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CONSTITUTIVE EQUATIONS IN NONLINEAR DAMAGE MECHANICS

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The classical method of solving problems concerning the delayed fracture of structural materials and elements is based on analysis of their stabilized stress-strain state and the use of parametric relations which include the time or number of cycles to failure. Over the last twenty years, researchers have developed another approach employing the concept of internal variables [1, 2, 22] along with a system of evolutionary equations and failure criteria. Entropy, internal energy, empirical temperature, internal time, strain-hardening, and damage are the quantities most often chosen as the internal variables. In the present study, we examine methods of constructing adequate evolutionary equations for a damage parameter that cannot be measured directly.

1. Introduction. From a phenomenological standpoint, the delayed fracture of structural materials and elements is regarded as a process involving the nucleation and growth of various types of discontinuities. "Crack mechanics" ("fracture mechanics") and "continuous damage mechanics" are used to quantitatively evaluate such processes.

Fracture mechanics was formulated on the basis of the works of A. A. Griffith [30] and J. R. Irwin [32] and is based on analysis of the conditions of growth of one or several sharp cracks in a loaded body. Fracture mechanics has since been theoretically validated and is widely used in engineering practice.

Continuous damage mechanics was constructed on the basis of the work of L. M. Kachanov [18] and is based on analysis of the conditions of evolution of a set of microscopic defects that are continuously distributed over the entire volume of a loaded body. Continuous damage mechanics offers at least as much promise as fracture mechanics, and perhaps more. For example, only the former can be used to analytically evaluate the moment of nucleation of a macrocrack (crack) [2, 18]. In many published studies — especially abroad — damage mechanics has been referred to as "continuum fracture mechanics."

In constructing the initial relations of damage mechanics, Kachanov [18] essentially took the concept of "effective stress" proposed by Hoff [31] to model viscous fracture under creep conditions and applied it to the region of brittle fracture. He hypothesized that cavities develop over time within a certain macroscopic volume of a material. The growth of these cavities decreases the effective cross section of the material while its geometry remains unchanged. To describe this phenomenon, Kachanov introduced the damage parameter ω. Thus, in the general case of a complex stress state, we can write

$$(\Delta S_j)_{ef} = (\delta_{ij} - \omega_{ij}) \Delta S_j, \tag{1.1}$$

where ΔS_j is an initial oriented elementary area; $(\Delta S_j)_{ef}$ is the true oriented area; δ_{ij} is the Kronecker symbol.

The second-rank tensor ω_{ij} determined by Eq. (1.1) is referred to as the damage tensor. In the given case, the components of the tensor of the effective stresses (σ_{ij}) are specified by means of the relation

$$(\sigma_{ij})_{ef} = \frac{\sigma_{ik}}{\sigma_{kj} - \omega_{kj}}, \tag{1.2}$$

where σ_{ik} is the tensor of the initial stresses.

Within the framework of the damage mechanics approach, the solution of the problem of the delayed fracture of a body under a load reduces to establishing the relationship between the components of the stress tensor σ_{ij}, strain tensor ε_{ij} (or strain-rate tensor $\dot{\varepsilon}_{ij}$), and damage tensor ω_{ij}. This basically amounts to closing the classical system of resolvent mechanics equations with the appropriate evolutionary damage equation so that

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$$\frac{\partial}{\partial x_j}\sigma_{ij} + X_j = 0; \qquad \sigma_{ij}\gamma_j = P_{i\nu} \quad (ij = 1,2,3); \tag{1.3}$$
$$(\varepsilon_{ik,jl} + \varepsilon_{lj,ik} - (\varepsilon_{kj,il} + \varepsilon_{il,jk})) = 0;$$
$$F(\sigma_{ij}, \varepsilon_{ij}, \varepsilon_{ij}, q_c) = 0;$$
$$d\omega_{ij} = \Phi(\sigma_{ij}, t, \omega_{ij})\,dt.$$

Here, P_i is the vector of force density on the surface of the body; X_j are the body forces; ν_j is a unit normal; q_c is the set of material constants ($c = 1, 2, \ldots, n$).

Two basic problems generally arise in the course of solving system (1.3). The first pertains to the relationship between the initial resolvent equations and the damage parameter. One way of resolving this issue is to use physical equations $F(\cdot) = 0$ for effective stress values $(\sigma_{ij})_{ef}$ in accordance with (1.2). In this case in particular, the damage to the material might have an effect on the material constants and we might have (for example) the following for the running values of the elastic constants (C_{ij})

$$(C_{ij})_{ef} = (\delta_{ij} - \omega_{ij})C_{ij}, \tag{1.4}$$

where C_{ij} are values of the elastic constants for the undamaged material.

