POLARIZATION PROPERTIES OF BOSON RADIATION

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The polarization properties of boson radiation in a magnetic field have been studied, and expressions obtained for the integral intensity of boson and fermion radiation in a magnetic field which are suitable for random energies. The investigation of boson radiation in an external field is of limited practical interest; however, from the methodological point of view it is very instructive and has already been investigated by several authors [1, 2, 4]. Also of undoubted interest is the comparison of boson and fermion (electron) radiation and hence the explanation of the "role of spin" in radiation.

§1. PARTICLE RADIATION INTENSITY IN AN EXTERNAL FIELD AND WAVE FUNCTIONS OF A BOSON IN A CONSTANT AND UNIFORM MAGNETIC FIELD

The particle radiation intensity in an external field has the form [3]

$$W = \sum_n \int_0^\pi \sin\Theta \, d\Theta \, W(n, n', \Theta), \tag{1}$$

where n, n' are quantum numbers characterizing the initial and final states, while the quantity W(n, n', Θ) is given by the expressions

$$W(n, n', \Theta) = \frac{ce^2 \varkappa^2}{1 + \frac{\partial K'}{\partial \varkappa}} \cdot S, \tag{2}$$

$$S = ([\bar{\boldsymbol{\alpha}} \boldsymbol{\varkappa}^{0+}] [\bar{\boldsymbol{\alpha}}^+ \boldsymbol{\varkappa}^0]) (N+1). \tag{3}$$

The matrix element $\bar{\alpha}$ has the form

$$\bar{\boldsymbol{\alpha}} = \int \psi_{n'}^+ e^{-i\vec{\varkappa} \boldsymbol{r}} \boldsymbol{\alpha} \psi_n d^3 x. \tag{4}$$

Formula (3) is identical with (28, 28) in [3] when the vector identity ([ab] [cd]) = (ac) (bd) – (ad) (bc) is allowed for and $\varkappa^0 = \boldsymbol{\varkappa}/\varkappa$ is also considered, N being the number of photons in the initial state.

As shown in [4], for bosons formula (3) takes the form

$$S = \frac{1}{\hbar^2 K K'} ([\bar{\boldsymbol{P}} \boldsymbol{\varkappa}^{0+}] [\bar{\boldsymbol{P}}^+ \boldsymbol{\varkappa}^0]) (N+1), \tag{5}$$

$$\bar{\boldsymbol{P}} = \int \psi_{n'}^+ e^{-i\vec{\varkappa} \boldsymbol{r}} \boldsymbol{P} \psi_n d^3 x, \quad \boldsymbol{P} = -i\hbar\nabla - \frac{e}{c}\boldsymbol{A}. \tag{6}$$

For a description of the radiation polarization, the following orthogonal unit vectors are introduced [11]:

$$\boldsymbol{\beta}_2 = \frac{[\boldsymbol{\varkappa}^0 \boldsymbol{j}]}{\sqrt{1 - (\boldsymbol{\varkappa}^0 \boldsymbol{j})^2}}, \quad \boldsymbol{\beta}_3 = \frac{\boldsymbol{\varkappa}^0 (\boldsymbol{\varkappa}^0 \boldsymbol{j}) - \boldsymbol{j}}{\sqrt{1 - (\boldsymbol{\varkappa}^0 \boldsymbol{j})^2}}, \tag{7}$$
$$\boldsymbol{\beta}_\kappa^+ = \boldsymbol{\beta}_\kappa, \ (\kappa = 2, 3), \ |\boldsymbol{j}| = 1,$$

describing the two components of linear polarization; we may also introduce the two vectors

$$\boldsymbol{\beta}_l = \frac{1}{\sqrt{2}} (\boldsymbol{\beta}_2 + il\boldsymbol{\beta}_3), \ (l = \pm 1), \ \boldsymbol{\beta}_l^+ = \boldsymbol{\beta}_{-l}, \tag{8}$$

describing the two components of the circular polarization of radiation.

