HYPOELASTIC FORM OF EQUATIONS IN NONLINEAR ELASTICITY THEORY

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It is shown that the complete system of equations of elasticity theory for an isotropic medium admits a unique representation in the hypoelastic form (the tensor of the rate of change of stresses is a linear function of the tensor of strain rates with coefficients depending on the invariants of the stress tensor). It is necessary to this end that the hypothesis be satisfied on the determination of strains by stresses which are unknown. Any arbitrariness in the choice of the coefficients of the hypoelastic relation may result in the thermodynamic identity being infringed.

To describe deformation processes of a medium one usually uses such tensor expressions as strain tensor, stress tensor, strain-velocity tensor, tensor of the rate of change of stresses. In constructing a determining system of equations of motion for the medium a relation is established to complete the system between any tensor quantities as shown above. The most often used forms of such a relation are now enumerated [1].

Hyperelastic Medium. The existence is assumed of an elastic potential (of inner energy for adiabatic processes) which depends on the strain tensor by means of which the stress tensor is determined.

Elastic Medium. The stress tensor is given as a function of the strain tensor.

Hypoelastic Medium. The tensor of the rate of change of stresses is given as a linear function of the tensor of strain velocities the coefficients depending on the stress tensor.

It can be shown that with such a classification an elastic medium is also a hypoelastic one, and the hyperelastic medium is elastic as well as a hypoelastic one. Moreover, if one starts with the thermodynamic identity which must hold for a hyperelastic medium (similarly as in the case of gaseous media the thermodynamic identity $dE = -pdv + TdS$ takes place) then such a hyperelastic medium uniquely determines the elastic and the hypoelastic relations.

An adiabatic deformation is considered of a hyperelastic isotropic medium by assuming that the inner energy depends on three independent invariants of the tensor of the Cauchy strains g_{ij} and on the entropy S. The independent invariants chosen by us are

$$K_1 = g_{11} + g_{22} + g_{33}$$
$$K_2 = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} + \begin{vmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{vmatrix} + \begin{vmatrix} g_{33} & g_{31} \\ g_{13} & g_{11} \end{vmatrix}$$
$$K_3 = \det \| g_{ij} \|$$

If the notation $h_i = -1/2 \ln g_i$ is used where g_i are the principal values of the tensor g_{ij} then the thermodynamic identity is

$$\rho dE\,(h_1,\ h_2,\ h_3,\ S) = \sigma_1 dh_1 + \sigma_2 dh_2 + \sigma_3 dh_3 + \rho T dS \tag{1}$$

In the above ρ is the density of the medium, σ_i are the principal values of the stress tensor, T is the temperature (see, for example, [2]). The tensor $\| h_{ij} \| = -1/2 \ln \| g_{ij} \|$ is called the Hankey strain

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tensor. As regards the tensor g_{ij} in a coordinate system independent of the principal axes of the strain tensor, the equation is obtained for the stress tensor, namely

$$\sigma_{ij} = -2\rho \frac{\partial E}{\partial g_{i\alpha}} g_{\alpha j}$$

These Murnaghan formulas give the stress tensor in terms of the strain tensor, that is, they are elastic relations for the hyperelastic medium under consideration. The density is calculated by the formula $\rho = \rho_0 K_3^{1/3}$ where ρ_0 is the density of the unstrained medium. In our further considerations matrix notation is used; thus by using the notation

$$\Sigma = \| \sigma_{ij} \|, \quad G = \| g_{ij} \|, \quad \frac{\partial E}{\partial G} = \left\| \frac{\partial E}{\partial g_{ij}} \right\|$$

one obtains the Murnaghan formulas in the matrix notation,

$$\Sigma = -2\rho \frac{\partial E}{\partial G} G \tag{2}$$

The use of the thermodynamic identity (1) whose corollaries are the Murnaghan formulas (2) enables one to formulate a complete system of differential equations for the nonlinear elasticity theory,

$$\rho\, dE/dt - \sigma_{ij} \partial u_i / \partial x_j = 0 \tag{3}$$

$$\rho du_i/dt - \partial \sigma_{ij} / \partial x_j = 0 \tag{4}$$

$$dg_{ij}/dt + g_{i\alpha} \partial u_\alpha / \partial x_j + g_{j\alpha} \partial u_\alpha / \partial x_i = \varphi_{ij} \tag{5}$$