The other problem is related to identification of the damage parameter itself and construction of the evolutionary equation in (1.3) on this basis. The difficulties encountered in this case [2, 7, 23, 24, 33] suggest that damage is best interpreted as an "internal variable" of the process. Neither direct nor indirect measurements are very useful here. The main problem is the lack of clarity as to the macroscopic significance of the damage parameter and, thus, the correlation between measurements made by different methods. The damage parameter is an integral characteristic and thus cannot be expected to contain all of the necessary information on damage even if precisely measured.

A more promising approach is to construct the evolutionary equations for the damage function on the basis of empirically substantiated hypotheses and specify a system of control experiments to check the adequacy of the relations that are obtained. However, nearly all of the evolutionary equations constructed to date and the problems solved with them [2-7, 10, 11, 15-25, 33] are based on the hypothesis of linear damage summation — which is valid only in certain cases. The goal of the present study is to discuss the principles behind the construction of nonlinear damage mechanics and the main regions of its application.

2. Linearity and Nonlinearity in Damage Mechanics. Two definitions of damage have been formulated thus far in fracture mechanics [7, 26]. One approach introduces the notion of "partial times," while in the other approach (already discussed above) damage is regarded as an internal variable. Several differences in the treatment of linearity and nonlinearity in the damage process follow from the existence of the two definitions.

2.1. Identification of Damage. Proceeding on the basis of the identification of damage with "partial time," we can write the following for the increment of the surface

$$\Delta\omega = \frac{\Delta t_i}{t_R(\sigma_i)} \rightarrow d\omega = \frac{dt}{t_R(\sigma)}, \tag{2.1}$$

where Δt_i is the time of action of the i-th load; $t_R(\sigma_i)$ is the time to failure under the i-th load.

Assigning damage as an internal variable involves the formulation of suitable evolutionary equations for this quantity. These equations can be generalized by means of a differential equation of the form

$$d\omega = f_t(\sigma(t)\omega(t), C_i)\,dt \tag{2.2}$$

with the initial condition and fracture condition in the form

$$\omega(t)\big|_{t=0} = 0 \text{ and } \omega(t)\big|_{t=t_R} = 1, \tag{2.3}$$

where C_i is a set of coefficients which can be determined empirically; t_R is the time to rupture. All of the solutions obtained henceforth are for the unidimensional case, since they are also the initial equations in the construction of the constitutive relations for complex stress states.

The function f_t in (2.2) is usually specified using a power stress function

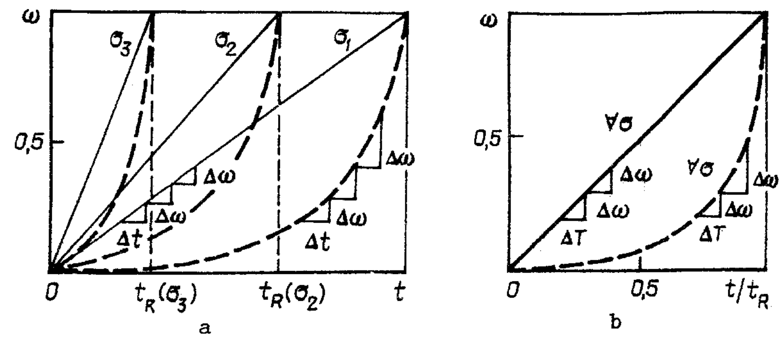

Fig. 1

$$\frac{d\omega}{dt} = D\sigma^k, \tag{2.4}$$

the Kachanov equation [18]

$$\frac{d\omega}{dt} = D\left(\frac{\sigma}{1-\omega}\right)^k, \tag{2.5}$$

the Rabotnov equation [22]

$$\frac{d\omega}{dt} = D\left(\frac{\sigma}{1-\omega}\right)^k \frac{1}{\omega^{-\beta}}, \tag{2.6}$$

the Shesterikov equation [13]

$$\frac{d\omega}{dt} = D\left(\frac{\sigma}{1-\omega^r}\right)^k \tag{2.7}$$

and the Lemaitre equation [33]

$$\frac{d\omega}{dt} = D\left(\frac{\sigma}{1-\omega}\right)^k \frac{1}{(1-\omega)^q}. \tag{2.8}$$

Here, k, r, β, q, and D are empirical coefficients. A graphical interpretation of the damage kinetics as given by Eqs. (2.4-2.8) is shown in Fig. 1 in the scales of physical (a) and corrected (b) time ($\sigma_3 > \sigma_2 > \sigma_1$).

