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Water Table Fluctuation in Response to Transient Recharge from a Rectangular Basin

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Abstract. A problem of water-table fluctuation in a finite two-dimensional aquifer system in response to transient recharge from an overlying rectangular area is studied. An analytical solution is obtained by using the method of finite Fourier transform to predict the transient position of the water-table. The solution for constant rate of recharge is shown as a special case of the present solution. Effects of variation in the rate of recharge on the growth of two-dimensional groundwater mound is illustrated with the help of a numerical example.

Key words. Transient recharge, water-table, analytical solution, aquifer, prediction.

Notation

A	half width of the aquifer [L]
B	half length of the aquifer [L]
D	half width of the recharge basin [L]
e	specific yield
h	varying water-table height [L]
h_0	initial water-table height [L]
$\bar{h}$	weighted mean of the depth of saturation [L]
K	hydraulic conductivity [LT^{-1}]
L	half length of the recharge basin [L]
$P(t)$	time varying rate of recharge [LT^{-1}]
$P_1 + P_0$	initial rate of time varying recharge [LT^{-1}]
P_1	final rate of time varying recharge [LT^{-1}]
t	time of observation [T]
x, y	coordinate axes
α	decay constant [T^{-1}]

1. Introduction

Artificial recharging of groundwater is being practised in many regions in order to maintain water balance in an aquifer system. Generally, the recharge is applied from rectangular or circular basins. The problems of water-table fluctuations in response to recharge from such basins have been studied by many workers like Baumann (1952), Glover (1960), Hantush (1967), Hunt (1971), and Rao and Sarma (1981a, b). In all these studies, the rate of recharge is considered as constant. However, the rate of recharge has the same time dependence as the infiltration rate. The infiltration rate has been approximated by combination of an exponential function and a constant term (Horton, 1940, Bear, 1979).

Zomorodi (1991) has shown that in the case of time-varying recharge, the analytical solutions which are based on the assumption of the constant rate of recharge give erroneous results. In this paper, we extend the work of Rao and Sarma (1981a)

