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EQUILIBRIUM AND STABILITY OF A NONLINEAR-ELASTIC PLATE WITH A TAPERED DISCLINATION

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In solid-state physics, it is important to study dislocations and disclinations in two-dimensional bodies (plates, films, and so forth), as well as in three-dimensional bodies. Methods of analyzing such problems and specific solutions are given, for example, in [1-3]. One of these defects arises in the study, introduced in [4], of a tapered disclination, a stress–strain state in a cylinder which is made up of a large number of thin disks which contain disclinations. In this case, the disks do not remain plane, but are transformed into either conical funnels or complex curved surfaces with a saddle-shaped configurations. These were observed in numerical simulation of disclinations using molecular dynamics methods [4]. This has generated interest in the study of similar models using the methods of elasticity theory. The linear theory of dislocations in shells is explained in [5], and its nonlinear aspects, in [6].

In this work, the problem of equilibrium and stability of a nonlinear-elastic plate with tapered disclinations is studied. The analysis is based on nonlinear equations of the theory of Love plates and shells, formulated in [7]. In the framework of momentless theory, it is established that in the case of positive disclination, two axisymmetric equilibrium shapes are possible: equilibrium plane or conic surfaces. A nonaxisymmetric nonplane equilibrium form for the momentless plate with negative discilination is determined. The equilibrium equations of nonlinear momentum theory admit a "plane" solution (deflection identically equal to zero) for all values of the disclination parameter, which coincides with the solution from momentless theory. A numerical investigation of its stability has been done.

1. We consider a plate of thickness h, having the form of a ring $c \le r \le d$, $0 \le \phi \le 2\pi$, $-h/2 \le \zeta \le h/2$ (r, ϕ, ζ are cylindrical coordinates). We adopt the Kirchoff–Love hypothesis, and further, by plate, we will understand its average surface. The formation of a disclination in the plates is given by the relations

Rostov-on-Don. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, No. 3, pp. 157-163, May-June, 1992. Original article submitted April 12, 1991.

 0021-8944/92/3303-0464$12.50

$$R = R(r),\ \Phi = \varkappa\varphi,\ Z = Z(r). \tag{1.1}$$

The quantities R, Φ, Z, relative to the deformed state, describe the distance between the axis and a point on the mean surface of the plate, the polar angle and the vertical displacement, respectively. The case $\kappa < 1$ (negative disclination) corresponds to deformation of the ring which occurs after it is cut along the line $\phi = 0$ and a wedge with cone angle $2\pi(1 - \kappa)$ is introduced into the cut. The case $\kappa > 1$ (positive disclination) corresponds to removal of a sector $2\pi\kappa^{-1} \le \phi \le 2\pi$ from the ring, with subsequent joining of the edges. According to (1.1), the mean surface of the plate after deformation is a surface of revolution.

The equations for equilibrium of the plate in the absence of surface loading have the form [7]

$$\nabla \cdot \mathbf{S} = 0,\ \mathbf{S} = \boldsymbol{\sigma} + \boldsymbol{\mu} \cdot \nabla'\mathbf{N} + (\nabla \cdot \boldsymbol{\mu}) \cdot \nabla'\mathbf{r}\mathbf{N}, \tag{1.2}$$

