DEFORMATION ENERGY FUNCTION OF LARGE HUMAN BLOOD VESSELS

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A function of the specific energy of deformation, selected in the form of a number of exponents, is proposed. It describes well the stress–strain state of anisotropic human blood vessel at large deformations. The constants of the material included in the deformation energy function are determined by experiments for a monoaxial tensioning, along the main anisotropy axes. As an example, they were found for the human abdominal aorta, taken during an autopsy (male, age 29 years), by approximation of the experimental data on a computer by the method of least squares.

1. At present, an increasing number of investigations deal with a study of the mechanical properties of walls of human blood vessels [1-8], which play an important role in blood circulation processes [9-12]. It was found that these properties change with age [5-8] and involve definite changes in the blood circulation system, for example, change in the diffusion processes through the wall of the vessel, and decrease in the blood supply to the wall itself [13]. A knowledge of the mechanical properties of the vessels and the distribution of stresses and deformations in the vessel wall over a wide age group, will make it possible to find the characteristic features of the changes taking place in the blood circulation process and to study the reasons for the most prevalent disease of the blood circulation organs, atherosclerosis.

Various models have been proposed [13-17] and different theories used [18-27] to describe both the static and the dynamic properties of arterial vessels. In [18, 19] the Laplace law was used to calculate the vessels under a constant internal and external pressure. In [20] an attempt was made to apply the small deformations theory to finite deformations of the blood vessel under an internal pressure. In [21-23] the classic elasticity theory was used, and it was assumed that the vessel wall is isotropic. However, in [21, 22] it was found that under pressure the arteries can change their radius by 200% with respect to their initial value, and during each cardiac cycle this change has the value of 5-14% [28, 29]. Therefore, an accurate description of the behavior of the blood vessel can only be made on the basis of the finite deformations theory [30]. In most studies using this theory, for the sake of simplicity, the vessel wall was assumed to be isotropic [13, 24-27].

In [1] the arterial walls were considered to be anisotropic, and their properties were characterized by six elastic constants. The deformation energy function was expanded into an exponential series in terms of invariants, and the unknown coefficients were determined by means of a second order curve, plotted from the experimental data. The deformation energy and the energy of the change in volume cannot be derived from the proposed expression for the energy function. In [31] it was found that the blood vessels of a dog are anisotropic in the undeformed state. Partial derivatives of the deformation energy have been determined. But from these derivatives it is impossible to obtain a complete expression for the deformation energy function.

In the present work, a method is proposed for the calculation of an orthotropic blood vessel from a selected deformation energy function. The unknown coefficients of the specific deformation energy function were determined on a computer by approximation of the experimental curves obtained when specimens of a wall of an aorta were tensioned along the main axes (Fig. 1).

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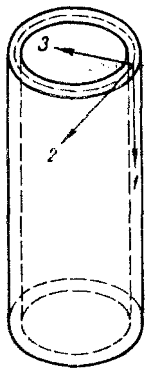

Fig. 1. Direction of the main anisotropy axes in a human blood vessel.

2. The position of a certain point P_x of a nonlinearly elastic orthotropic material in an undeformed state in a fixed orthogonal coordinate system $i = 1, 2, 3$ is determined by three coordinates x_i. After the deformation, the point is in a new position P_y with respect to this coordinate system, which is characterized by three other derivatives designated as $y_i = y_i(x_1, x_2, x_3, t)$.

We introduce a general curvilinear coordinates system θ_i in such a way that $x_i = x_i(\theta_1, \theta_2, \theta_3)$, where $x_i(\theta_1, \theta_2, \theta_3)$ is a single-valued function with continuous derivatives of all orders. The deformed state is determined by the dependence $y_i = y_i(\theta_1, \theta_2, \theta_3, t)$, where $y_i(\theta_1, \theta_2, \theta_3, t)$ is a single-valued function with continuous derivatives of any order, with respect to both the coordinates θ_i and the time t.