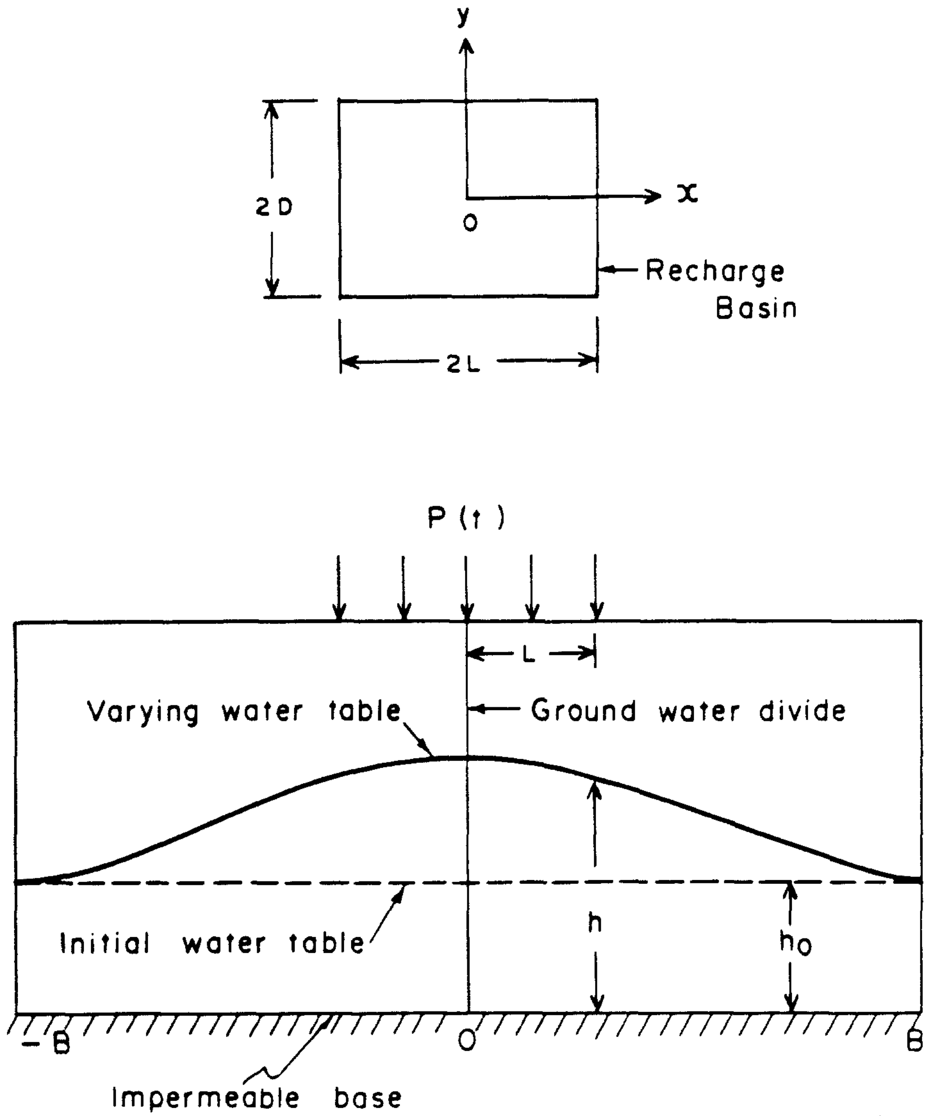

Fig. 1. Diagrammatic representation of the flow systems.

to the variable recharge case and obtain an analytical solution which can be used to predict the spatio-temporal variation of the water-table in response to time-varying recharge from a rectangular basin.

2. Formulation and Solution

The aquifer system under consideration is shown in Figure 1. An unconfined aquifer of length 2A and width 2B is receiving recharge from an overlying rectangular

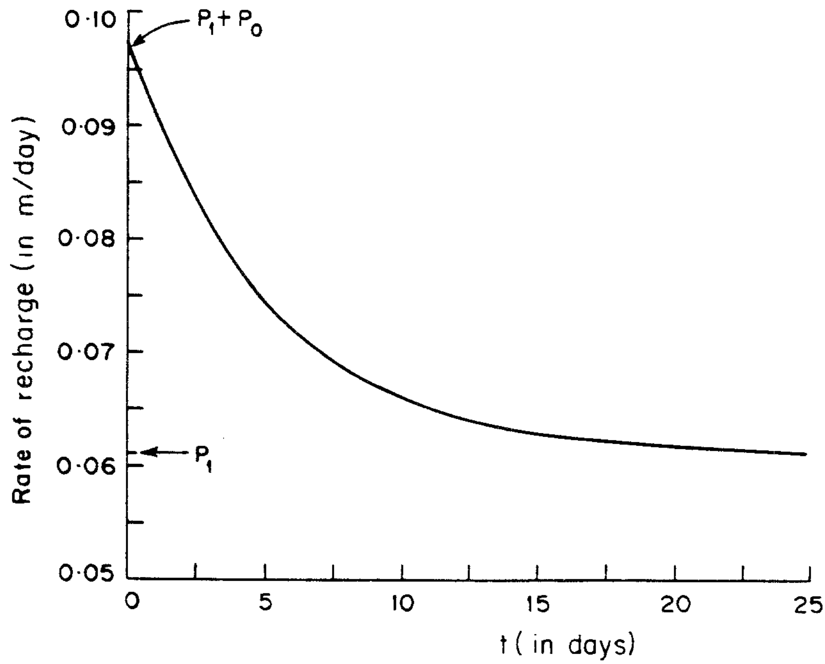

Fig. 2. Time-varying rate of recharge.

basin having the length and width as 2L and 2D, respectively. The rate of recharge is considered as exponentially decaying with time from a large value, $P_1 + P_0$ to a small value P_1 and thereafter remains constant (Figure 2). The origin of the coordinate axis is taken at the centre of the recharge basin. Because of the symmetry, only the positive quarter of the aquifer system is considered in further investigations.