where $\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \frac{1}{r}\mathbf{e}_\phi \frac{\partial}{\partial \phi}$ is the two-dimensional nabla operator of the undeformed state; $\mathbf{e}_r$, $\mathbf{e}_\phi$, $\mathbf{e}_\zeta$ are the unit vectors for the cylindrical coordinates in this state ($\mathbf{e}_\zeta = \mathbf{n}$ is the normal to the undeformed surface of the plate); $\boldsymbol{\sigma} = \partial W/\partial \mathbf{U} \cdot \mathbf{A}$ is the tensor of membrane forces (Of the Piola type); $\boldsymbol{\mu} = -\partial W/\partial \mathbf{K} \cdot \nabla \mathbf{P}$ the bending moment tensor (of the Piola type); $\mathbf{P} = R\mathbf{e}_R + Z\mathbf{e}_Z$ is the radius-vector of a point in the plate in the deformed state; $\mathbf{e}_R$, $\mathbf{e}_\Phi$, $\mathbf{e}_Z$ the unit vectors of the cylindrical coordinates in this state; $\mathbf{N}$ is the normal to the deformed surface of the plate; $\mathbf{K} = -\nabla'\mathbf{N} \cdot (\nabla\mathbf{P})^T$; $\mathbf{U} = [\nabla\mathbf{P} \cdot (\nabla\mathbf{P})^T]^{1/2}$; $\mathbf{A}$ is the rotation tensor ($\nabla\mathbf{P} = \mathbf{U} \cdot \mathbf{A}$); $\mathbf{r}$ is the radius-vector of a point in the undeformed plate; ∇' is the two-dimensional nabla operator of the deformed state. As the function of specific potential energy of deformation $W(\mathbf{U}, \mathbf{K})$, we take

$$W = \frac{1}{2} E_1 [\nu\, \mathrm{tr}^2 \boldsymbol{\varepsilon} + (1 - \nu)\, \mathrm{tr}\, \boldsymbol{\varepsilon}^2] + \frac{1}{2} E_2 [\nu\, \mathrm{tr}^2 \mathbf{K} + (1 - \nu)\, \mathrm{tr}\, \mathbf{K}^2]. \tag{1.3}$$

Here, $E_1 = Eh/(1 - \nu^2)$; $E_2 = Eh^3/12(1 - \nu^2)$; E is Young's modulus; ν is Poisson's ratio for the plate material; $\boldsymbol{\varepsilon} = \mathbf{U} - \mathbf{g}$; $\mathbf{g} = \mathbf{e}_r\mathbf{e}_r + \mathbf{e}_\phi\mathbf{e}_\phi$ is the two-dimensional unit tensor. Expression (1.3) coincides with the energy function in linear plate theory, if for $\boldsymbol{\varepsilon}$ and $\mathbf{K}$ we take the corresponding linear tensors $(1/2)(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ and $\nabla\nabla w$ (these are explained in the designations for the constants E and ν given above).

The nonzero components of the force and momentum tensors, corresponding to (1.1), are written as

$$\sigma_{rR} = E_1 \left[R' + \nu \frac{\varkappa R}{r} \frac{R'}{\psi} - (\nu + 1) \frac{R'}{\psi}\right], \tag{1.4}$$

$$\sigma_{\varphi\Phi} = E_1 [\varkappa R/r + \nu\psi - \nu - 1],\ \sigma_{rZ} = \frac{Z'}{R'} \sigma_{rR};$$

$$\mu_{rR} = -\frac{E_2}{\psi}\left(\eta R' + \nu \frac{\varkappa^2}{r^2} R'Z'R\right),$$

$$\mu_{\varphi\Phi} = -\frac{E_2}{\psi}\left(\frac{\varkappa^3}{r^3} R^2 Z' + \nu \frac{\varkappa}{r} \eta R\right),\ \mu_{rZ} = \frac{Z'}{R'} \mu_{rR}, \tag{1.5}$$

where $\psi = \sqrt{R'^2 + Z'^2}$; $\eta = R'Z'' - Z'R''$; the primes denote differentiation with respect to r.

The tensor $\mathbf{S}$ has the following representation:

$$\mathbf{S} = S_{rR}(r)\mathbf{e}_r\mathbf{e}_R + S_{rZ}(r)\mathbf{e}_r\mathbf{e}_Z + S_{\varphi\Phi}(r)\mathbf{e}_\varphi\mathbf{e}_\Phi. \tag{1.6}$$