2.2. Determination of Linearity. Linearity as it pertains to damage accumulation traditionally signifies conditions whereby equal amounts of damage are accumulated during equal time intervals and the total time to rupture under variable loading is equal to 1. The condition of linearity is given analytically by Eq. (2.1) and the relation

$$\sum_{i=1}^{n} \frac{\Delta t_i}{t_R(\sigma_i)} = 1 \rightarrow \int_0^{t_R} \neq \frac{dt}{t_R(\sigma)} = 1, \tag{2.9}$$

which is proper and is known as the hypothesis of linear damage summation.

Linearity conditions (2.1) and (2.9) agree fully with evolutionary equation (2.4), which gives the damage kinetics as depicted by the solid lines in Fig. 1. However, it has been established experimentally that the linear hypothesis is invalid for the overwhelming majority of materials and loading regimes [24-27]. The error obtained here may reach two orders or more.

According to evolutionary equations (2.5-2.8), the damage kinetics are nonlinear (dashed lines in Fig. 1) and different amounts of damage are accumulated during equal time intervals. There are no differences between Eqs. (2.5-2.8) and (2.4) from the viewpoint of satisfying condition (2.9), since they lead to a similar result in regard to evaluating endurance for variable loading regimes. In particular, we obtain the following for the damage parameter from (2.5)

$$\omega = 1 - \left(1 - \frac{t}{t_R(\sigma)}\right)^{\frac{1}{1+k}}, \tag{2.10}$$

By analogy with (2.1) and (2.9), we can obtain the following from (2.10)

$$d\omega = \frac{1}{1+k}\left(1 - \frac{t}{t_R(\sigma)}\right)^{-\frac{1}{1+k}} \frac{dt}{t_R(\sigma)} \rightarrow \int_0^{t_R} d\omega = 1. \tag{2.11}$$

The kinetic damage diagrams given by Eqs. (2.6-2.8) will differ from that given by Eq. (2.5) only in the curvature. These diagrams will also conform to linearity condition (2.9).

2.3. Concept of Nonlinearity. We can conclude from the above analysis that nonlinearity as it pertains to damage summation should be taken to mean conditions under which different amounts of damage are accumulated during equal time intervals, with the total damage for nonsteady loading differing from unity. The nonlinearity condition is written analytically in the form

$$d\omega = \frac{1}{t_R(\sigma)}\psi_t(dt) \Rightarrow d\omega = \psi_T\left(\frac{dt}{t_R(\sigma)}\right); \tag{2.12}$$

$$\omega_\Sigma\Big|_{t=t_R} = \sum_{i=1}^{k} \int_0^{t_i} \omega_i(t)\,dt \neq 1,$$

where ψ_t and ψ_T are nonlinear functions.

3. Principal Methods of Assigning Nonlinear Damage Accumulation. All of the known methods of constructing evolutionary equations satisfying nonlinearity condition (2.12) can be placed in one of three main groups: equations with inseparable variables; equations obeying the principle of "separability"; equations conforming to the "similarity" hypothesis.

3.1. Equations with Inseparable Variables. In this approach, Eq. (2.2) is written in such a way that its right side cannot be represented as a product of two functions when one function depends only on stress and the other depends only on damage [as was the case, for example, in (2.5-2.8)]. In this case, initial equation (2.2) can be represented in the form

$$d\omega = f_\lambda(\sigma(t), \omega(\sigma, t))\,dt, \tag{3.1}$$

where the damage function will be independent of the entire loading history.

One example of the use of this approach is the evolutionary damage equation derived by Novozhilov [21]

$$d\omega = f_\lambda(d_1(\lambda), d_2(\lambda), \ldots)\,d\lambda, \tag{3.2}$$

where λ is a parameter characterizing the path of the failure process; $q_1(\cdot)$, $q_2(\cdot)$ are quantities which depend on λ and affect the failure process. Equation (3.2) can be integrated only when the loading history is given.

Serious methodological problems must be overcome to make practical use of the approach based on equations with inseparable variables.

3.2. Principle of Separability. An analysis of certain experimental results [23, 33] and some well-known hypotheses [19] shows that the only requirement [9, 14] for satisfaction of nonlinearity condition (2.12) is

$$d\omega = f_T(\sigma(t), \omega(T))\,dt, \tag{3.3}$$

where $T = t/t_R(\sigma)$ is the corrected (normalized) time.