According to [30], the large strain tensor in the curvilinear system of coordinates is determined by the dependence

$$\gamma_{ij} = \frac{1}{2}(G_{ij} - g_{ij}),$$

where $G_{ij} = (\partial y^m/\partial\theta^i)\,\partial y^m/\partial\theta^j$ and $g_{ij} = (\partial x^m/\partial\theta^i)\,\partial x^m/\partial\theta^j$ are covariant metric tensors of a deformed and original body, respectively; $m = 1, 2, 3$. When the curvilinear coordinates are selected in such a way that they coincide with orthogonal coordinates which determine the position of the point in the original body, the large strain tensor is designated by Green as e_{ij} and is expressed in the following form:

$$e_{ij} = \frac{\partial\theta^r}{\partial x^i}\frac{\partial\theta^s}{\partial x^j}\gamma_{rs} = \frac{1}{2}\left(\frac{\partial y^m}{\partial x^i}\frac{\partial y^m}{\partial x^j} - \delta_{ij}\right). \tag{1}$$

The strain tensor can also be determined by the displacement components with respect to the axes i, j = 1, 2, 3 in an undeformed material (the Lagrange formula):

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right).$$

The invariants of the strain tensor are expressed in metric tensors, components of deformations tensor, or the degree of the main elongation λ_i:

$$I_1 = g^{ij}G_{ij} = 3 + 2e_{ii} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2;$$

$$I_2 = g_{ij}G^{ij}I_3 = 3 + 4e_{ii} = 2(e_{ii}e_{jj} - e_{ij}e_{ij}) = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2;$$

$$I_3 = \frac{\det[G_{ij}]}{\det[g_{ij}]} = \det[\delta_{ij} + 2e_{ij}] = \lambda_1^2\lambda_2^2\lambda_3^2;$$

where $G^{ij} = (\partial\theta^i/\partial y^m)\,\partial\theta^j/\partial y^m = D_G^{ij}/\det[G_{ij}]$, and $g^{ij} = (\partial\theta^i/\partial x^m)\,\partial\theta^j/\partial x^m = D_g^{ij}/\det[g_{ij}]$ are contravariant metric tensors of the deformed and original body, respectively; D_G^{ij} and D_g^{ij} are the algebraic supplements of the elements G_{ij} and g_{ij} in the determinants $\det[G_{ij}]$ and $\det[g_{ij}]$, respectively; δ_{ij} is a unit tensor.

The stressed state at a certain point of the body, with the deformation energy function W, related to unit volume of an undeformed material, is characterized in a curvilinear system of coordinates by a symmetric contravariant stress tensor, determined per unit area of the deformed body:

$$\tilde{\sigma}^{kl} = \frac{1}{2\sqrt{I_3}}\left(\frac{\partial W}{\partial e_{ij}} + \frac{\partial W}{\partial e_{ji}}\right)\frac{\partial\theta^k}{\partial x^i}\frac{\partial\theta^l}{\partial x^j}. \tag{2}$$

For a noncompressed material, where $I_3 = 1$,

$$\tilde{\sigma}^{kl} = \frac{1}{2}\left(\frac{\partial W}{\partial e_{ij}} + \frac{\partial W}{\partial e_{ji}}\right)\frac{\partial\theta^k}{\partial x^i}\frac{\partial\theta^l}{\partial x^j} + pG^{kl} \qquad (i, j, k, l = 1, 2, 3),$$

where p is a scalar function of coordinates θ.

The "physical" components of the stresses related to a fixed orthogonal system of coordinates in a deformed body are determined by the interrelation $\sigma_{rs} = (\partial y^r/\partial\theta^k)(\partial y^s/\partial\theta^l)\,\tilde{\sigma}^{kl}$, while the stress tensor, measured per unit area of the undeformed material, is expressed in the form

$$\sigma_0^{kl} = \sqrt{I_3}\,\tilde{\sigma}^{kl}. \tag{3}$$

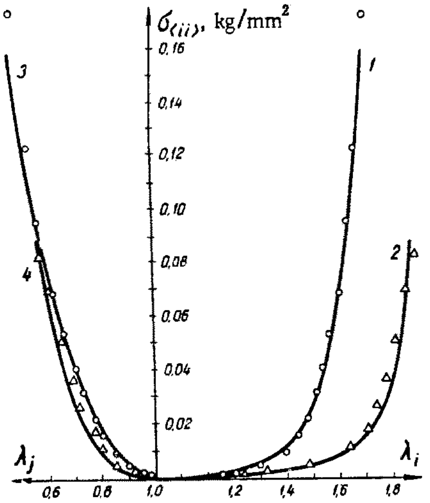

Fig. 2. Interrelations $\sigma_{\langle ii \rangle} - \lambda_i$ and $\sigma_{\langle ii \rangle} - \lambda_j$ (i, j = 1, 2; i ≠ j) for a human abdominal aorta (male, 29 years): 1) $\sigma_{11} - \lambda_1$; 2) $\sigma_{22} - \lambda_2$; 3) $\sigma_{11} - \lambda_2$; 4) $\sigma_{22} - \lambda_1$.