Substituting (1.6) into (1.2), we obtain a system of equilibrium equations:

$$S'_{rR} + (S_{rR} - \varkappa S_{\varphi\Phi})/r = 0; \tag{1.7}$$

$$S'_{rZ} + S_{rZ}/r = 0. \tag{1.8}$$

Assuming that the plate edges are free, we write the boudnary conditions in the form [7]

$$S_{rR}(r) = 0,\ r = c,\ d; \tag{1.9}$$

$$S_{rZ}(r) = 0,\ r = c,\ d; \tag{1.10}$$

$$\mu_{rR}(r) = 0,\ r = c,\ d. \tag{1.11}$$

From (1.8), taking (1.10) into account, there follows directly

$$S_{rZ}(r) \equiv 0. \tag{1.12}$$

First we examine the momentless case, for which the constant E_2 in (1.3) is formally set to zero. Then $\mathbf{S} = \boldsymbol{\sigma}$ and (1.12) takes the form

$$Z'\left[1 + \frac{1}{\sqrt{R'^2 + Z'^2}}\left(\nu\varkappa\frac{R}{r} - \nu - 1\right)\right] \equiv 0.$$

Two variants are possible.

1. $Z' \equiv 0$. Using this in the expressions in (1.4) for σ_{rR} and $\sigma_{\phi\Phi}$ and substituting these into (1.7) and (1.9), we arrive at a boundary-value problem determining the function R(r):

$$\begin{gathered} R'' + R'/r - \varkappa^2 R/r = (1-\varkappa)(1-\nu)/r, \\ R' + \nu\varkappa R/r = 1 + \nu,\ r = c,\ d. \end{gathered} \tag{1.13}$$

We represent the solution as

$$R(r) = \frac{d}{1+\varkappa}\left[(1-\nu)C_1\rho^{\varkappa} + (1+\nu)C_2\rho^{-\varkappa} + (1+\nu)\rho\right], \tag{1.14}$$

where

$$C_1 = \frac{1-\rho_0^{\varkappa+1}}{1-\rho_0^{2\varkappa}};\ C_2 = \frac{\rho_0^{2\varkappa} - \rho_0^{\varkappa+1}}{1-\rho_0^{2\varkappa}};\ \rho = \frac{r}{d};\ \rho_0 = \frac{c}{d}.$$

In particular, in the case of a solid disk ($\rho_0 = 0$), the main stresses to (1.14) (the main forces averaged over the plate thickness) are expressed in terms of the tensor $\boldsymbol{\sigma}$ by the relations [7] $\sigma_R = \mathbf{e}_R \cdot J^{-1}(\nabla\mathbf{P})^T \cdot \boldsymbol{\sigma} \cdot \mathbf{e}_R$, $\sigma_\Phi = \mathbf{e}_\Phi \cdot J^{-1}(\nabla\mathbf{P})^T \cdot \boldsymbol{\sigma} \cdot \mathbf{e}_\Phi$ $(J = \det(\nabla\mathbf{P} + \mathbf{nN}))$. Taking (1.4) into account, these take the form

$$\sigma_R = E_1\frac{(1-\nu^2)(\rho^{\varkappa-1} - 1)}{(1-\nu)\rho^{\varkappa-1} + (1+\nu)},\quad \sigma_\Phi = E_1\frac{(1-\nu^2)(\varkappa\rho^{\varkappa-1} - 1)}{(1-\nu)\varkappa\rho^{\varkappa-1} + (1+\nu)}.$$

It can be verified that if in the "plane" solution obtained, the well-known change of constants is made which is related to the change from a plane stressed state to plane strain ($\nu \to \nu/(1-\nu)$), then we have the solution found in [8] for the problem of a disclination in a cylinder of semilinear material. As in [8], the main stresses cannot have singularities on the disclination axis in the case of a solid disk. At the same time, they have a logarithmic singularity, as in linear theory [3].

2. $1 + (\nu\kappa R/r - \nu - 1)/\sqrt{R'^2 + Z'^2} \equiv 0$. In this case σ_{rR} is also equal to zero. Consequently, due to (1.7), $\sigma_{\phi\Phi} \equiv 0$ as well. To determine R(r) and Z(r), we obtain a system of equations of the form

$$\sqrt{R'^2 + Z'^2} + \nu\varkappa R/r = 1 + \nu,\ \nu\sqrt{R'^2 + Z'^2} + \varkappa R/r = 1 + \nu,$$

where

$$R(r) = r/\varkappa,\ Z(r) = \sqrt{\varkappa^2 - 1}\ r/\varkappa + \text{const}. \tag{1.15}$$

This solution exists only for positive disclination ($\kappa > 1$). Equation (1.15) corresponds to a (truncated) cone.