Proceeding on the basis of (3.3), we can represent nonlinearity condition (2.12) in the form

$$\frac{d\omega}{dT}\Big|_{\sigma=\text{const}} = \text{var} \quad \text{and} \quad \frac{d\omega}{d\sigma}\Big|_{T=\text{const}} = \text{var}. \tag{3.4}$$

In accordance with (3.3) and (3.4), we find that the only those evolutionary equations which give stress-dependent kinetic diagrams in the corrected time scale lead to nonlinear damage-accumulation models. In other words, the corresponding kinetic diagrams should be stratified with respect to the stress parameter, i.e., be separable.

It is not hard to show that that evolutionary equations (2.4)-(2.8), satisfying the hypothesis of linear damage summation, do not satisfy condition (3.3) because

$$\frac{d\omega}{dT} = 1, \tag{3.5}$$

$$\frac{d\omega}{dT} = \frac{1}{(1+k)}\frac{1}{(1-\omega)^k}, \tag{3.6}$$

$$\frac{d\omega}{dT}=\frac{\omega^{\beta}}{(1-\omega)^{k}}\int_{0}^{1}\frac{(1-\omega)^{k}d\omega}{\omega^{\beta}}d\omega, \tag{3.7}$$

$$\frac{d\omega}{dT}=\frac{1}{(1-\omega')^{k}}\int_{0}^{1}(1-\omega')^{k}d\omega, \tag{3.8}$$

$$\frac{d\omega}{dT}=\frac{1}{(1+k+q)}\frac{1}{(1-\omega)^{k+q}}. \tag{3.9}$$

In addition, it can be seen that all of them are invariant to the stress level.

3.3. Similarity Hypothesis. Compared to the linear formulation in [2], the initial relations of this hypothesis are obtained with the use of broader assumptions regarding the law which governs the change in the damage measure over time. We have the following for the damage measure in this case

$$d\omega=f_{T}(T,C_{1},C_{2},\ldots)dt, \tag{3.10}$$

where nonlinearity in the sense of (2.12) is assigned by varying the coefficients C_i in the structure of function f_T.

Practical use of the similarity hypothesis in the form (3.10) involves calculating residual time to rupture with single and multiple alternations of dissimilar damaging processes [5-7, 23, 25].

In another interpretation of the similarity hypothesis, loading history might be accounted for with the use of integral operators of the difference type. The evolutionary equation for the damage function in this case can be represented in the form [20]

$$d\omega=f_{t}(\omega)\int_{0}^{t}F(t-\tau)\varphi(\sigma)d\tau dt, \tag{3.11}$$

where $F(\cdot)$ and $\varphi(\cdot)$ are unknown functions. Practical realization of the determining equations in the form (3.11) is also limited by the possibilities for specifying the unknown functions.

4. Nonlinear Damage Model Based on the Separability Principle. Of all of the methods examined above for constructing damage equations satisfying nonlinear conditions, only the method based on the separability principle has been used in practice [9, 12, 14, 26-29]. Such use entails selection and substantiation of the structure of the initial evolutionary equation, identification of the damage function, and formulation of the failure criterion.

4.1. Initial Evolutionary Equations. A detailed analysis was made in [13, 14] of the structures of the unknowns of different evolutionary damage equations and the conditions for using them to construct nonlinear models. In particular, it was shown that the initial equation should contain an additional nonlinear damage function with a nonlinearity index dependent on stress. In this case, differential equation (2.2) is written in the form

$$d\omega=f_{t}'(\sigma(t),\omega(t),C_{1},C_{2})f_{t}''(\omega(t),C_{3}(\sigma))dt, \tag{4.1}$$

where $C_3(\sigma)$ is a coefficient which depends on the stresses.

The structure of evolutionary equations (2.6-2.8) conforms to initial equation (4.1).

4.2. Criterion of Delayed Fracture. In the general case of delayed static and cyclic loading, nonlinearity condition (3.3) can be represented by the relation

$$d\omega=f_{T}(\sigma(t),\omega(T),\frac{dn}{dt},\dot{\sigma},\ldots)dT, \tag{4.2}$$

which also accounts for the loading frequency dn/dt and the rate of change in the stresses $\dot{\sigma}$ ($n = ft$).

To construct constitutive damage equations satisfying nonlinearity condition (4.2), we introduce two damage parameters and assign corresponding integral measures [12, 13, 24-27]. One of them characterizes the instantaneous (independent of time) component of damage Ω_σ, while the other characterizes the temporal (time-dependent) component Ω_T. Each component is identified with a quantity corresponding to specific energy.

