INVERSE PROBLEMS IN HEAT CONDUCTION

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UDC 536.24.02

Formulations and principles for obtaining stable solutions of various inverse problems applicable in the physics of heat and in heat engineering are discussed.

1. In the fields of the physics of heat and heat engineering, as in many other fields of scientific-industrial activity, experimental investigations are highly automated. Previously acceptable "manual" methods for the treatment of data are no longer in a condition to cope with the abundant volume of information. On the other hand, this information contains more precise data concerning the objects studied because of the increased accuracy of observation. The problem is to learn to extract the data by using all the information furnished by observations.

The development of computer technology offers a basis for the solution of this problem. For rational use of a computer for the specified purpose, however, it is necessary to devote particular attention to the correctness of the mathematical formulation of the problem and to the development of stable algorithms for the solution. This primarily refers to the inverse problems with which this paper is concerned. By "inverse" is meant a problem in which the studied object or determined quantity is inaccessible (or difficultly accessible) by direct investigation, and conclusions about its properties are reached on the basis of indirect measurements of quantities which are the effects of the properties sought for.

We assume that a cause-and-effect relation has been established between the property sought for and the observed quantity. We then arrive at a mathematical formulation of the inverse problem.

A mathematical problem is labelled correctly formulated if it satisfies the following three conditions:

1) a solution of the problem exists for any input data;

2) the solution is unique;

3) the solution is stable with respect to small perturbations of the input data.

A problem which does not satisfy at least one of these conditions is labelled incorrectly formulated.

Inverse problems belong among the incorrectly formulated problems. If the "solution" is carried out on a computer, one often observes that when the final approximation of the problem is more exact, the approximation to the solution is obtained with a greater error and has nothing in common with the actual solution.

Nevertheless, the practice of physical research has for a long time given rise to a need to "solve" incorrectly formulated problems. Their solution was ordinarily based on simplified models of an object or phenomenon where the latter are characterized by a small number of parameters. For example, the method of trial and error in the solution of such problems is well known. It is natural that only extremely approximate representations of the properties of an object or of the characteristics of a defined quantity can be obtained by this method. Such an approach to the solution of a problem does not use all the potentialities furnished by modern levels of observational accuracy and by the development of computer technology.

These potentialities can be used for a study of new formulations of inverse problems and in the employment of recently proposed stable algorithms for their solution. The development of such formulations

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 29, No. 1, pp. 7-12, July, 1975. Original article submitted February 10, 1975.

and algorithms is not merely a mathematical problem, but a general theoretical problem, the solution of which is only possible through the close collaboration of physicists and mathematicians.

We consider in somewhat greater detail and in rather general form the correctness of the mathematical formulation of inverse problems in heat conduction.

2. Mathematics is a means for the cognition of natural phenomena in order to control them. The cognition process is always associated with the construction of some model, the correctness of which is verified by practical means.

Let z be a sought-for property of a model and u (an effect of z) be a quantity which is compared with an actually observed quantity. The direct problem with respect to the determination of u for a given z is represented by the expression

$$A(z) = u, \tag{1}$$

where it is assumed that the solution of this problem is unique (a completely defined u corresponds to each z). For example, if z is a boundary temperature mode, a unique value of the temperature (u) at any point within the heated body corresponds to this mode. In this case, the operator A is a well-known integral expression for a one-dimensional system with constant thermal properties.

The observed quantity $\tilde{u}$ is usually given with some error $\|\hat{u} - u\| \le \delta$, where $\hat{u}$ is the exact value of the "observable."

We are interested in the inverse problem: to find z from given $\tilde{u}$. In the "classical" formulation tacitly assumed in practice, one is talking of the solution of the operator equation (1), where $u = \tilde{u}$. We further assume that the law for the correspondence $z \to u$ (the operator A) is adequate for the actual relationship between these quantities. Then the specified formulation is incorrect primarily for the reason that there may be no solution of Eq. (1) for a given $\tilde{u}$; $\tilde{u} \neq Az$ for every z, and it is not difficult to point to examples of such a situation. Furthermore, the question of the uniqueness of the solution of this inverse problem requires special study.

Finally, perturbations of the z sought for as large as desired may correspond in Eq. (1) to small perturbations of $\hat{u}$, i.e., to a given $\tilde{u}$; that is, the solution is unstable with respect to small perturbations of the input data. Thus the search for an exact solution of Eq. (1) with approximate input data is an incorrectly formulated problem for this reason.

A more natural formulation of the inverse problem which considers the fact that $\tilde{u}$ does not necessarily correspond precisely to the model selected for z consists of a determination of all elements z satisfying the inequality

$$\|Az - \tilde{u}\| \leqslant \delta. \tag{2}$$

Indeed, all z satisfying Eq. (2) are equivalent from the viewpoint of the selection of a model for a given error level δ of the input data and in the absence of any additional information about the desired model.