An interesting feature of solution (1.15) is its "universality." It is easy to show that in this case $\mathbf{U} = \mathbf{g}$, that is, $\boldsymbol{\varepsilon} = 0$, which means that the forces are also identically equal to zero. Thus for a momentless shell of nonlinear-elastic material, any type of equilibrium equations will be identically satisfied.

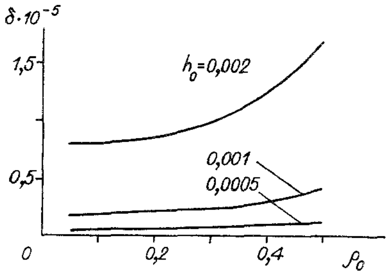

Fig. 1

2. It was estabished above that for $\kappa > 1$, the boundary-value problem (1.7)-(1.10) has two solutions in the momentless approximation. It can be shown that both of them are stable with respect to axisymmetric perturbations, in the sense that there are no equilibrium shapes close to them. This is because each of the equations, linearized in the neighborhood of these solutions, has only a trivial solution. From the energy point of view, for any value of κ greater than one, preferentially leads back to (1.15), since the deformation energy in this case is in general equal to zero.

A more strict and precise result can be obtained by analyzing the equations based on nonlinear moment plate theory. It is easy to see that these equations also admit a "plane" solution (deflection is identically equal to zero), which agrees with the solution of momentless theory (1.15). This follows from the fact the moment tensor vanishes at $Z' \equiv 0$. To study the stability of the plane solution with respect to axisymmetric perturbations, we linearize the equilibrium equations in the neighborhood of this solution to obtain

$$\mathbf{P} = R(r)\mathbf{e}_R + \varepsilon\mathbf{w}, \tag{2.1}$$

$$\mathbf{w} = u(r)\mathbf{e}_R + w(r)\mathbf{e}_Z,$$

where R(r) is determined by (1.14). Substituting (2.1) into (1.4), (1.5) and the resultant equations into (1.7), (1.12), and then retaining terms of first order in ε, we obtain

$$\begin{gathered} v' = -\frac{1}{R'}\left(\nu\frac{\varkappa^2}{r^2}R - R''\right)v - \frac{1}{R'}\mu, \\ \mu' = \left[\frac{\varkappa^4}{r^4}\frac{R^2}{R'}(\nu^2-1) - l\left(R' + \nu\frac{\varkappa R}{r} - \nu - 1\right)\right]v + \frac{1}{r}\left(\nu\frac{\varkappa^2}{r^2}\frac{R}{R'} - 1\right)\mu \end{gathered} \tag{2.2}$$

($v = w'$, $\mu = \mu_r R/E_2$, $\ell = E_1/E_2$). Relations (1.11) form the boundary conditions for this system.

For the axisymmetric case considered here, the second boundary-value problem (to determine u(r)) is a linearized version of (1.14) that is, it has a unique solution.

Critical values of the disclination parameter κ are determined by the occurrence of nontrivial solutions to (2.2), (1.11). Calculations show that for $0 < \kappa < 1$, such nontrivial solutions do not exist, that is, the plate does not lose stability in an axisymmetric fashion. Figure 1 shows a plot of the critical value of the parameter $\delta = \kappa - 1$ as a function of the radius of the ring aperture ($\rho_0 = c/d$), for various thicknesses ($h_0 = h/d$) for $\kappa > 1$. Note that to a high degree of accuracy, δ can be considered as proportional to h_0^2. The modes of stability loss are surfaces which are close to conical.