For a compressed orthotropic material, the deformation energy function is a function of seven values [30]: e_{11}, e_{22}, e_{33}, e_{12}^2, e_{23}^2, e_{31}^2, and $e_{11}e_{22}e_{33}$. Since the last term can be represented by means of I_3, in general form we obtain:

$$W = W(e_{11}, e_{22}, e_{33}, e_{12}^2, e_{23}^2, e_{31}^2, I_3). \tag{4}$$

From (2), (3), and (4) the stress tensor becomes equal to

$$\sigma_0^{kl} = \frac{\partial W}{\partial e_{11}} \frac{\partial \theta^k}{\partial x^1} \frac{\partial \theta^l}{\partial x^1} + \frac{\partial W}{\partial e_{22}} \frac{\partial \theta^k}{\partial x^2} \frac{\partial \theta^l}{\partial x^2} + \frac{\partial W}{\partial e_{33}} \frac{\partial \theta^k}{\partial x^3} \frac{\partial \theta^l}{\partial x^3} + \\ + e_{12} \frac{\partial W}{\partial (e_{12}^2)} \left(\frac{\partial \theta^k}{\partial x^1} \frac{\partial \theta^l}{\partial x^2} + \frac{\partial \theta^k}{\partial x^2} \frac{\partial \theta^l}{\partial x^1} \right) + e_{23} \frac{\partial W}{\partial (e_{23}^2)} \left(\frac{\partial \theta^k}{\partial x^2} \frac{\partial \theta^l}{\partial x^3} + \\ + \frac{\partial \theta^k}{\partial x^3} \frac{\partial \theta^l}{\partial x^2} \right) + e_{31} \frac{\partial W}{\partial (e_{31}^2)} \left(\frac{\partial \theta^k}{\partial x^3} \frac{\partial \theta^l}{\partial x^1} + \frac{\partial \theta^k}{\partial x^1} \frac{\partial \theta^l}{\partial x^3} \right) + 2I_3 \frac{\partial W}{\partial I_3} G^{kl}. \tag{5}$$

3. To find the correlation between the stresses and the elongation at a monoaxial tensioning of the wall of a sample of a human blood vessel, we shall consider the problem of an orthotropic parallelepipedon, subjected to a uniform deformation [30]: $y_i = C_{ij}x_j$, while the coefficients C_{ij} are constant. From formula (1) it follows that $e_{ij} = \frac{1}{2}(C_{mi}C_{mj} - \delta_{ij})$.

In the particular case when the uniform deformation consists in a regular tensioning, at which the degrees of principal elongations in the direction of axes i are equal to λ_i, we have:

$$C_{ii} = \lambda_i; \quad C_{ij} = 0 \quad (i \neq j); \\ e_{ii} = \frac{1}{2}(\lambda_i^2 - 1); \quad e_{ij} = 0 \quad (i \neq j). \tag{6}$$

The stress components are determined from (5), if we assume that the reference system θ_i coincides with a perpendicular Cartesian system of coordinates x_i. Then, these stresses are physical components of stress σ_{ij}^0 relative to the system under consideration. For compressible materials, if we take $G_{ij} = G^{ij} = \delta_{ij}$, the stress is

$$\sigma_{ij}^0 = \lambda_i \lambda_j \frac{1}{2} \left(\frac{\partial W}{\partial e_{ij}} + \frac{\partial W}{\partial e_{ji}} \right) + 2I_3 \frac{\partial W}{\partial I_3} \delta_{ij},$$

and the stress per unit volume of the deformed body

$$\sigma_{ij} = \frac{1}{\sqrt{I_3}} \lambda_i \lambda_j \frac{1}{2} \left(\frac{\partial W}{\partial e_{ij}} + \frac{\partial W}{\partial e_{ji}} \right) + 2\sqrt{I_3} \frac{\partial W}{\partial I_3} \delta_{ij}. \tag{7}$$