We further assume that the energy expended on failure is a constant equal to the energy corresponding to one-time static failure W_R. Thus,

$$\Omega_{\sigma}+\Omega_{T}=W_{R}\Rightarrow\frac{\Omega_{\sigma}}{W_{R}}+\frac{\Omega_{T}}{W_{R}}=1, \tag{4.3}$$

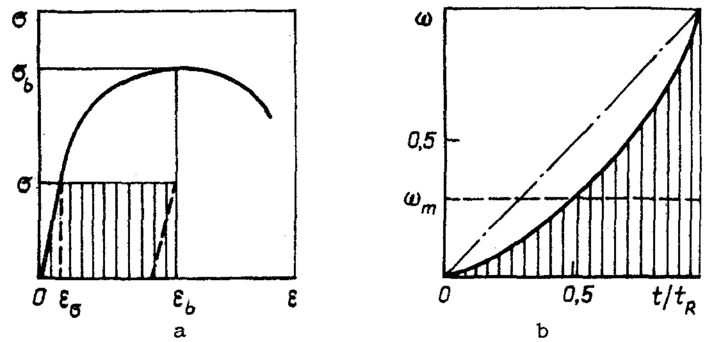

Fig. 2

where the following relation is valid for W_R

$$W_R = \int_0^{\varepsilon_b} \varphi_0(\varepsilon)\, d\varepsilon. \tag{4.4}$$

Here, $\varphi_0(\cdot)$ is the equation which gives the instantaneous strain diagram in the coordinates σ—ε; ε_R is the strain corresponding to the moment of failure according to this diagram.

The quantity Ω_σ is also determined from the σ—ε diagram (the hatched region in Fig. 2a), and in the case of static loading we have [9]

$$\Omega_\sigma = \int_0^{\varepsilon_\sigma} \varphi_0(\varepsilon)\, d\varepsilon + (\varepsilon_b - \varepsilon_\sigma)\sigma, \tag{4.5}$$

In the case of cyclic loading, with allowance for the loading rate [27]

$$\Omega_\sigma = \sigma \varepsilon_b - \int_0^{\sigma} \frac{\partial f_0(\sigma)}{\partial \sigma}\, d\sigma, \tag{4.6}$$

where σ is the applied stress; ε_σ is the strain corresponding to this stress; ε_b is the strain corresponding to the ultimate strength σ_b; f_0 is the inverse of the function φ_0.

The quantity Ω_T is determined from the normalized damage diagram (the hatched region in Fig. 2b) and is given by the mean value ω_m of the function ω on the interval $0 \le T \le 1$. Thus,

$$\frac{\Omega_T}{W_R} = \omega_m \Big|_0^{t_R} = \frac{\int_0^{t_R} \omega(t)\, dt}{t_R} = \int_0^1 \omega(T)\, dT, \tag{4.7}$$

Here, for cyclic loading it is best to replace the corrected time T by the corrected number of cycles N.

Allowing for Eqs. (4.4)-(4.7) for static loads and performing some simple transformations, we can obtain the condition expressing the balance between the instantaneous and time-dependent components of damage (4.3) in the form

$$\left(1 + \int_0^1 \omega\left(\frac{t}{t_R(\sigma)}\right) d\left(\frac{t}{t_R(\sigma)}\right)\right) \int_0^{\varepsilon_b} \varphi_0(\varepsilon)\, d\varepsilon - \int_{\varepsilon_\sigma}^{\varepsilon_b} (\varphi_0(\varepsilon) - \sigma)\, d\sigma = 1, \tag{4.8}$$

For cyclic loads, the analogous condition has the form

$$\int_0^1 \omega\left(\frac{n}{n_R(\sigma)}\right) d\left(\frac{n}{n_R(\sigma)}\right) + \frac{\varepsilon_b - f_0(\sigma)}{\varepsilon_b} = 1. \tag{4.9}$$

Equations (4.8) and (4.9) can be interpreted as energy criteria of delayed fracture.

4.3. Constitutive Equations of the Model. The construction of constitutive equations on the basis of the criteria formulated above reduces to selection and concretization of functions $\omega(\cdot)$ such that conditions (4.8) or (4.9) are satisfied.