Formula (3) now becomes

$$S = S_2 + S_3 = \sum_{\kappa=2}^{3} (\bar{\boldsymbol{\alpha}} \boldsymbol{\beta}_\kappa^+) (\bar{\boldsymbol{\alpha}}^+ \boldsymbol{\beta}_\kappa) =$$
$$= S_{-1} + S_1 = \sum_{l=-1}^{1} (\bar{\boldsymbol{\alpha}} \boldsymbol{\beta}_l^+) (\bar{\boldsymbol{\alpha}}^+ \boldsymbol{\beta}_l). \tag{9}$$

For computing the matrix elements of (4) or (6) it is necessary to know the wave functions of particles in a magnetic field. For bosons, the wave functions in a constant and uniform magnetic field are to be found in [5] and [6]. In the cylindrical coordinate system r, φ, z they take the form

$$\psi_{n, l, \kappa_z} = \frac{1}{\sqrt{L_z}} e^{-icK_n t + i\kappa_z z} \cdot e^{-il\varphi} \sqrt{\frac{\gamma}{\pi}} I_{n, s}(\gamma r^2), \tag{10}$$

where

$$\gamma = \frac{eH}{2c\hbar}, \quad E_n = c\hbar K_n = c\hbar \sqrt{\kappa_0^2 + \kappa_z^2 + 4\gamma(n + 1/2)}. \tag{10a}$$

The functions are normalized to unity:

$$\int \psi_{n', l', \kappa_z'}^+ \psi_{n, l, \kappa_z} d^3 x = \delta_{n, n'} \cdot \delta_{l, l'} \cdot \delta_{\kappa_z, \kappa_z'}. \tag{11}$$

The functions $I_{n, s}(x)$ required here are quite familiar in the literature.

Using functions (10) it is easy to compute the matrix elements in (6).

Simple computations lead to the following results (it is assumed that $K_z = 0$ in the initial state):

$$\bar{P}_x = i\hbar \sqrt{\gamma} e^{i(l-l')\varphi} I_{s, s'}(x) \times$$
$$\times (e^{i\varphi} \sqrt{n+1} I_{n+1, n'}(x) - e^{-i\varphi} \sqrt{n} I_{n-1, n'}(x)),$$
$$\bar{P}_y = \hbar \sqrt{\gamma} e^{i(l-l')\varphi} I_{s, s'}(x) \times$$
$$\times (e^{i\varphi} \sqrt{n+1} I_{n+1, n'}(x) + e^{-i\varphi} \sqrt{n} I_{n-1, n'}(x)),$$
$$\bar{P}_z = 0, \quad \varphi = \varphi' - \frac{\pi}{2}, \quad x = \frac{\varkappa^2 \sin^2\Theta}{4\gamma}, \tag{12}$$

$$\kappa_z' = -\varkappa \cos\Theta. \tag{13}$$

Here the spherical coordinates

$$\varkappa(\varkappa\cos\varphi'\sin\Theta,\ \varkappa\sin\Theta\sin\varphi',\ \varkappa\cos\Theta) \quad (13a)$$

are introduced for the vector $\varkappa$.

§ 2. SPECTRAL DISTRIBUTION OF BOSON RADIATION INTENSITY

Substituting the expressions obtained into formula (9) and allowing for (5), we obtain, after simple manipulation,

$$S_2 = \frac{1}{KK'}\gamma I^2_{SS'}(x)\times$$
$$\times\{\sqrt{n+1}\,I_{n+1,n'}(x) - \sqrt{n}\,I_{n-1,n'}(x)\}^2, \quad (14)$$

$$S_3 = \frac{1}{KK'}\gamma I^2_{S,S'}(x)\cos^2\Theta\times$$
$$\times\{\sqrt{n+1}\,I_{n+1,n'}(x) + \sqrt{n}\,I_{n-1,n'}(x)\}^2, \quad (15)$$

$$S_l = \frac{1}{2}(S_2 + S_3 - ilN), \quad (16)$$

$$N = -\frac{2i}{KK'}\cdot\gamma I^2_{S,S'}(x)\cos\Theta\times$$
$$\times\{(n+1)\,I^2_{n+1,n'}(x) - nI^2_{n-1,n'}(x)\}. \quad (17)$$

From this we may conclude that for a boson there is no circular polarization, since the correlation term has the form

$$\int_0^\pi \sin\Theta d\Theta\ f(\cos^2\Theta)\cos\Theta = 0. \quad (18)$$

This conclusion agrees with the general assumption according to which photons with circular polarization can radiate only particles with oriented spin. The absence of circular polarization of boson radiation is one characteristic difference from fermion radiation.

