduce a nearness $\mathcal{N}^*$ on cL generated by $\{c_*(C) : C \in \mathcal{N}\}$, where c_* is the right adjoint of c as before, i.e., $c_*(x) = (x, \Sigma_x)$. Using these notions, we have the following.

PROPOSITION 9. (1) *$(cL, \mathcal{N}^*)$ is a nearness frame.*

(2) *$c : (cL, \mathcal{N}^*) \to (L, \mathcal{N})$ is a dense surjection.*

Proof. (1) Since X consists of Cauchy filters, for any $C \in \mathcal{N}$, $\bigcup\{\Sigma_x : x \in C\} = \Sigma_C = X$; hence $c_*(C)$ is a cover of cL. Since c_* preserves meets, $\{c_*(C) : C \in \mathcal{N}\}$ is a filter base in $(Cov(cL), \leqslant)$. Moreover, $x \lhd y$ in $(L, \mathcal{N})$ implies $c_*(x) \lhd c_*(y)$ in $(cL, \mathcal{N}^*)$, for $c_*(z) \wedge c_*(x) = c_*(z \wedge x) \neq 0$ implies $z \wedge x \neq 0$. Furthermore, $c_*(x) = (x, \Sigma_x) = \bigvee\{c_*(a) : a \lhd x\}$, because X consists of regular Cauchy filters and therefore $\Sigma_x = \bigcup\{\Sigma_a : a \lhd x\}$. Since $c_*(L)$ generates cL, $\mathcal{N}^*$ is a nearness on cL.

(2) It is immediate from the definition of $\mathcal{N}^*$. □

THEOREM 10. *The nearness frame $(cL, \mathcal{N}^*)$ is Cauchy complete.*

Proof. By Remark 8, it remains to show that every regular Cauchy filter on $(cL, \mathcal{N}^*)$ is convergent. Take any regular Cauchy filter Ψ on cL and take any basic cover S of cL, i.e., $S = \{(x, \Sigma_x) : x \in A\}$ for some $A \subseteq L$. Since c is a dense surjection, $c(\Psi)$ is also a regular Cauchy filter on $(L, \mathcal{N})$ and hence $c(\Psi) \in X$. Since $\bigvee S = (\bigvee A, \Sigma_A) = (e, X)$, $c(\Psi) \in \Sigma_A$; hence $c(\Psi) \cap A \neq \phi$. Pick $a \in c(\Psi) \cap A$, then $c_*(a) = (a, \Sigma_a) \in \Psi \cap S$. Thus by Remark 8, Ψ is convergent. □

DEFINITION 11. For any nearness frame $(L, \mathcal{N})$, $c : (cL, \mathcal{N}^*) \to (L, \mathcal{N})$ or simply $(cL, \mathcal{N}^*)$ is called the *Cauchy completion* of $(L, \mathcal{N})$.

Since the Cauchy completion $c : (cL, \mathcal{N}^*) \to (L, \mathcal{N})$ of a nearness frame $(L, \mathcal{N})$ is a dense surjection, the following is immediate.

COROLLARY 12. *Every complete nearness frame is Cauchy complete.*

DEFINITION 13 ([4]). Let $(M, \mathcal{M})$ and $(L, \mathcal{N})$ be nearness frames. A frame homomorphism $h : M \to L$ is said to be a *Cauchy homomorphism* on $(M, \mathcal{M})$ to $(L, \mathcal{N})$ if for any regular Cauchy filter F on $(L, \mathcal{N})$, there is a regular Cauchy filter G on $(M, \mathcal{M})$ with $G \subseteq h^{-1}(F)$.

The proof of the following can be found in [4].

LEMMA 14. (1) *Every dense surjection is a Cauchy homomorphism.*
(2) *Every frame homomorphism with a Cauchy complete codomain is a Cauchy homomorphism.*

Using the above, we now have our main theorem.

