THE TEMPERATURE FIELD IN A CRYSTAL DURING HEATING BY RADIATION

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An accurate solution is given of the problem of the temperature field in a crystal when radiative heat transfer is taking place on its surface and the thermophysical characteristics depend on temperature range below the Debye temperature ($T < \Theta_D$).

In the growth and annealing of crystals it is important to avoid sudden temperature changes, since these cause irregular crystal growth, give rise to defects, and do not completely remove the residual stresses in the grown crystal. The quality of a crystal depends to a considerable extent on the temperature field in it [5].

The heat diffusion within a crystal results from the nonstationary thermal conductivity, which in the simplest case is described by the following system of differential equations:

$$C\rho \frac{\partial T}{\partial \tau} = \operatorname{div}(\lambda \operatorname{grad} T), \tag{1}$$

$$T(x,\ 0) = f_1(x), \tag{2}$$

$$\frac{\partial T(0,\ \tau)}{\partial x} = 0, \tag{3}$$

with the boundary condition

$$\lambda \frac{\partial T(R,\ \tau)}{\partial x} + \sigma T^4(R,\ \tau) = q_c(\tau). \tag{4}$$

Here $\lambda = \lambda(T)$, $C = C(T)$. Substitution of

$$\eta(x, \tau) = \int_0^T \lambda(T)\, dT$$

reduces differential equation (1) to the form

$$\frac{\partial \eta(x,\ \tau)}{\partial \tau} = a(\eta) \frac{\partial^2 \eta(x,\ \tau)}{\partial x^2}, \tag{1'}$$

while the initial conditions (2) and the symmetry condition (3) are, respectively,

$$\eta(x,\ 0) = f_2(x), \tag{2'}$$

$$\frac{\partial \eta(0,\ \tau)}{\partial x} = 0. \tag{3'}$$

For many crystals at temperatures below the Debye temperature the thermophysical characteristics are proportional to the cube of the absolute temperature [4], i.e.,

$$\lambda = \varkappa_1 T^3, \tag{A}$$

$$C = \varkappa_2 T^3. \tag{B}$$

Under conditions (A) and (B) a = const. Taking account of (A), the boundary condition (4) can be written

$$\frac{\partial \eta(R,\ \tau)}{\partial x} = q_c(\tau) - \omega \eta(R, \tau); \tag{4'}$$

here $\omega = 4\sigma/\varkappa_1$.

In solving the system (1′)–(4′) it is possible to use the integral transform given by the expression [1, 3]

$$\Phi(p,\ \tau) = \int_0^R \eta(x,\ \tau) \cos px\, dx, \tag{C}$$

where p is the positive root of the transcendental equation $p \cdot \operatorname{tg} pR = \omega$.

The numerical values of the roots are given in [6]. Integration by parts, taking account of (3′) and (4′), gives

$$\int_0^R \cos px \frac{\partial^2 \eta(x,\ \tau)}{\partial x^2} dx = q_c(\tau) \cos pR - p^2 \Phi(p,\ \tau). \tag{5}$$

By applying transformation (C) to both parts of equation (1′) and taking into account (5), we obtain the ordinary differential equation with respect to the transform

$$\frac{d\Phi(p,\ \tau)}{d\tau} + ap^2 \Phi(p,\ \tau) = aq_c(\tau) \cos pR, \tag{6}$$

the solution of which

$$\Phi(p,\ \tau) = \exp(-ap^2\tau) \Big[c(p) + + a \cos pR \int_0^\tau q_c(t) \exp(ap^2 t)\, dt \Big], \tag{7}$$

c(p) is found from the transformed condition (2′)

$$c(p) = \Phi(p,\ 0) = \int_0^R f_2(x) \cos px\, dx.$$

Then the solution in transforms is

$$\Phi(p,\ \tau) = \exp(-ap^2\tau) \int_0^R f_2(x) \cos px\, dx + + a \cos pR \exp(-ap^2\tau) \int_0^\tau q_c(t) \exp(ap^2 t)\, dt. \tag{8}$$

Passing from the transform to the original, by means of the conversion formula [2, 3],

$$\eta(x, \tau) = 2\sum_{n=1}^{\infty} \frac{p_n^2 + \omega^2}{R(p_n^2 + \omega^2) + \omega} \cos p_n x \, \Phi(p_n, \tau), \quad (9)$$

we obtain

$$\eta(x, \tau) = 2\sum_{n=1}^{\infty} \frac{p_n^2 + \omega^2}{R(p_n^2 + \omega^2) + \omega} \cos p_n x \cdot$$

$$\cdot \exp(-a p_n^2 \tau) \int_0^R f_2(x) \cos p_n x dx +$$

$$+ 2a \sum_{n=1}^{\infty} \frac{p_n^2 + \omega^2}{R(p_n^2 + \omega^2) + \omega} \cos p_n R \cdot \cos p_n x \cdot$$

$$\cdot \exp(-a p_n^2 \tau) \int_0^{\tau} q_c(t) \exp(a p_n^2 t) dt. \quad (10)$$