Subsequent operations involving formulas (14) and (15) are not possible without certain approximations. Notably the functions $I_{n,n}(x)$ are approximated by Macdonald functions as follows [7]:

$$I_{n,n'}(x) = \frac{\sqrt{\varepsilon_1}}{\pi\sqrt{3}}K_{1/3}\left(\frac{2}{3}\sqrt[4]{x_0^2 nn'}\,\varepsilon_1^{3/2}\right),$$

$$\varepsilon_1 = 1 - \frac{x}{x_0},\quad x_0 = (\sqrt{n} - \sqrt{n'})^2. \quad (19)$$

We introduce the variable ν',

$$\nu = n - n' = \nu'\left(1 - \frac{\nu'}{4n}\beta^2\sin^2\Theta\right), \quad (20)$$

and define y in terms of ν':

$$y = \frac{2}{3}\frac{\nu'}{1 - \frac{\nu'}{2n}}\varepsilon_0^{3/2},\quad \varepsilon_0 = 1 - \beta^2; \quad (21)$$

then, introducing the characteristic parameter ξ,

$$\xi = \frac{3}{2}\frac{\hbar}{Rmc}\left(\frac{E}{mc^2}\right)^2, \quad (22)$$

where $R = \sqrt{n}/\gamma$ is the radius of a stable orbit, we obtain the following expressions for the linear radiation polarization components:

$$W_2 = \frac{9}{8\pi^2}\cdot\frac{ce^2}{R^2}\cdot$$
$$\cdot\varepsilon_0^{-3/2}\int_0^\infty dy\int_0^\pi \sin\Theta\,d\Theta\frac{y^2}{(1+\xi y)^3}\varepsilon^2 K^2_{2/3}(z), \quad (23)$$

$$W_3 = \frac{9}{8\pi^2}\cdot\frac{ce^2}{R^2}\cdot$$
$$\cdot\varepsilon_0^{-3/2}\int_0^\infty dy\int_0^\pi \sin\Theta\,d\Theta\frac{y^2}{(1+\xi y)^3}\varepsilon\cos^2\Theta\,K^2_{1/3}(z), \quad (24)$$

where

$$z = \frac{1}{2}\left(\frac{\varepsilon}{\varepsilon_0}\right)^{3/2}y,\quad \varepsilon = 1 - \beta^2\sin^2\Theta.$$

Formulas (23) and (24) are summed over S' and the summation over ν is replaced by integration with respect to y. Integration over the angles is done using multiply evaluated integrals (see, for example, [3]).

Upon integrating we obtain the spectral distribution of the boson radiation intensity

$$W_2 = A\int_0^\infty dy\frac{y}{(1+\xi y)^3}\left\{3K_{2/3}(y) - \int_y^\infty K_{1/3}(x)\,dx\right\}, \quad (25)$$

$$W_3 = A\int_0^\infty dy\frac{y}{(1+\xi y)^3}\left\{K_{2/3}(y) - \int_y^\infty K_{1/3}(x)\,dx\right\}, \quad (26)$$

where

$$A = \frac{3\sqrt{3}}{8\pi}\frac{ce^2}{R^2\varepsilon_0^2}. \quad (26a)$$

§ 3. INTEGRAL INTENSITY OF BOSON AND ELECTRON RADIATION

Integration over the spectrum in (25) and (26) can easily be realized for the two limit cases when $\xi \ll 1$ or $E \ll E_{1/2}$ and $\xi \gg 1$ or $E \gg E_{1/2}$, where $E_{1/2}$ is the characteristic energy

$$E_{1/2} = mc^2\left(\frac{2}{3}\frac{mcR}{\hbar}\right)^{1/2}. \quad (27)$$

a) $\xi \ll 1$. Expanding the integrands in series in powers of ξ and taking into account the value of the integral

$$\int_0^\infty y^p K_\mu(y)\,dy = 2^{p-1}\Gamma\left(\frac{p+1-\mu}{2}\right)\Gamma\left(\frac{p+1+\mu}{2}\right), \quad (28)$$