To analyze nonaxisymmetric forms of stability loss, we must set $u = u(r, \phi)$, $w = w(r, \phi)$. Restricting the problem to the case of bending-stability loss, we assume that $u \equiv 0$. Linearized quantities are denoted by dots:

$$\mathbf{S}^{\cdot} = \frac{d}{d\varepsilon}\mathbf{S}(\mathbf{P} + \varepsilon\mathbf{w})\,|_{\varepsilon=0}, \mathbf{w} = w(r, \varphi)\,\mathbf{e}_Z.$$

Calculations give

$$\mathbf{S}^{\cdot} = S_1(r, \varphi)\mathbf{e}_r\mathbf{e}_Z + S_2(r, \varphi)\mathbf{e}_\varphi\mathbf{e}_Z, \tag{2.3}$$

where

$$S_1 = \sigma_{rR}\frac{w'}{R'} + \frac{1}{R'}\left(\mu_{rR}^{\cdot\prime} + \frac{1}{r}\mu_{rR}^{\cdot} - \frac{\varkappa}{r}\mu_{\varphi\Phi}^{\cdot} + \frac{1}{r}\mu_{\varphi R,\varphi}^{\cdot}\right); \tag{2.4}$$

$$S_2 = \sigma_{\varphi\Phi}\frac{w_{,\varphi}}{\varkappa R} + \frac{r}{\varkappa R}\left(\mu_{r\Phi}^{\cdot\prime} + \frac{1}{r}\mu_{r\Phi}^{\cdot} + \frac{\varkappa}{r}\mu_{\varphi R}^{\cdot} + \frac{1}{r}\mu_{\varphi\Phi,\varphi}^{\cdot}\right);$$

$$\mu_{rR}^{\cdot} = -E_2\left[R'w'' + \left(\nu\frac{\varkappa^2}{r^2}R - R''\right)w' + \nu\frac{R'^2}{r^2}w_{,\varphi\varphi}\right];$$

$$\mu_{\varphi\Phi}^{\cdot} = -E_2\left[\nu\frac{\varkappa R}{r}w'' + \frac{\varkappa R}{rR'}\left(\frac{\varkappa^2}{r^2}R - \nu R''\right)w' + \frac{\varkappa R}{r^3}w_{,\varphi\varphi}\right];$$

$$\mu_{r\Phi}^{\cdot} = -E_2(1-\nu)\frac{\varkappa}{r^2}\left(Rw'_{,\varphi} - R'w_{,\varphi}\right);\ \mu_{\varphi R}^{\cdot} = \frac{rR'}{\varkappa R}\mu_{r\Phi}^{\cdot}.$$

The notation $f' = \partial f/\partial r$, $f_{,\phi} = \partial f/\partial\phi$ has been used in (2.4). The quantities σ_{rR}, $\sigma_{\phi\Phi}$, and R pertain to the "plane" stressed state, whose stability is being studied. Expression (2.3) corresponds to the equilibrium equation

$$S_1' + S_1/r + S_{2,\varphi}/r = 0, \tag{2.5}$$

which, with the help of (2.4) can be written in the form of a system of four equations in terms of the functions w, $v = w'$, $\mu_{\dot{r}R}$, and S_1:

$$w' = v,\quad v' = -\frac{\nu}{r^2}w_{,\varphi\varphi} + \frac{1}{R'}\left(R'' - \frac{\nu\varkappa^2}{r^2}R\right)v - \frac{1}{R'}E_2^{-1}\mu_{rR}^{\cdot}, \tag{2.6}$$

$$\mu_{rR}^{\cdot\prime} = \left[(\nu^2-1)\frac{\varkappa^2}{r^4}R - (1-\nu)\frac{R'^2}{Rr^2}\right]E_2 w_{,\varphi\varphi} + \left[(\nu^2-1)E_2\frac{\varkappa^4}{r^4}\frac{R^2}{R'} - \sigma_{rR}\right]v +$$

$$+ E_2(1-\nu)\frac{R'}{r^2}v_{,\varphi\varphi} + \left(\frac{\nu\varkappa^2 R}{r^2 R'} - \frac{1}{r}\right)\mu_{rR}^{\cdot} + R'S_1,$$