The groundwater movement in the homogeneous and isotropic phreatic aquifer system is described under the Dupuit assumption (Bear, 1979) by the following initial boundary-value problem

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{2}{K} P(x, y, t) = \frac{1}{a} \frac{\partial H}{\partial t}, \tag{1}$$

$$\frac{\partial H}{\partial x}(B, y, t) = \frac{\partial H}{\partial y}(x, A, t) = 0, \tag{2}$$

$$P(x, y, t) = \begin{cases} P(t), & x \leq L; y \leq D, \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

$$H(x, y, 0) = 0, \tag{4}$$

in which

$$P(t) = P_1 + P_0 \exp(-\alpha t), \quad H = h^2 - h_0^2 ,$$

Other symbols are defined in the list of Notations. $\bar{h}$ is approximated by $0.5(h + h_0)$. Marino (1967) has discussed a method of successive approximation for the computation of $\bar{h}$. He has also shown for a one-dimensional form of Equation (1) that this assumption describes the water-table fluctuation up to 6% deviation for a water-table rise as high as $h_0/2$.

This boundary-value problem is solved by the method of finite Fourier transform following Rao and Sarma (1981a). The finite Fourier cosine transform for the function $H(x, y, t)$ is defined as:

$$F_c[H(x, y, t)] = S(m, n, t)$$
$$= \int_0^B \int_0^A H(x, y, t) \operatorname{Cos}\left(\frac{m\pi x}{B}\right) \operatorname{Cos}\left(\frac{n\pi y}{A}\right) \mathrm{d}y \, \mathrm{d}x , \qquad (5)$$

in which m and n are integers. By transforming Equation (1) using (5) with conditions (2) and (3), we get

$$\frac{\mathrm{d}S(m, n, t)}{\mathrm{d}t} + \lambda \, S(m, n, t) = g(m, n) \, P(t) , \qquad (6)$$

$$\lambda = a\pi^2 \, (m^2/B^2 + n^2/A^2)$$

and

$$g(m, n) = \left(\frac{2aAB}{Kmn\pi^2}\right) \operatorname{Sin}\left(\frac{m\pi L}{B}\right) \operatorname{Sin}\left(\frac{n\pi D}{A}\right)$$

The solution of Equation (6), subject to the initial condition (4), is given by

$$S(m, n, t) = g(m, n) \exp(-\lambda t) \int_0^t P(\tau) \exp(\lambda \tau) \, \mathrm{d}\tau . \qquad (7)$$

$H(x, y, t)$ is obtained from $S(m, n, t)$ by using the inverse Fourier cosine transform (Sneddon, 1974) as

$$H(x, y, t) =$$
$$= \frac{1}{AB} S(0, 0, t) + \frac{2}{AB} \sum_{m=1}^{\infty} S(m, 0, t) \operatorname{Cos}\left(\frac{m\pi x}{B}\right) +$$
$$+ \frac{2}{AB} \sum_{n=1}^{\infty} S(0, n, t) \operatorname{Cos}\left(\frac{n\pi y}{A}\right) +$$
$$+ \frac{4}{AB} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} S(m, n, t) \operatorname{Cos}\left(\frac{m\pi x}{B}\right) \operatorname{Cos}\left(\frac{n\pi y}{A}\right) . \qquad (8)$$

Expressions for $S(0, 0, t)$, $S(m, 0, t)$, $S(0, n, t)$, $S(m, n, t)$ can be obtained from

Equation (7). After substituting these expressions into Equation (8), we get the following desired expression for the variable water-table height.