we obtain the asymptotic series

$$\left.\begin{aligned} W_2 &= \frac{A}{3}\sum_{n=0}2^n(n+1)(3n+7)\times\\ &\times\Gamma\left(\frac{3n+8}{6}\right)\Gamma\left(\frac{3n+4}{6}\right)(-\xi)^n,\\ W_3 &= \frac{A}{3}\sum_{n=0}2^n(n+1)\times\\ &\times\Gamma\left(\frac{3n+8}{6}\right)\Gamma\left(\frac{3n+4}{6}\right)(-\xi)^n \end{aligned}\right\} \quad (29)$$

or, allowing for only two terms, we have

$$
\left.\begin{aligned}
W_2 &= W^{(cl)}\left\{\frac{7}{8}-\frac{25\sqrt{3}}{12}\xi+\frac{162\pm 6}{9}\xi^2+\ldots\right\},\\
W_3 &= W^{(cl)}\left\{\frac{1}{8}-\frac{5\sqrt{3}}{24}\xi+3\cdot\frac{6\pm 2}{9}\xi^2+\ldots\right\},\\
W &= W^{(cl)}\left\{1-\frac{55\sqrt{3}}{24}\xi+\frac{60\pm 4}{3}\xi^2+\ldots\right\},\\
W^{(cl)} &= \frac{2}{3}\frac{ce^2}{R^2}\left(\frac{E}{mc^2}\right)^4.
\end{aligned}\right\} \quad (30)
$$

The plus sign in formulas (30) refers to the corresponding quantities for electrons [8] and the minus sign refers to bosons. Thus, for small energies, the radiation defference for bosons and electrons appears only in terms $\sim\hbar_2$ and this term for the total intensity has the form

$$
6\left[\frac{\hbar}{mcR}\left(\frac{E}{mc^2}\right)^2\right]^2 b^{(0,\,1/2)}, \quad (31)
$$

where $b^0 = 7$ for bosons and $b^{1/2} = 8$ for electrons (see also [9]).

A general exception is that at small energies, bosons radiate somewhat less than electrons, both components of linear polarization decreasing by the same value, $4/3\ \xi^2 W^{(cl)}$.

b) $\xi \gg 1$. In this case the integrals are also easy to obtain and we have

$$
\left.\begin{aligned}
W_2 &= \frac{34\pm 7}{64}W^{(glob)},\\
W_3 &= \frac{16\pm 7}{64}W^{(glob)},\\
W^{(glob)} &= \frac{8}{27}\cdot\frac{ce^2}{R^2\varepsilon_0^2}\cdot 2^{2/3}\Gamma(2/3)\,\xi^{-4/3},\\
W &= 2\cdot\frac{25\pm 7}{64}W^{(glob)},
\end{aligned}\right\} \quad (32)
$$

where the plus sign refers to electrons and the minus sign to bosons. It is clear from this that at high energies both components also decrease by the same quantity $7/32\ W^{(glob)}$, but in contrast to the small-energy case this reduction appears in the main term and significantly changes the radiation picture. Boson radiation is almost one-half that of electron radiation and the "spin role" in radiation becomes extremely important [4].

§ 4. EXACT EVALUATION OF SPECTRUM INTEGRALS

Integrals (25) and (26) and others similar to them can be evaluated using special functions associated with the Bessel functions. In particular ([10] p. 128, formula (5)),

$$
\begin{aligned}
L_\mu(x) &= \int_0^\infty \frac{K_\mu(y)\,dy}{x+y} = \frac{\pi^2}{2\sin^2\mu\pi}\left\{\left[I_\mu(x)+I_{-\mu}(x)\right]-\right.\\
&\left.-\left[e^{-i\frac{\mu\pi}{2}}J_\mu(ix)+e^{i\frac{\mu\pi}{2}}J_{-\mu}(ix)\right]\right\} \quad |\mu|<1.
\end{aligned} \quad (33)
$$

For the functions $L_\mu(x)$ the following relation exists:

$$
\left.\begin{aligned}
L'_\mu(x) &= L_{1-\mu}(x)-\frac{\mu}{x}L_\mu(x)-\frac{\pi}{2\sin\frac{\mu\pi}{2}}\frac{1}{x}\\
L'_{1-\mu}(x) &= L_\mu(x)-\frac{1-\mu}{x}L_{1-\mu}(x)-\frac{\pi}{2\cos\frac{\mu\pi}{2}}\frac{1}{x}.
\end{aligned}\right\} \quad (34)
$$

whence $L_\mu^{(n)}(x)$ is expressed in terms of $L_\mu(x)$, $L_{1-\mu}(x)$, and $1/x$.