THEOREM 15. *The category* **CCNFrm** *of Cauchy complete nearness frames is coreflective in the category of* **NFrmC** *of nearness frames and Cauchy homomorphisms.*

Proof. Take any nearness frame $(L, \mathcal{N})$ and let $c_L : (cL, \mathcal{N}^*) \to (L, \mathcal{N})$ be the Cauchy completion of $(L, \mathcal{N})$. Since c_L is a dense surjection, c_L is a Cauchy homomorphism. Take any Cauchy complete nearness frame $(M, \mathcal{M})$ and a Cauchy homomorphism $h : (M, \mathcal{M}) \to (L, \mathcal{N})$. We define $\bar{h} : M \to cL$ by $\bar{h}(a) = \bigvee\{(h(x), \Sigma_{h(x)}) : x \lhd a\}\,(a \in M)$. Since $a = \bigvee\{x : x \lhd a\}$ for any $a \in M$, $\bar{h}(a) = (h(a), \bigcup\{\Sigma_{h(x)} : x \lhd a\})$ and therefore $c_L \circ \bar{h} = h$. In order to prove that $\bar{h} : M \to cL$ is a frame homomorphism, it is enough to show that $h_1 : M \xrightarrow{\bar{h}} cL \xrightarrow{p_2} P(X)$ is a homomorphism, where X is the set of regular Cauchy filters on L as before and p_2 is the restriction of the second projection $L \times P(X) \to P(X)$ to cL, because $c_L : cL \to L$ and $p_2 : cL \to P(X)$ form an initial source. Since $e \lhd e$ and $\{x : x \lhd a\} \cap \{y : y \lhd b\} = \{z : z \lhd a \wedge b\}$ for $a, b \in M$, h_1 preserves finite meets. Take any $S \subseteq M$ and any $F \in h_1(\bigvee S) = \bigcup\{\Sigma_{h(x)} : x \lhd \bigvee S\}$. Since h is a Cauchy homomorphism, there is a regular Cauchy filter G with $G \subseteq h^{-1}(F)$. We note that there is x_0 such that $x_0 \lhd \bigvee S$ and $h(x_0) \in F$; therefore $\bigvee S \in G$, for G is a Cauchy filter. Thus there is $s_0 \in S \cap G$, because $(M, \mathcal{M})$ is Cauchy complete. Since G is a regular Cauchy filter, there is $y \in G$ with $y \lhd s_0$ so that $h(y) \in F$ and $y \lhd s_0$. Thus $F \in \bigcup\{\Sigma_{h(z)} : z \lhd s_0\} = h_1(s_0)$. In all, $h_1(\bigvee S) \subseteq \bigvee\{h_1(s) : s \in S\}$ and the other inclusion is trivial. Therefore h_1 preserves arbitrary joins. Furthermore, $\bar{h}$ is a Cauchy homomorphism by (2) of Lemma 14 and it is unique, for c_L is dense and hence monic. Thus $c_L : cL \to L$ is the **CCNFrm**-coreflection of L in **NFrmC**. □

3. Cauchy Completions of Strong Nearness Frames

In this section, we show that the Cauchy completion gives rise to a coreflection in the category **SNFrm** of strong nearness frames and uniform homomorphisms.

DEFINITION 16 ([4]). A nearness frame $(L,\mathcal{N})$ is said to be *strong* if for any $C\in\mathcal{N}$, $\check{C}=\{x\in L: x\lhd y$ for some $y\in C\}$ also belongs to $\mathcal{N}$.

Remark 17. A uniform frame is an almost uniform frame which implies a strong nearness frame ([2]).

For a filter F on a nearness frame $(L,\mathcal{N})$, F° denotes the filter $\{x\in L: a\lhd x$ for some $a\in F\}$.

LEMMA 18. *Suppose that $(L,\mathcal{N})$ is a strong nearness frame.*

(1) *If F is a Cauchy filter, then F° is a Cauchy filter.*
(2) *F is a regular Cauchy filter if and only if it is a minimal Cauchy filter.*
(3) *If F is a Cauchy filter, then F° is a unique regular Cauchy filter contained in F.*
(4) *$(L,\mathcal{N})$ is Cauchy complete if and only if every Cauchy filter is convergent.*

Proof. (1) Take any $C\in\mathcal{N}$, then $\check{C}=\{x\in L: x\lhd y$ for some $y\in C\}\in\mathcal{N}$ and hence $F\cap\check{C}\neq\phi$. Pick $x_0\in F\cap\check{C}$, then there is $y_0\in C$ with $x_0\lhd y_0$, which implies $y_0\in F^\circ\cap C$. Thus F° is again a Cauchy filter.