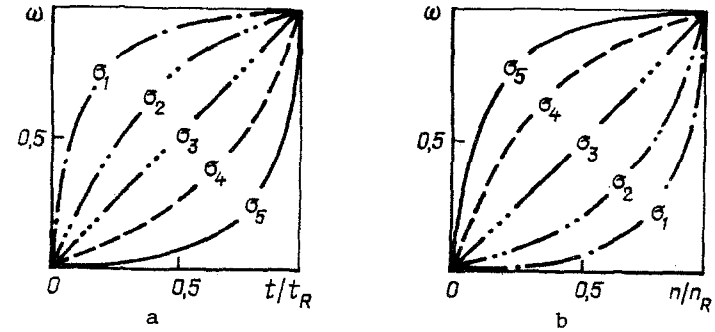

Fig. 3

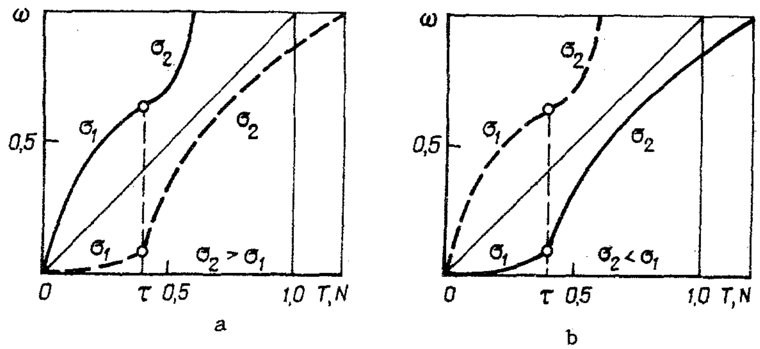

Fig. 4

Let us examine a specific variant of a nonlinear damage-accumulation model which employs either evolutionary equation (2.8) or, in the corrected time scale, the equation

$$\frac{d\omega}{dT}=\frac{1}{(1+k+q(\sigma))}\;\frac{1}{(1-\omega)^{k+q(\sigma)}}. \tag{4.10}$$

Here, in contrast to (3.9), we assume that the index q depends on the stresses — in accordance with (4.1).

Inserting (4.10) into (4.8) or (4.9) and varying σ, we concretize the dependence of q on σ. As a result, the nonlinear damage model is given by a system of two equations including the initial evolutionary equation and the function q(σ). Then for static loading we obtain [9, 14, 26]

$$\frac{d\omega}{dt}=D\left(\frac{\sigma}{1-\omega}\right)^{k}\frac{1}{(1-\omega)^{q(\sigma)}},$$
$$q(\sigma)=\frac{\int\limits_{0}^{\varepsilon_b}\varphi_0(\varepsilon)\,d\varepsilon}{\int\limits_{\varepsilon_\sigma}^{\varepsilon_b}(\varphi_0(\varepsilon)-\sigma)\,d\varepsilon}-(2+k), \tag{4.11}$$

while for cyclic loading [27]

$$\frac{d\omega}{dn}=D\left(\frac{\sigma}{1-\omega}\right)^{k}\frac{1}{(1-\omega)^{q(\sigma)}},\quad q(\sigma)=\frac{\varepsilon_b}{f_0(\sigma)}-(2+k), \tag{4.12}$$

which, as can be seen, satisfies nonlinearity conditions (3.3) and (3.4).

A graphical interpretation of solutions obtained on the basis of models (4.11) and (4.12) is shown in Fig. 3, a and b, respectively. As can be seen, the kinetic damage diagrams are stratified with respect to the stress parameter ($\sigma_5 > \sigma_4 > \sigma_3 > \sigma_2 > \sigma_1$). The time-dependent component of damage typically decreases with an increase in stress in the case of static loading, while the opposite is seen for cyclic loading.

4.4. Base Experiment. To solve problems based on nonlinear models (4.11) and (4.12), it is necessary to have two groups of characteristics. The first group includes the coefficients k, $q(\sigma)$, and D and characterizes the resistance of a material to damage accumulation. The coefficients k and D are determined from standard stress-rupture or fatigue tests [9, 14, 27].

The second group includes the characteristics of short-term strength $\varphi_0(\sigma)$, σ_b, and ε_b. These characteristics are determined from standard tensile σ—ε curves.

The nonlinearity index $q(\sigma)$ is determined from known values of k and D and the characteristics $\varphi_0(\sigma)$, σ_b, and ε_b.

4.5. Principal Mechanical Effects. The stress dependence of damage kinetics given by nonlinear models (4.11) and (4.12) makes it possible to account for the loading history in problems involving calculation of residual life under nonsteady loading. As an example, let us examine the classical problem of the additional loading ($\sigma_2 > \sigma_1$) and partial unloading ($\sigma_2 < \sigma_1$) of a material. The subscripts denote the sequences of stress application. Solutions can be obtained by using the simplest possible geometric constructions (Fig. 4). The regime involving additional loading is shown in Fig. 4a, while the regime involving partial unloading is shown in Fig. 4b.