For a noncompressed material, if we assume that $I_3 = 1$, we obtain

$$\sigma_{ij} = \lambda_i \lambda_j \frac{1}{2} \left(\frac{\partial W}{\partial e_{ij}} + \frac{\partial W}{\partial e_{ji}} \right) + p\delta_{ij}. \tag{8}$$

From the Green strain tensor (6), we find the relationship between $de_{\langle ii \rangle}$ and $d\lambda_i$: $de_{ii} = \lambda_i d\lambda_i$, and hence expressions (7) and (8) for a monoaxial tensioning can be written in the form:

$$\sigma_{ii} = \frac{1}{\lambda_j \lambda_k} \frac{\partial W}{\partial \lambda_i} + 2\sqrt{I_3} \frac{\partial W}{\partial I_3}; \tag{9}$$

$$\sigma_{ii} = \lambda_i \frac{\partial W}{\partial \lambda_i} + p \qquad (i, j, k = 1, 2, 3;\ i \neq j \neq k). \tag{10}$$

The indices included between the angular brackets are mononomials with fixed values.

If it is assumed that the wall of the vessel is incompressible, the stress arising in samples cut from a blood vessel along the main axes can be determined at a monoaxial tensioning from formula (10). In this case we shall determine the hydrostatic pressure p during tensioning along axis 1 from the condition $\sigma_{22} = \sigma_{33} = 0$. The solution of the last two equations (10) under conditions of a monoaxial tensioning gives an expression for the hydrostatic pressure:

$$p=-\frac{1}{2}\left[\lambda_2\frac{\partial W}{\partial\lambda_2}+\frac{1}{\lambda_1\lambda_2}\left(\frac{\partial W}{\partial\lambda_1}\frac{\partial\lambda_1}{\partial\lambda_3}+\frac{\partial W}{\partial\lambda_2}\frac{\partial\lambda_2}{\partial\lambda_3}\right)\right]. \tag{11}$$

Thus, the hydrostatic pressure is found by tensioning the sample along axis 2.

4. We shall determine the constants of the material included in the deformation energy function of a human blood vessel. The vessel is considered to be a thin-walled cylindrical tube made of an orthotropic uncompressible uniform material. The assumption of a small thickness of the wall means that the stresses $\sigma_{33}=0$. In an axisymmetrical blood vessel, the axial tensioning and the internal pressure cannot lead to shear deformations [1]. Hence, the deformation energy is a function of three degrees of principal elongations [31], and expression (4) can be written as

$$W=W(\lambda_1,\lambda_2,\lambda_3). \tag{12}$$

In general, W is a sum of the deformation energy and the energy of change in volume [32]. It was found that the deformation energy function for blood vessels is exponential [33-35]. On this basis we select an expression for W of the compressible material in the general form

$$W=A(e^Q-1)+E(e^R-1)+G[(I_3-1)e^{H(I_3-1)}-I_3]+G,$$

where $Q=B(\lambda_1-\lambda_2)^2+C(\lambda_2-\lambda_3)^2+D(\lambda_3-\lambda_1)^2$; $R=F(\lambda_1-\lambda_2)^2+K(\lambda_2-\lambda_3)^2+L(\lambda_3-\lambda_1)^2$; A, B, C, D, E, F, K, L, G, H are constants of the material, determined experimentally.

For compressible material, if we take $I_3=1$ and $\lambda_3=1/\lambda_1\lambda_2$, we obtain the specific deformation energy function in the form:

$$W=A(e^Q-1)+E(e^R-1), \tag{13}$$

where $Q=B(\lambda_1-\lambda_2)^2+C\left(\lambda_2-\frac{1}{\lambda_1\lambda_2}\right)^2+D\left(\frac{1}{\lambda_1\lambda_2}-\lambda_1\right)^2$; $R=F(\lambda_1-\lambda_2)^2+K\left(\lambda_2-\frac{1}{\lambda_1\lambda_2}\right)^2+L\left(\frac{1}{\lambda_1\lambda_2}-\lambda_1\right)^2$.