$$S_1' = \left[E_2(1-\nu)\frac{1}{Rr^2}\left(\frac{R'}{r} - R'' - \frac{R'^2}{R}\right) - \frac{\sigma_{\varphi\Phi}}{\varkappa rR}\right]w_{,\varphi\varphi} + E_2(1-\nu)\frac{1}{r^4}w_{,\varphi\varphi\varphi\varphi} +$$

$$+ E_2(1-\nu)\frac{1}{Rr^2}\left[\left(R'' + \frac{\varkappa^2 R}{r^2}\right)\frac{R}{R'} + R' - \frac{R}{r}\right]v_{,\varphi\varphi} - \frac{1}{R'r^2}\mu_{rR,\varphi\varphi}^{\cdot} - \frac{1}{r}S_1.$$

The boundary conditions for the free edges of the plate in the nonaxisymmetric case have the form [7]

$$\mathbf{e}_r\cdot\mathbf{S} + \frac{\partial}{\partial s}\left(\mu_{r\Phi}\frac{1}{\varkappa}\mathbf{N}\right) = 0,\quad \mu_{rR} = 0\quad (r = c, d)$$

($\partial/\partial s$ is the tangential derivative). After linearization, we obtain

$$S_1 - E_2(1-\nu)\frac{1}{r^3}\left(w'_{,\varphi\varphi}R - w_{,\varphi\varphi}R'\right) = 0,\ \mu_{rR}^{\cdot} = 0\quad (r = c, d). \tag{2.7}$$

We seek a nontrivial solution to boundary-value problem (2.6), (2.7) in the form

$$w(r, \varphi) = w_r(r)\sin(n\varkappa\varphi + \varphi_0). \tag{2.8}$$

Substituting (2.8) into (2.6), (2.7), we obtain the linear homogeneous boundary-value problem for a system of ordinary differential equations. The values of κ for which it has a nontrivial solution are determined numerically. In particular, calculations show that for $\kappa > 1$, stability loss occurs in an axisymmetric fashion: the critical values of δ for the nonaxisymmetric form ($n \geq 1$) are an order of magnitude larger. Figure 2 shows a plot of δ as a function of aperture radius for $\kappa < 1$. Nonaxisymmetric stability loss occurs here for $n = 2$.

It is clear from a comparison of the plots in Figures 1 and 2, that disclinations of different signs have virtually the same effect on plate stability. Comparison of these results with those obtained in [9] on the stability of a cylinder with a tapered disclination

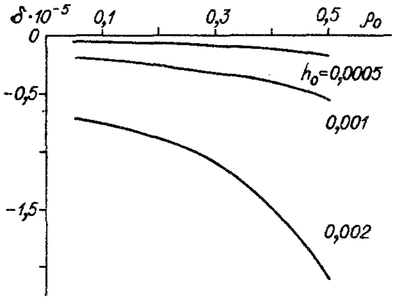

Fig. 2

shows that the bending-stability loss in the plate, in the range of parameters studied, occurs for those values of δ which are a few orders of magnitude smaller than for plane-stability loss in the cylinder.

3. As shown above, the equilibrium surface for positive disclination is obtained, within the framework of momentless theory, by an axisymmetric isometric transformation (bending) of the undeformed plate. We will construct such a transformation in the case of negative disclination. It is clear that, unlike (1.1), it will not be axisymmetric.