In such a formulation, inverse problems are no different than ordinary problems, since the input data are given with a certain error in all cases and the "solution" of a problem with approximate data is always some "region" of values of z.

Further, if the diameter of the specified region is sufficiently small, as is the case by reason of stability in correctly formulated problems, the formulation of the problem (2) can be changed; in this case one can select any element z satisfying the inequality (2) as a solution.

However, for incorrectly formulated problems, the diameter of the region of values of z corresponding to (2), generally speaking, is unbounded. Consequently the problem (2) is quite undetermined and in this case it is impossible to select any element satisfying the inequality (2) as an approximate solution of this inverse problem, since it may be quite far from $\hat{z}$, which best characterizes the model.

There then arises the problem of selecting an approximation from the elements satisfying (2). The selection can be made on the basis of additional information about the desired model. Primarily this information can be used for a priori limitations (quantitative limitations in practical problems) on the region of values of z [1]. These limitations reduce the diameter of the region of values accessible to the inequality (2) and thus the problem may be brought into the class of correctly formulated problems. It is well known how such an approach is realized in the trial-and-error method where limitations in the form of inequalities are imposed on z.

Along with this, it is possible to use additional information of a qualitative nature for the selection of a stable approximation with respect to a solution of the inverse problem from among the elements satisfying the inequality (2). A particular example of this is information about the smoothness of the solution sought for (if z is a function of a real variable). Furthermore, the resultant approximation z_δ satisfies the following principle: If $\hat{z}$ is a model for which $A\hat{z} = \hat{u}$ and $\|\hat{u} - \tilde{u}\| \le \delta$, then $z_{\hat{\delta}} \to \hat{z}$ when $\delta \to 0$. Approximations to a solution of the inverse problem which satisfy this principle are called regularized approximations and algorithms for their construction are called regularizing algorithms [2, 3].

We shall not discuss regularizing algorithms for the solution of incorrectly formulated problems in greater detail, since they will be the subject of separate papers. We point out that z_δ is a solution of a certain supplementary mathematical problem which takes into account additional information about the desired solution.

Together with the specified principle, there are other possible principles for the selection of an approximation from the family of equivalences corresponding to the inequality (2) which are based on the statistical characteristics of the model.

3. Inverse problems, particularly in the physics of heat and in heat engineering, can be classified with respect to level of complexity in the following manner:

a) The Development of a System for the Analysis of Experimental Results. This problem is understood in a broad sense; the idea of "analysis" includes both primary analysis, for example, statistical analysis of data, as well as interpretation of the data from the viewpoint of constructing a model of the object under study. A system of analysis for which the input information is just experimental data was developed recently at Moscow University [4]. The incorporation of such systems solves the problem of increasing productivity.

b) The Problem of System and Instrument Construction. In problems of this kind, supplementary a priori information may include requirements associated with the opportunities for technical realization. In this case, $\tilde{u} = \hat{u}$ in the formulation of the problem (2), where $\hat{u}$ is a desired effect of the model sought for.

An exact solution of a problem such as (1) (where $u = \hat{u}$) may also be lacking in this case. There are examples of solutions of such problems by means of regularizing algorithms [5].

In these problems one can use "quasi-inversion" algorithms [6] together with regularizing algorithms. An appropriate algorithm does not assume convergence of the selected approximation z_δ to any element of the family of equivalences, which may be a redundant requirement in the sense of the problem. However, the formulation of the problem given by the authors of the quasi-inversion method does not also take into account conditions for the practical realization of the selected model, which may be important for the solution of an inverse problem of this kind.

c) The Problem of Recognition of the Characteristics of a Process Controlling Observed Phenomena. In the physics of heat, one can place in this class both problems involving the determination of boundary temperatures or thermal modes and problems involving the determination of the thermal characteristics of materials in operating systems. We consider below some examples of previously solved problems of this kind.

4. A typical problem of the type c) is the problem concerning the climatic history of the planet, the mathematical formulation of which has been given [7] along with a proof of the uniqueness of the solution. Its subject is a study of the temperature variation on the surface of the earth in previous times from measured data at a given time for the temperature at various depths within the earth. In the same class of problems is the reconstruction of the boundary temperature conditions for a heated body from temperature measurements at some internal point. A regularizing algorithm is used for the solution of such a problem [8] within the confines of a one-dimensional model with constant thermal characteristics. If one includes as supplementary information the requirement for maximum smoothness of the approximation selected from the family of equivalences according to (2), it turns out that the boundary temperature is reproduced independently of the mathematical character of the regime with the accuracy of reproduction being comparable to the accuracy of measurement.