As an example, the constants A, B, C, D, E, F, K and L were determined for a human abdominal aorta of age group A (25-30 years) [8]. The samples were cut in the direction of the main axes 1 and 2. From (10), (11), and (13), we obtained expressions for the stresses arising in a strip of an aorta wall. The stress arising in elongated sample along axis 1 is determined from

$$\sigma_{11}=3A\left[B(\lambda_1^2-\lambda_1\lambda_2)+C\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_1^2\lambda_2^2}\right)+D\left(\lambda_1^2-\frac{1}{\lambda_1^2\lambda_2^2}\right)\right]e^Q+$$
$$+3E\left[F(\lambda_1^2-\lambda_1\lambda_2)+K\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_1^2\lambda_2^2}\right)+L\left(\lambda_1^2-\frac{1}{\lambda_1^2\lambda_2^2}\right)\right]e^R. \tag{14}$$

Here λ_1, λ_2 are the degrees of longitudinal and lateral elongations of the sample in the direction of axes 1 and 2, respectively: $\lambda_1=l/l_0>1$; $\lambda_2=b/b_0<1$; l_0, b_0 are the length and the width of the sample before loading. The stress arising in a tensioned sample in the direction of axis 2 is determined from

$$\sigma_{22}=3A\left[B(\lambda_2^2-\lambda_2\lambda_1)+C\left(\lambda_2^2-\frac{1}{\lambda_2^2\lambda_1^2}\right)+D\left(\frac{1}{\lambda_2}-\frac{1}{\lambda_2^2\lambda_1^2}\right)\right]e^Q+$$
$$+3E\left[F(\lambda_2^2-\lambda_2\lambda_1)+K\left(\lambda_2^2-\frac{1}{\lambda_2^2\lambda_1^2}\right)+D\left(\frac{1}{\lambda_2}-\frac{1}{\lambda_2^2\lambda_1^2}\right)\right]e^R, \tag{15}$$

where λ_1, λ_2 are the degrees of lateral and longitudinal elongations in the direction of axes 1 and 2, respectively.

The constants of the material included in the deformation energy function were determined from Eqs. (14) and (15) by a combined approximation of the experimental data by the method of least squares. Fig. 2 shows the experimental points obtained on samples of an abdominal aorta of a human male who died at the age of 29 years, and the stress–elongation curves, calculated from Eqs. (14) and (15) at the optimum values of the coefficients of the specific energy of deformation found: A = 0.000123; B = 0.160929; C = −0.5; D = 0.002183; E = 0.001798; F = 1.485844; K = −0.651317; L = 0.328281. The determination of the constants of the material from the experimental data on monoaxial elongation makes it possible to solve the problem of the distribution of stresses and deformations in the wall of a blood vessel subjected to an internal pressure and axial tensioning, taking into account the orthotropy of the material.

The increment of stresses at any level of Cauchy deformation for nonlinearly elastic orthotropic material in the direction of the main anisotropy axes is determined from the interrelation

$$d\sigma_{ii}=A_{iijj}d\varepsilon_{jj} \qquad (i, j=1, 2, 3) \tag{16}$$

However, the increment of stress $\sigma_{\langle ii \rangle}=f(\lambda_1, \lambda_2, \lambda_3)$ can be expressed in the form:

$$d\sigma_{ii}=\lambda_i\left(\frac{\partial\sigma_{ii}}{\partial\lambda_i}\right)\frac{\partial\lambda_i}{\lambda_i}+\lambda_j\left(\frac{\partial\sigma_{ii}}{\partial\lambda_j}\right)\frac{\partial\lambda_j}{\lambda_j}+\lambda_k\left(\frac{\partial\sigma_{ii}}{\partial\lambda_k}\right)\frac{\partial\lambda_k}{\lambda_k} \tag{17}$$

$$(i, j, k=1, 2, 3,\ i\neq j\neq k),$$

where $d\lambda_1/\lambda_2$ represents an increment of Cauchy deformation $d\varepsilon_{\langle ii \rangle}$. It follows from (16), (17) that the stiffness components of the material are determined by the expression

$$A_{iijj}=\lambda_j\frac{\partial\sigma_{ii}}{\partial\lambda_j}$$

$$(i, j=1, 2, 3).$$

Hence, the stiffness characteristics of the material are found by differentiating expressions for stresses represented in terms of degrees of principal elongations λ_i, and the constants of the material included in the deformation energy function.