$$
\begin{aligned}
&h^2(x, y, t) = \\
&= h_0^2 + \frac{2aLDt}{KAB}\left[P_1 + P_0\,\frac{1-\exp(-\alpha t)}{\alpha t}\right] + \\
&+ \frac{4aD}{\pi KA}\sum_{m=1}^{\infty}\frac{1}{m}\,\mathrm{Sin}\left(\frac{m\pi L}{B}\right)\mathrm{Cos}\left(\frac{m\pi x}{B}\right)\times \\
&\times\left[P_1\,\frac{1-\exp(-at\pi^2m^2/B^2)}{a\pi^2m^2/B^2} + P_0\,\frac{\exp(-\alpha t)-\exp(-at\pi^2m^2/B^2)}{a\pi^2m^2/B^2-\alpha}\right] + \\
&+ \frac{4aL}{\pi KB}\sum_{n=1}^{\infty}\frac{1}{n}\,\mathrm{Sin}\left(\frac{n\pi D}{A}\right)\mathrm{Cos}\left(\frac{n\pi y}{A}\right)\times \\
&\times\left[P_1\,\frac{1-\exp(-at\pi^2n^2/A^2)}{a\pi^2n^2/A^2} + P_0\,\frac{\exp(-\alpha t)-\exp(-at\pi^2n^2/A^2)}{a\pi^2n^2/A^2-\alpha}\right] + \\
&+ \frac{8a}{K\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\,\mathrm{Sin}\left(\frac{m\pi L}{B}\right)\mathrm{Sin}\left(\frac{n\pi D}{A}\right)\mathrm{Cos}\left(\frac{m\pi x}{B}\right)\mathrm{Cos}\left(\frac{n\pi y}{A}\right)\times \\
&\times\left[P_1\,\frac{1-\exp(-\lambda t)}{\lambda} + P_0\,\frac{\exp(-\alpha t)-\exp(-\lambda t)}{\lambda-\alpha}\right].
\end{aligned} \tag{9}
$$

3. Solution for Constant Recharge Rate

By substituting $\alpha = 0$, Equation (9) reduces to

$$
\begin{aligned}
&h^2(x, y, t) = \\
&= h_0^2 + \frac{2aLDtP}{KAB} + \frac{4PDB^2}{KA\pi^3}\sum_{m=1}^{\infty}\frac{1}{m^3}\,\mathrm{Sin}\left(\frac{m\pi L}{B}\right)\mathrm{Cos}\left(\frac{m\pi x}{B}\right)\times \\
&\times\left[1-\exp(-at\pi^2m^2/B^2)\right] + \frac{4PLA^2}{KB\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\,\mathrm{Sin}\left(\frac{n\pi D}{A}\right)\mathrm{Cos}\left(\frac{n\pi y}{A}\right)\times \\
&\times\left[1-\exp(-at\pi^2n^2/A^2)\right] + \frac{8PA^2B^2}{K\pi^4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\,\frac{1}{m^2A^2+n^2B^2}\times \\
&\times\left[1-\exp(-a\pi^2(m^2A^2+n^2B^2)\,t/A^2B^2)\right]\times \\
&\times\,\mathrm{Sin}\left(\frac{m\pi L}{B}\right)\mathrm{Sin}\left(\frac{n\pi D}{A}\right)\mathrm{Cos}\left(\frac{m\pi x}{B}\right)\mathrm{Cos}\left(\frac{n\pi y}{A}\right).
\end{aligned} \tag{10}
$$