We consider a transformation of the form $R = R(r, \phi)$, $\Phi = \kappa\phi$, $Z = Z(r, \phi)$. The condition that it be isometric reduces to

$$\mathbf{U} = \mathbf{g}. \tag{3.1}$$

The problem of finding functions $R(r, \phi)$ and $Z(r, \phi)$, which satisfy (3.1), does not contain dimensional parameters. This determines the following form of the unknown functions

$$R(r, \varphi) = rf(\varphi),\ Z(r, \varphi) = rg(\varphi). \tag{3.2}$$

Using (3.2), relation (3.1) reduces to three scalar equations for determining $f(\phi)$ and $g(\phi)$:

$$f^2 + g^2 = 1,\ \ f\frac{df}{d\varphi} + g\frac{dg}{d\varphi} = 0,\ \ \left(\frac{df}{d\varphi}\right)^2 + \varkappa^2 f^2 + \left(\frac{dg}{d\varphi}\right)^2 = 1. \tag{3.3}$$

The second of equations (3.3) follows directly from the first. Therefore, it can be discarded. Furthermore, solving the first of equations (3.3) for $g(\phi)$ and substituting the result in the third equation, we have

$$(df/d\varphi)^2 = (1 - f^2)(1 - \varkappa^2 f^2),$$

which has the solutions:

$f^2 \equiv 1$ is the constant solution, which does not satisfy (3.3);

$f^2 \equiv \kappa^{-2}$ is the solution which describes the twisting of the plate into a cone ($\kappa > 1$) and is studied above:

$$f = \operatorname{sn}(\varphi + C). \tag{3.4}$$

The disclination parameter κ ($\kappa < 1$) is the modulus of the elliptic function sn in (3.4). In this case, $g^2(\phi) = \operatorname{cn}^2(\phi + C)$; C is a constant which can be determined from the continuity condition for the surface of the deformed plate; R and Z must be continuous, 2π-periodic functions of the coordinate Φ, that is

$$R(r, 0) = R(r, 2\pi/\varkappa); \tag{3.5}$$

$$Z(r, 0) = Z(r, 2\pi/\varkappa). \tag{3.6}$$

Let us consider the case of small negative disclinations ($\kappa \lesssim 1$). By small values of $1 - \kappa$ is meant that the period of the function sn ϕ with modulus κ is larger than $4\pi/\kappa$:

$$K(\varkappa) > \pi/\varkappa \tag{3.7}$$

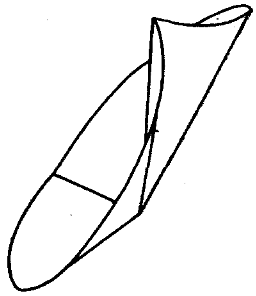

Fig. 3

($K(\kappa)$ is the complete elliptical integral of the first kind [10]). Requirement (3.7) corresponds to the interval for variation of κ ($\kappa \in (0.9858; 1)$). In this case, from (3.5) we obtain $\mathrm{sn}\, C = \mathrm{sn}(C + 2\pi/\kappa)$, whence $C = K - \pi/\kappa$ for the constant C in (3.4).

Requirement (3.7) serves as the condition that the function $f(\phi)$ (and thus $R(r, \phi)$ as well) will be greater than zero on the interval $[0, 2\pi/\kappa]$. Relations (3.6) can be satisfied if the function $g(\phi) = |\mathrm{cn}(\phi + C)|$ is chosen for $g(\phi)$. This function is continuous and at the ends of the interval $[0, 2\pi/\kappa]$ takes on the unique values and satisfies (3.3) everywhere, except at the points $\phi = 0$ and $\phi = \pi/\kappa$, where the derivative of $g(\phi)$ is discontinuous.

The surface corresponding to transformation (3.2) has two cusps (at $\Phi = 0$ and $\Phi = \pi$), a result frequently encountered in the analysis of problems of the deformation of momentless shells on the basis of the theory of surface bending [11]. A picture of this surface for $\kappa = 0.99$ is shown in Fig. 3.

It is worth noting that this problem has infinitely many solutions, since the continuity conditions for the surface in the deformed state of the type (3.5), (3.6) can be imposed not only for $\Phi = 0$ and 2π, but also for any other value α (and $\alpha + 2\pi$). Thus, the position of the cusp on this surface is not fixed, and any two lines $\Phi = \alpha$ and $\Phi = \alpha + \pi$, $\alpha \in [0, 2\pi)$ can serve in tis capacity.

The author is deeply grateful to L. M. Zubov for his attention to this work.

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