LITERATURE CITED

1. E. G. Tickner and A. H. Sacks, Biorheology, 4, 151 (1967).
2. J. Greenfield and D. Patel, Circul. Res., 10, 778 (1962).
3. E. M. Krokosky and T. A. Krouskop, J. Biomed. Mater. Res., 4, No. 4, 525 (1970).
4. E. R. Gozna, A. E. Marble, A. J. Shaw, and D. A. Winter, Cardiovasc. Res., 7, No. 2, 261 (1973).
5. M. R. Roach and A. C. Burton, Can. J. Biochem. Physiol., 37, 557 (1959).
6. H. Bader, Circul. Res., 20, 354 (1967).
7. B. M. Learoyd and M. G. Taylor, Circul. Res., 18, 278 (1966).
8. B. A. Purinya, G. L. Vilka, Yu. K. Vilks, V. A. Kas'yanov, and É. E. Tseders, Mekhan. Polim., No. 1, 129 (1974).
9. S. A. Regirer, in: Hydrodynamics of Blood Circulation [in Russian], Moscow (1971), p. 242.
10. Yu. N. Pavlovskii, S. A. Regirer, and I. M. Skobeleva, in: Science Progress [in Russian], Moscow (1970), p. 7.
11. V. M. Khayutin, Vessel-Motoric Reflexes [in Russian], Moscow (1964).
12. W. E. Bogley, Phys. Med. Biol., 16, No. 4, 663 (1971).
13. B. R. Simon, A. S. Kobayashi, D. E. Strandess, and C. A. Wiederhielm, Trans. ASME Ser. D, 93, No. 2, 25 (1971).
14. Yu. Ya. Kislyakov, Biofizika, 16, 1093 (1971).
15. A. S. Vol'mir and M. S. Gershtein, Izv. Akad. Nauk Armen. SSR, Mekhanika, 19, No. 1, 8 (1966).
16. A. S. Vol'mir and M. S. Gershtein, Mekhan. Polim., No. 2, 373 (1970).
17. E. M. Krokosky and T. A. Krouskop, J. Biomed. Mater. Res., 4, No. 4, 503 (1970).
18. T. Azuma and S. Oka, Amer. J. Physiol., 221, No. 5, 1310 (1971).
19. S. Oka and T. Azuma, Biorheology, 7, 109 (1970).
20. J. Hildebrandt, Angiologica, 7, 257 (1970).
21. D. H. Bergel, J. Physiol., 156, 445 (1961).
22. P. B. Dobrin and A. A. Rovick, Amer. J. Physiol., 217, 1644 (1969).
23. D. J. Patel, F. M. Defreitas, J. C. Greenfield, and D. L. Fry, J. Appl. Physiol., 18, 1111 (1963).
24. I. Opatowski, J. Appl. Physiol., 23, No. 5, 772 (1967).
25. J. Hildebrandt, H. Fukaya, and C. Martin, Biophys. J., 9, 781 (1969).
26. J. D. C. Crisp, Microvasc. Res., 1, 35 (1968).
27. W. H. Hoppmann and L. Wan, J. Biomechanics, 3, 593 (1970).
28. D. J. Patel, A. J. Mallos, and D. L. Fry, J. Appl. Physiol., 16, 293 (1961).
29. J. O. Arndt, J. Klauske, and F. Mersche, Pflulgers Arch., 301, 230 (1968).
30. A. Green and J. Adkins, Large Elastic Deformations, Oxford Univ. Press (1971).
31. J. M. Doyle and P. B. Dobrin, Microvasc. Res., 3, No. 4, 400 (1971).
32. M. Shinozuka and A. M. Freudenthal, J. Appl. Polym. Sci., 9, 2565 (1965).

33. Y. C. Fung, Amer. J. Physiol., 213, No. 6, 1532 (1967).
34. Y. C. Fung, Appl. Mech. Rev., 21, No. 1, 1 (1968).
35. P. F. Gou, J. Biomechanics, 3, 547 (1970).