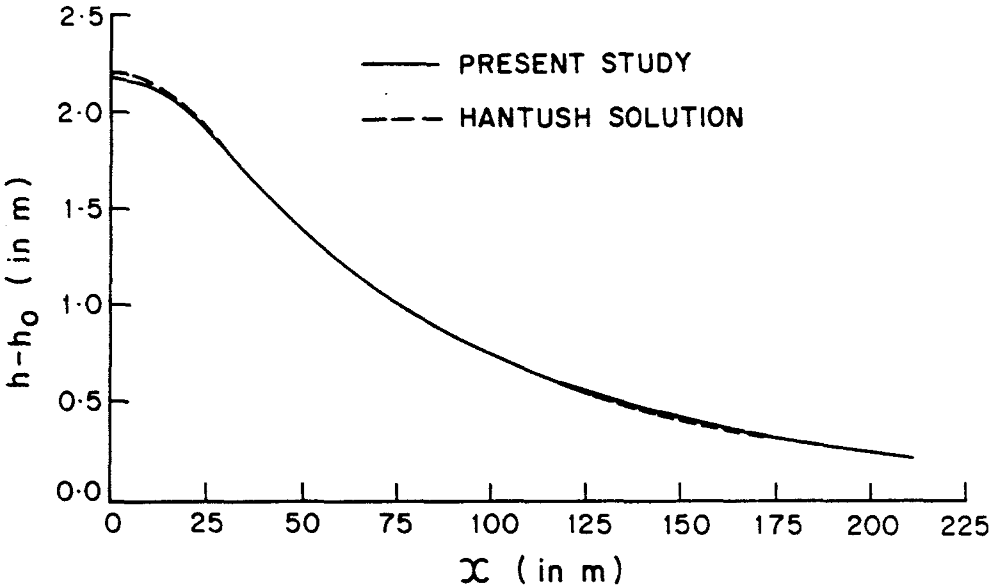

Fig. 3. Comparison of water-table profiles obtained by using the present solution and the Hantush solution.

in which $P = P_1 + P_0$. Equation (10) has two more terms, *i.e.* second and third, than Equation (17) of Rao and Sarma (1981a), which we obtained by using the inverse Fourier cosine transform (Equation (8)).

4. Numerical Results and Discussion

Warner *et al.* (1989) have shown that Rao and Sarma's (1981a) solution gives erratic results. Here, we first consider their numerical example to compare the results of the present solution with the Hantush (1967) solution. In this example, $h_0 = 60.96$ m, $L = D = 30.48$ m, $e = 0.20$, and $K = 3.048$ m/d.

As Hantush solution is for an infinite aquifer, to simulate the condition of infinite aquifer we have taken the length and width of the finite aquifer as 100 times of the length and width of the recharge basin. Using Equation (10), the water-table profile for $t = 20$ days is computed and compared in Figure 3 with the profile obtained by using the Hantush (1967) solution for a rectangular recharge basin. It is evident from the figure that both profiles are in very good agreement.

Now, in order to see the effect of the variation in the rate of recharge on the growth of the water-table, we consider the numerical example used by Rao and Sarma (1981a) in which

$$h_0 = 4.877 \text{ m}, \quad L = D = 44.958 \text{ m}, \quad e = 0.089\,,$$

$$K = 31.699 \text{ m/d}, \quad P_1 = 0.061 \text{ m/d}, \quad P_0 = 0.0365 \text{ m/d}\,.$$