(2) Suppose that F is a minimal Cauchy filter, then F° is a Cauchy filter contained in F and therefore $F=F^\circ$. Thus F is a regular Cauchy filter. The converse is in Proposition 5.

(3) Suppose that G is a Cauchy filter and $G\subseteq F^\circ$. For any $x\in F^\circ$, there is $a\in F$ with $a\lhd x$. Since G is a Cauchy filter, $a^*\in G$ or $x\in G$. Since $G\subseteq F^\circ\subseteq F$, $a^*\notin G$; hence $x\in G$. Thus F° is a minimal Cauchy filter, i.e., a regular Cauchy filter. If H is a regular Cauchy filter with $H\subseteq F$, then $F^\circ\subseteq H$ ([4]) and hence $H=F^\circ$.

(4) It is immediate from the fact that a filter containing a convergent filter on a frame is again convergent. □

Using Proposition 6 and the above lemma, one has the following:

COROLLARY 19. *Suppose that $(M,\mathcal{M})$ is a strong nearness frame, then every uniform homomorphism $h:(M,\mathcal{M})\to(L,\mathcal{N})$ is a Cauchy homomorphism.*

LEMMA 20. *Let $h:(M,\mathcal{M})\to(L,\mathcal{N})$ be a dense surjection.*

(1) *If $(L,\mathcal{N})$ is strong, then $(M,\mathcal{M})$ is strong.*
(2) *Suppose that $(K,\mathcal{L})$ is a strong nearness frame and $k:K\to M$ is a frame homomorphism. If $h\circ k:(K,\mathcal{L})\to(L,\mathcal{N})$ is uniform, then $k:(K,\mathcal{L})\to(M,\mathcal{M})$ is also uniform.*

Proof. (1) The proof can be found in [4].

(2) Take any $C\in\mathcal{L}$, then $\check{C}\in\mathcal{L}$. For any $x\in\check{C}$, there is $y\in C$ with $x\lhd y$. Since $x\lhd y$, $x\prec y$ and hence $k(x)\prec k(y)$. Since h is dense, $h_*(h(k(x)))\prec$

$k(y)$, where h_* denotes the right adjoint of h. Thus $h_*(h(k(\check{C})))$ refines $k(C)$. Since $h \circ k$ is uniform, $h(k(\check{C})) \in \mathcal{N}$ and therefore $h_*(h(k(\check{C}))) \in \mathcal{M}$, so that $k(C) \in \mathcal{M}$. Thus k is uniform. □

Collecting the above, we have the following.

THEOREM 21. *The category* **CCSNFrm** *of Cauchy complete strong nearness frames is coreflective in the category* **SNFrm** *of strong nearness frames and uniform homomorphisms.*

Proof. For any strong nearness frame $(L, \mathcal{N})$, let $c_L : (cL, \mathcal{N}^*) \to (L, \mathcal{N})$ be the Cauchy completion of $(L, \mathcal{N})$. Then $(cL, \mathcal{N}^*) \in$ **CCSNFrm** by (1) of Lemma 20, and $c_L : (cL, \mathcal{N}^*) \to (L, \mathcal{N})$ is uniform, for c_L is a dense surjection. Take any uniform homomorphism $h : (M, \mathcal{M}) \to (L, \mathcal{N})$ such that $(M, \mathcal{M}) \in$ **CCSNFrm**, then by Corollary 19, h is a Cauchy homomorphism. Thus there is a unique Cauchy homomorphism $\bar{h} : (M, \mathcal{M}) \to (cL, \mathcal{N}^*)$ with $c_L \circ \bar{h} = h$. By (2) of Lemma 20, $\bar{h}$ is also a uniform homomorphism. Since c_L is monic, $c_L : (cL, \mathcal{N}^*) \to (L, \mathcal{N})$ is the **CCSNFrm**-coreflection of $(L, \mathcal{N})$. □

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