It can be seen that with the transition from a lower stress to a higher stress (Fig. 4a), the total time to failure will be less than unity in static loading (solid lines) but more than unity in cyclic loading (dashed lines). The opposite reaction is seen with the transition from a higher stress to a lower stress (Fig. 4b). Qualitatively speaking, the estimates obtained here are in complete accord with the well-known empirical data in [19].

5. Application to Creep and Fatigue Problems. As has already been noted, damage mechanics is used to solve many problems of practical importance. The most promising applications are for problems involving calculation of residual lifetime under nonsteady loading [2, 7, 9, 14, 19, 26, 27] when there is an interaction between two dissimilar damaging processes [3-7, 10, 23, 25, 28, 29, 33]. Other areas in which damage mechanics is very useful include dynamic fracture-mechanics problems [18, 33, 34], particularly with respect to the mechanics of fatigue failure [2, 5, 8, 11, 12, 15, 16]. We will examine several examples from our previous studies.

5.1. Lifetime under Stepped Loading. In a two-step static loading regime, we obtain the following if we assume that $\omega(\sigma_1) \equiv \omega(\sigma_2)$ and perform some simple transformations [9, 26]

$$\frac{t_1}{t_R(\sigma_1)} + \left(\frac{t_2}{t_R(\sigma_2)}\right)^{\gamma(\sigma)} = 1, \tag{5.1}$$

where we have the following, respectively, for the additional-loading and unloading conditions

$$\gamma(\sigma) = \frac{1 + k + q(\sigma_1)}{1 + k + q(\sigma_2)} < 1; \quad \gamma(\sigma) = \frac{1 + k + q(\sigma_1)}{1 + k + q(\sigma_2)} > 1, \tag{5.2}$$

since $q(\sigma_2) > q(\sigma_1)$ at $\sigma_2 > \sigma_1$ and $q(\sigma_2) < q(\sigma_1)$ at $\sigma_2 < \sigma_1$.

The results of calculations (dashed lines) performed with Eq. (5.1) are compared in Fig. 5a with experimental data (points) for alloy ÉI437B at $\theta = 750°C$, $\sigma_1 = 300$ MPa, and $\sigma_2 = 400$ MPa.

To evaluate lifetime for two-step cyclic loading, we used Eq. (4.12) to obtain a relation [27] similar to (5.1) in structure but corresponding to additional-loading conditions $\gamma(\sigma) > 1$ and unloading conditions $\gamma(\sigma) < 1$. The results of the calculations are shown in Fig. 5b for alloy ÉI437B at $\theta = 800°C$, $\sigma_1 = 200$ MPa, and $\sigma_2 = 300$ MPa.

5.2. Interaction of Creep and Fatigue. Such interaction normally includes processes occurring when creep and fatigue take place at the same time or alternately [5-7, 10, 25, 28, 29]. The initial evolutionary equation for total damage ω_Σ in this case is written in the form [7, 28, 29]

$$\frac{d\omega_\Sigma}{dt} = \frac{\partial}{\partial t}(\sigma, \omega_\Sigma) + \frac{\partial}{\partial n}(\sigma, \omega_\Sigma), \tag{5.3}$$

where it is assumed that the quantity ω_Σ is determined by summing the creep damage ω_c and fatigue damage ω_f.

When a material first undergoes creep for the time t_c, after we perform some simple transformations we obtain the following [28, 29] for the time to failure $t_{R\Sigma}$ from (5.3) with allowance for (4.11) and (4.12)

$$t_{R\Sigma} = t_c + t_{Rf}(\sigma)\left(1 - \frac{t_c}{t_{Rc}(\sigma)}\right)^{\frac{1+k_f+q_f(\sigma)}{1+k_c+q_c(\sigma)}}, \tag{5.4}$$