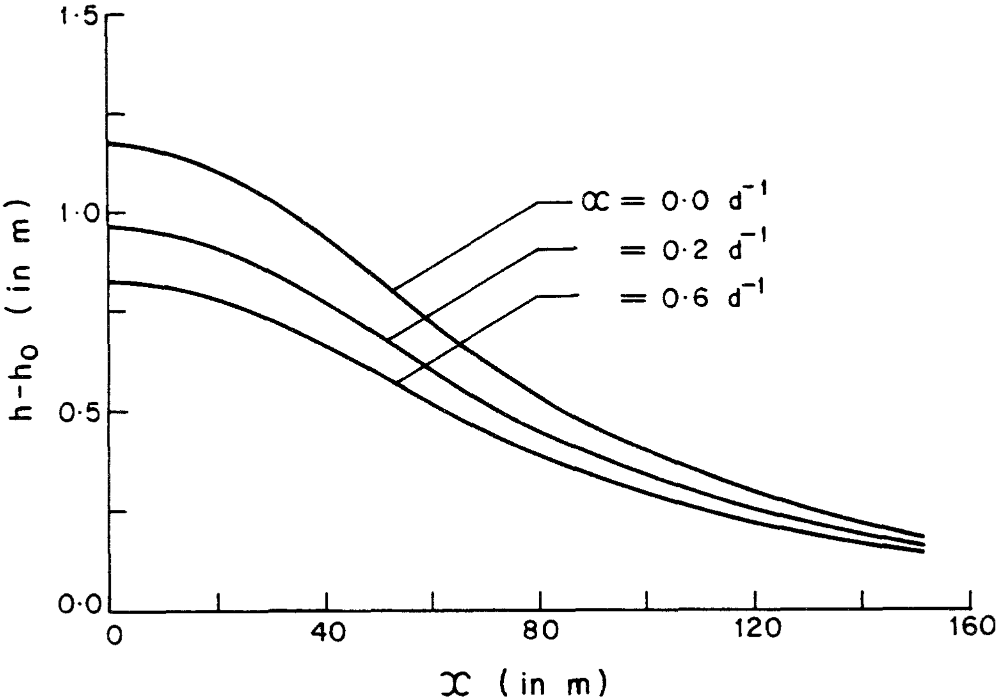

Fig. 4. Comparison of water-table profiles corresponding to time-varying and constant rates of recharge.

In the computation, the numerical values of A and B are taken as 100 times L. Two values of the decay constant are taken. These are $\alpha = 0.2$ and $0.6\ d^{-1}$.

The water-table profiles corresponding to the constant rate of recharge and time-varying rates of recharge are computed for $t = 5.15$ day by using Equations (10) and (9), respectively, and are compared in Figure 4. The constant rate of recharge is taken as being equal to the initial rate of time-varying recharge, i.e. $P_1 + P_0$. It is clear from the figure that the rise of the water-table for the constant rate of recharge ($\alpha = 0\ d^{-1}$) is more than the rise of the water-table for time-varying recharge ($\alpha = 0.2$ and $0.6\ d^{-1}$). The magnitude of difference between two successive profiles is maximum at the centre of the recharge basin and it decreases with distance away from the centre. In the case of time-varying recharge, the rise of the water-table for $\alpha = 0.2\ d^{-1}$ is more than the rise for $\alpha = 0.6\ d^{-1}$. This is because, for a larger value of α, the decay of recharge rate will be faster than that for a smaller value. As a result, the cumulative recharge during a given period for a larger value of α is comparatively less than that for a smaller value of α.

Time history of the growth of the water-table at the centre of the recharge basin for a constant rate of recharge ($\alpha = 0$) and for time-varying rates of recharge with $\alpha = 0.2$ and $0.6\ d^{-1}$ are shown in Figure 5. As expected, the growth of the water-table for $\alpha = 0.2\ d^{-1}$ is faster than that for $\alpha = 0.6\ d^{-1}$ at the beginning of