The same paper also showed the inefficiency of the extrapolation method for the solution of this problem which, even for "exact" input data, leads to a systematic error of up to 50% (for the models discussed in the paper).

Problems similar to this one were discussed in a number of subsequent papers which are partially represented in the program of this seminar [9, 10]. In particular, using a similar formulation [10] and information about the smoothness of the solution, associates of Moscow University in collaboration with a scientific-industrial organization solved a problem involving the reproduction of the thermal flux at the surface of a body from the same data. It was solved for a more complex model of a system including a given nonlinear temperature dependence of the thermal characteristics of the body (k, c, ρ); one of the subsequent reports will discuss the problem in greater detail.

In the problems mentioned above, one was talking about the determination (with the help of regularizing algorithms) of some controlling function. Thermal parameters of a material were assumed known (constant or a given function of temperature), determined, for example, by a laboratory study of the thermal conductivity of the material.

However, the method of laboratory study of the parameters mentioned is not always acceptable and not necessarily optimal. On the other hand, the use of regularizing algorithms makes it possible to determine these thermal parameters efficiently from a dynamic experiment (on the basis of data similar to that given above) so that the results of theory and experiment are in good agreement [inequality (2)]. Such a problem was solved by us in a related field [11].

The study of the model of a phenomenon can be carried out systematically in two stages on the whole. First, thermal parameters characterizing the model are determined in a dynamic experiment. In this case, the data from actual experiments can be used as "initial approximations" to the parameter values. After the thermal characteristics have been determined, one can solve the problem involving a broader study of the model, particularly of control by the process.

As an illustration of a heat-engineering problem of this kind, we consider the ventilation problem [12].

Air which gives up heat to the walls is blown through a duct. In this case, it is reasonable to consider only the normal thermal conductivity of the wall, neglecting its thermal conductivity along the duct, and, on the other hand, to consider only longitudinal thermal conductivity for the air flow. By considering the development of the process in the initial stage, one can solve the problem in linear approximation. Under these assumptions, the equation of thermal balance in the duct has the form

$$\nu S c_0 \rho_0 \frac{\partial u}{\partial x} - Q = S c_0 \rho_0 \frac{\partial u}{\partial t}, \tag{3}$$

where u = u(x, t) is the temperature in the duct (x is the coordinate along the thermal flux); S is the cross-sectional area of the duct; c_0 and ρ_0 are the thermal constants of the air; Q is the amount of heat released; ν is the velocity of air flow.

Let v = v(x, z, t) be the temperature of the wall. We assume that this temperature at the internal boundary of the wall is equal to the temperature of the air in the duct, v(x, 0, t) = u(x, t); the initial temperature of the wall is stationary and we can assume it to be zero, taking readings from the appropriate value. In turn, we assume the air temperature at the beginning of the duct is given, u(0, t) = μ(t) (as is always done in the formalization of a model).

Neglecting the right side in the equation of balance, which is also reasonable from the sense of the problem (otherwise, one could arrive at the same result by a change in the time reference point), we finally arrive at the following system of equations:

$$\begin{gathered}
\nu S c_0 \rho_0 \frac{\partial u}{\partial x} - Q = 0, \\
Q = -kL \frac{\partial v}{\partial z}(x, 0, t), \\
\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial z^2}, \\
v(x, t) = v(x, 0, t), \\
u(0, t) = \mu(t), \\
v(x, z, 0) = 0,
\end{gathered} \tag{4}$$

where L is the perimeter of the duct; k and a^2 are thermal constants of the wall.

It is then easy to show that the function u(x, t) satisfies the conditions of the well-known problem for the equation of thermal conductivity,

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \quad u(0, t) = \mu(t), \quad u(x, 0) = 0, \tag{5}$$

where c and ρ are the corresponding thermal characteristics of the wall, $\alpha^2 = (L/\nu S)^2 a^2 (c\rho/c_0\rho_0)^2$ with the coefficient α being determined by a similarity criterion.

Thus we have arrived at a known problem of thermal conductivity which establishes within the confines of the selected model the dependence between the observed [u(x, t)] and unknown [α^2, μ(t)] quantities.

In the plan for the solution of the inverse problem, which is of interest to us, it is convenient to determine the set of thermal constants by analysis of the results of a dynamic experiment (for example, from the temperature in some cross section $x = x_0$). In turn, knowledge of the specified constants makes it possible to solve the problem of optimal control in respect to the determination of: 1) the rate of ventilation necessary to reduce the temperature of the wall to a given value and 2) the minimum expenditure of power for the amount of air blown through to achieve this goal.

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