If we let $x = 1/\xi$, then we easily obtain

$$
\begin{aligned}
D_\mu(x) &= \int_0^\infty \ln(1+\xi y)K_\mu(y)\,dy =\\
&= \int_0^x L_\mu(x)\,dx-\frac{\pi}{2\cos\frac{\mu\pi}{2}}\ln x + c_\mu,
\end{aligned} \quad (35)
$$

$$
\int_0^\infty \frac{K_\mu(y)\,dy}{1+\xi y} = xL_\mu(x), \quad (36)
$$

$$
\begin{aligned}
\int_0^\infty \frac{K_\mu(y)\,dy}{(1+\xi y)^2} &= -x^2L'_\mu(x) =\\
&= \mu xL_\mu(x)-x^2L_{1-\mu}(x)+\frac{\pi}{2\sin\frac{\mu\pi}{2}}x,
\end{aligned} \quad (37)
$$

etc. Here

$$
c_\mu = \int_0^\infty \ln y\cdot K_\mu(y)\,dy. \quad (38)
$$

In view of the fact that any integrals of the form

$$
I = \int_0^\infty \frac{y^m}{(1+\xi y)^n}K_\mu(y)\,dy,
$$

$$
J = \int_0^\infty \frac{y^m dy}{(1+\xi y)^n}\int_y^\infty K_\mu(x)\,dx
$$

are expressed by linear combinations of the integrals in (35), (36), (37), etc., formulas for the integral intensity of bosons and electrons may be expressed in terms of the special functions $L_{1/3}(x)$ and $L_{2/3}(x)$, where all the formulas hold for any values of the parameter ξ. These expressions have the form

a) for bosons,

$$
\left.\begin{aligned}
W_2 &= \frac{A}{3}x^2\{(1-3x^2)L_{2/3}(x)-2xL_{1/3}(x)+3\pi x\},\\
W_3 &= \frac{A}{9}x^2\{L_{2/3}(x)+3xL_{1/3}(x)-\pi\sqrt{3}\};
\end{aligned}\right\} \quad (39)
$$

b) for electrons,

$$
\begin{aligned}
W_2 &= \frac{A}{2}\left\{\left(\frac{4}{9}x^4+\frac{82}{81}x^2\right)L_{2/3}(x)+\right.\\
&\left.+\left(\frac{1}{3}x^5+\frac{67}{27}x^3\right)L_{1/3}(x)-\frac{\pi\sqrt{3}}{9}x^4-\frac{\pi}{3}x^3-\frac{67\pi\sqrt{3}}{81}x^2\right\},
\end{aligned} \quad (40)
$$

$$W_3 = \frac{A}{2}\left\{\left(\frac{22}{9}x^4 + \frac{46}{81}x^2\right)L_{2/3}(x) + \left(\frac{1}{3}x^5 + \frac{121}{27}x^3\right)\times\right.$$

$$\left.\times L_{1/3}(x) - \frac{\pi\sqrt{3}}{9}x^4 - \frac{7\pi}{3}x^3 - \frac{85\pi\sqrt{3}}{81}x^2\right\}. \qquad (40)$$

Cont'd

It is obvious that formulas (32) are obtained from these expressions when we allow for the fact that for small x

$$L_\mu(x) \approx \frac{2^\mu \pi \Gamma(\mu)}{2\sin\mu\pi}x^{-\mu}. \qquad (41)$$

Formulas (30) may also be obtained from (40) using an asymptotic series for $L_\mu(x)$ for large x (small ξ):

$$L_\mu(x) = -\sum_{n=0} 2^{n-1}\Gamma\left(\frac{n+1-\mu}{2}\right)\times$$

$$\times\Gamma\left(\frac{n+1+\mu}{2}\right)(-\xi)^{n+1}, \qquad (42)$$

However, it is simpler to proceed directly from expressions (25) and (26).

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