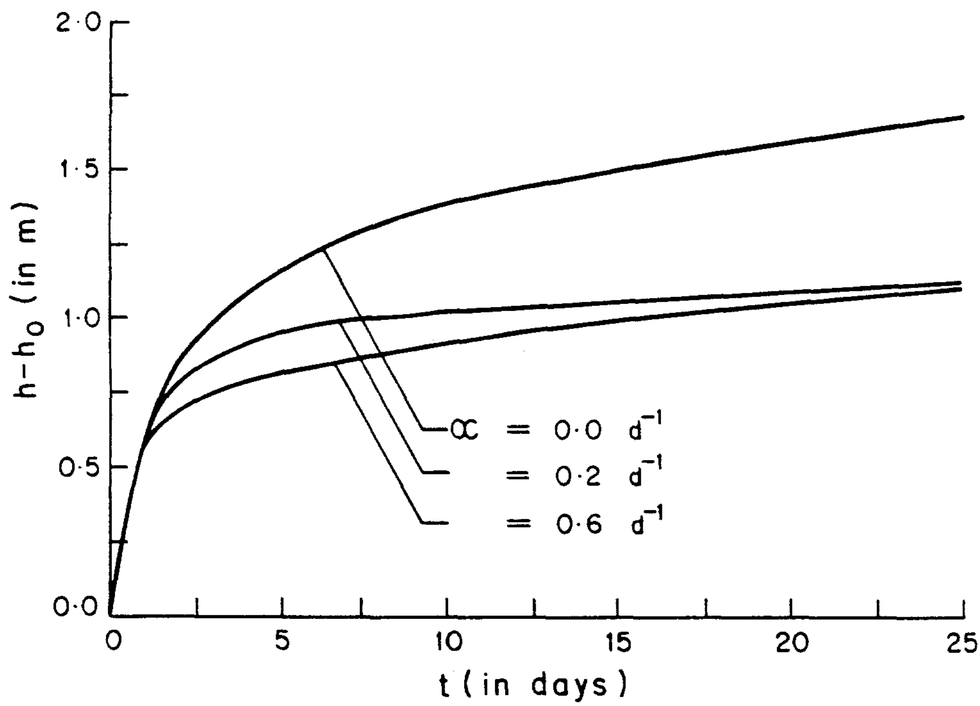

Fig. 5. Time history of the water-table growth at the centre of the recharge basin for different values of α.

the recharge. As a result, the difference between both curves increases with time. But after some time, it decreases and both curves approach the same level. This is because, that at a later time, the rates of recharge for $\alpha = 0.2$ and 0.6 d^{-1} maintain the same lower value of recharge rate, P_1. An example of water table contours at 5.15 day is shown in Figure 6 for an area of 175×175 m^2. These contours are computed for $\alpha = 0.6$ d^{-1} using Equation (9).

5. Conclusion

An analytical solution (Equation (9)) is presented to predict the spatio-temporal variation of a water-table in response to time-varying recharge from a rectangular basin. Numerical results reveal that the variation in the rate of recharge significantly affects the growth of the water-table. The magnitude of reduction in the growth of the water-table, due to a decrease in the rate of recharge, is maximum below the central part of the recharge basin and it decreases with distance away from the centre.

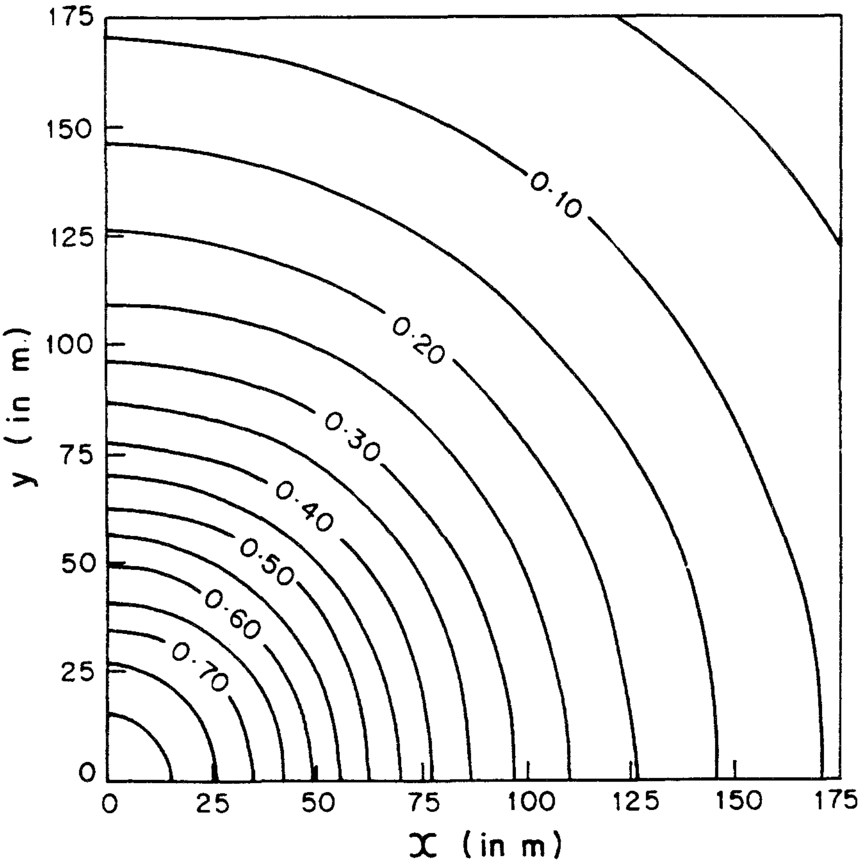

Fig. 6. Water-table contours at 5.15 day.

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