while when the material is first loaded in fatigue for the time t_f

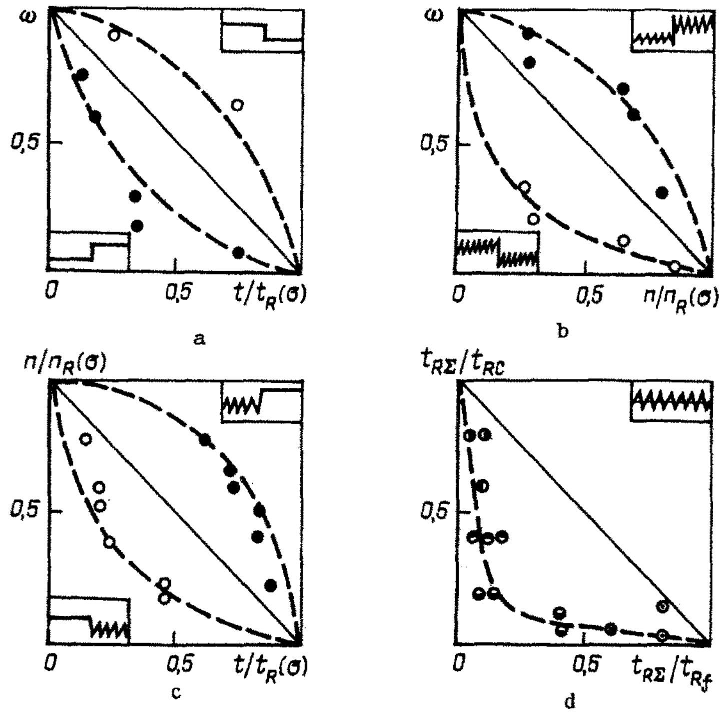

Fig. 5

$$t_{R\Sigma} = t_f + t_{Rc}(\sigma)\left(1 - \frac{t_f}{t_{Rf}(\sigma)}\right)^{\frac{1+k_c+q_c(\sigma)}{1+k_f+q_f(\sigma)}}. \tag{5.5}$$

Here, $t_{Rf}(\sigma)$ and $t_{Rc}(\sigma)$ are the times to failure from fatigue alone and from creep alone; k_c, q_c and k_f, q_f are coefficients determined on the basis of the rupture-strength curve and fatigue curve, respectively.

The problem of calculating lifetimes in the case of the simultaneous action of creep and fatigue is solved numerically, by stepwise integration of Eq. (5.3) [7, 25, 28, 29].

The results of calculations (dashed lines) performed with Eqs. (5.4) and (5.5) are shown in Fig. 5c with experimental data (points) for alloy ÉI867 at θ = 900°C in the regime creep (σ = 300 MPa) ⇒ fatigue (σ = 200 MPa) and the regime fatigue (σ = 200 MPa) ⇒ creep (σ = 300 MPa). Similar results are presented in Fig. 5d f or alloy ÉI867 at θ = 900°C when creep and fatigue act simultaneously and different stresses are applied.

On the whole, comparison of the theoretical and experimental data shows that the nonlinear models of damage accumulation constructed here adequately describe the actual processes that take place. This is illustrated first of all by the fact that the loading history can be accounted for on the basis of the models.

5.3. Growth of Fatigue Cracks. Damage mechanics can be used most naturally as the "driving force" in crack-growth problems not connected with an increase in the load. The explanation for this is fairly simple, in that the damage itself takes the form of microcracks which, upon merging, form a macrocrack. The feasibility of using damage mechanics to solve crack mechanics problems was demonstrated by Kachanov [18]. This idea has been further developed in many subsequent studies — particularly as it pertains to creep problems.

Less progress has been made in regard to the problem of fatigue failure. One possible approach was examined in [5, 8, 12]. In particular, the authors of [12] obtained the following for the rate of fatigue crack growth l_f at different points of isotropic plates subjected to axial loading [12]

$$\frac{dl_f}{dn} = \left(1 + \frac{1}{k_f}\right) C_f \frac{(\Delta K)^{k_f}}{(2\sqrt{\pi})^{k_f}[2\lambda(l_f)]^{\frac{k_f}{2}-1}}, \tag{5.6}$$

where ΔK is the amplitude of the stress-intensity factor; $\lambda(l_f)$ is the size of the plastic zone, dependent on the running length of the crack; k_f and C_f are coefficients determined from standard fatigue curves.

To a certain extent, model (5.6) makes it possible to generalize existing empirical relations, since the index k_f can take values from 2 (for ductile materials) to 10-12 (for brittle materials), depending on the law governing the change in $\lambda(\cdot)$.

Conclusion. In recent decades, damage mechanics has become an independent field of study within the mechanics of deformable solids. Many phenomena which previously had no explanation can now be regarded as consequences of damage initiation and development, and many unresolved problems have been reduced to qualitative and quantitative evaluations making use of damage functions. This applies in particular to embrittlement processes occurring during creep, the growth of microcracks and the formation of macrocracks, the effect of the medium, friction and wear processes, and fatigue phenomena. Damage functions have been used in creep theory to construct constitutive equations which allow description of all of the characteristic stages of the process and calculation of the time to failure for structural elements. Exploration of the relationship between damage mechanics and crack mechanics has proven very fruitful.

The use of damage mechanics shows the most promise in regard to calculation of the total and residual life of materials under nonsteady loading conditions, evaluation of the interaction of qualitatively different processes, and the solution of dynamic fracture mechanics problems.

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