Finally, the temperature distribution is given by the relationship

$$T(x, \tau) = \Big[2\sum_{n=1}^{\infty} \frac{p_n^2 + \omega^2}{R(p_n^2 + \omega^2) + \omega} \cos p_n x \cdot$$

$$\cdot \exp(-a p_n^2 \tau) \int_0^R f_1^4(x) \cos p_n x dx +$$

$$+ \frac{8a}{\varkappa_1} \sum_{n=1}^{\infty} \frac{p_n^2 + \omega^2}{R(p_n^2 + \omega^2) + \omega} \cdot \cos p_n R \cdot \cos p_n x \cdot$$

$$\cdot \exp(-a p_n^2 \tau) \int_0^{\tau} q_c(t) \exp(a p_n^2 t) dt \Big]^{1/4}. \quad (11)$$

Particular cases:

a) If $T(x, 0) = T_0 = \text{const}$, and the constant source of radiant heat $q_c(\tau) = q_c = \text{const}$, then

$$T(x, \tau) = \Big\{ 2T_0^4 \sum_{n=1}^{\infty} \frac{\omega}{R(p_n^2 + \omega^2) + \omega} \cdot \frac{\cos p_n x}{\cos p_n R} \cdot$$

$$\cdot \exp(-a p_n^2 \tau) + \frac{8q_c}{\varkappa_1} \sum_{n=1}^{\infty} \frac{1 + \left(\frac{\omega}{p_n}\right)^2}{R(p_n^2 + \omega^2) + \omega} \cdot$$

$$\cdot \cos p_n R \cdot \cos p_n x \, [1 - \exp(-a p_n^2 \tau)] \Big\}^{1/4}. \quad (12)$$

b) Linear variation of radiator temperature

$$q_c(\tau) = \sigma (T_{c0} + b\tau)^4,$$

$$T(x, \tau) = \Big\{ 2T_0^4 \sum_{n=1}^{\infty} \frac{\omega}{R(p_n^2 + \omega^2) + \omega} \cdot \frac{\cos p_n x}{\cos p_n R} \cdot$$

$$\cdot \exp(-a p_n^2 \tau) + \frac{8a\sigma}{b\varkappa_1} \sum_{n=1}^{\infty} \frac{p_n^2 + \omega^2}{R(p_n^2 + \omega^2) + \omega} \cos p_n R \cdot \quad (13)$$

$$\cos p_n x \left[\frac{(T_{c0} + b\tau)^4}{\frac{a p_n^2}{b}} + \sum_{\kappa=1}^{4} (-1)^{\kappa} \frac{(5-\kappa)!}{\left(\frac{a p_n^2}{b}\right)^{\kappa+1}} \cdot \right.$$

$$\cdot (T_{c0} + b\tau)^{4-\kappa} - \exp(-a p_n^2 \tau) \cdot \quad \text{(13) (cont'd)}$$

$$\left. \cdot \left(\frac{T_{c0}^4}{\frac{a p_n^2}{b}} + \sum_{\kappa=1}^{4} (-1)^{\kappa} \frac{(5-\kappa)!}{\left(\frac{a p_n^2}{b}\right)^{\kappa+1}} T_{c0}^{4-\kappa} \right) \right] \Big\}^{1/4}.$$

c) Exponential law of temperature of the radiating medium

$$q_c(\tau) = \sigma [T_{\max} - (T_{\max} - T_{c0}) \exp(-m\tau)]^4,$$

$$T(x, \tau) = \Big\{ 2T_0^4 \sum_{n=1}^{\infty} \frac{\omega}{R(p_n^2 + \omega^2) + \omega} \cdot \frac{\cos p_n x}{\cos p_n R} \cdot$$

$$\cdot \exp(-a p_n^2 \tau) + \frac{8a\sigma}{\varkappa_1} \sum_{n=1}^{\infty} \sum_{\kappa=0}^{4} \frac{p_n^2 + \omega^2}{R(p_n^2 + \omega^2) + \omega} \cos p_n R \cdot$$

$$\cdot \cos p_n x \, \frac{B_{\kappa} T_{\max}^{4-\kappa} (T_{\max} - T_{c0})^{\kappa}}{a p_n^2 - \kappa m} \cdot$$

$$\cdot [\exp(-\kappa m \tau) - \exp(-a p_n^2 \tau)] \Big\}^{1/4}. \quad (14)$$

Two- and three-dimensional problems can be solved by the above method. The problem presented for heating can be adapted for cooling by a simple change of variable [6].

Principal notation: $q = \varepsilon_n C_0 / 10^8$—apparent coefficient of radiative heat transfer; ε_n—the reduced emissivity of the system; C_0—black-body radiation constant; C—heat capacity; λ—thermal conductivity. $q_c(\tau)$—radiator heat flux; $T(x, \tau)$—variable absolute temperature; T_{c0}—absolute initial temperature of the radiating medium; T_{max}—maximum absolute temperature of radiator.

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