In the above $d/dt = \partial/\partial t + u_\alpha (\partial / \partial x_\alpha)$ denotes the derivative along the trajectory of the motion. Equation (3) is the law of energy conservation, Eq. (4) the momentum equation, Eq. (5) describes the time change of the strain tensor. The tensor φ_{ij} is the velocity tensor of plastic deformations. The model can be made complete with the aid of Maxwell viscosity which ensures relaxation of the tangential stresses. If the deformation of the medium takes place in the elastic domain only one should assume $\varphi_{ij} = 0$; therefore, subsequent results are valid for the elastic-medium model with no need for additional assumptions.

The actual form of φ_{ij} is not discussed since this was done in [2]. We only notice that φ_{ij} are the "first" terms of the differential equations, that is, they do not contain any derivatives.

One fact, however, should be noted; in view of the system of Eqs. (3)-(5) the thermodynamic identity (1) implies the law of increasing entropy [2],

$$dS/dt = \varkappa \geqslant 0 \tag{6}$$

Equation (6) can be obtained as a linear combination of Eqs. (3)-(5). The right-hand side $\varkappa$ then represents the corresponding combination of the right-hand sides φ_{ij} (if $\varphi_{ij} = 0$ then $\varkappa = 0$).

The system of equations of the nonlinear elasticity theory (3)-(5) can also be written in a hypoelastic form; to this end one has to proceed from Eqs. (5) which describe the time change of the strain tensor to equations which describe the time change of the stress tensor. A very general form which constitutes a basis for various models of hypoelastic relations can be found in the survey [3], namely

$$\begin{gathered} d\Sigma/dt + \Sigma U + U^* \Sigma = \alpha_0 I \operatorname{tr} W + \alpha_1 W + \alpha_2 \Sigma \operatorname{tr} W + \\ + \alpha_3 I \operatorname{tr}(\Sigma W) + {}^1/_2 \alpha_4 (\Sigma W + W\Sigma) + \alpha_5 \Sigma^2 \operatorname{tr} W + \alpha_6 \Sigma \operatorname{tr}(\Sigma W) + \\ + \alpha_7 I \operatorname{tr}(\Sigma^2 W) + {}^1/_2 \alpha_8 (\Sigma^2 W + W \Sigma^2) + \alpha_9 \Sigma^2 \operatorname{tr}(\Sigma W) + \alpha_{10} \Sigma \operatorname{tr}(\Sigma^2 W) + \alpha_{11} \Sigma^2 \operatorname{tr}(\Sigma^2 W) \\ U = \| u_{ij} \| = \| \partial u_i / \partial x_j \|, \quad W = {}^1/_2 (U + U^*) \end{gathered} \tag{7}$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{11}$ are any continuous scalar functions of the three principal invariants of the matrix $\Sigma : \operatorname{tr} \Sigma, \operatorname{tr} \Sigma^2, \operatorname{tr} \Sigma^3$. The model of the hypoelastic medium depends on the choice of the arbitrary coefficients α_i. The thermodynamics of such models is still at present not clear.

It will be shown below that Eqs. (5) and (6) together with the Murnaghan formulas (2) imply (if the plastic terms φ_{ij} are ignored) hypoelastic relations where the coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{11}$ are computed by specific formulas resulting from the thermodynamics of the equations.

Equation (5) is rewritten in matrix form as

$$dG/dt = -GU - U^* G + \Phi, \quad \Phi = \| \varphi_{ij} \| \tag{8}$$

Using the expressions for the invariants K_1, K_2, K_3 in terms of the g_{ij} it can be shown that

$$\partial K_1/\partial G = I, \quad \partial K_2/\partial G = K_1 I - G, \quad \partial K_3/\partial G = G^2 - K_1 G + K_2 I \tag{9}$$

Moreover, it follows from the Cayley–Hamilton theorem for the matrix G that any integral power of the matrix G can be represented by a second-degree polynomial in the matrix G with coefficients depending only on the invariants K_1, K_2, K_3.

By using (2) and (9) the Murnaghan formulas are transformed into

$$\Sigma = l_0 I + l_1 G + l_2 G^2 \tag{10}$$

$$l_0 = -2\rho_0 K_3^{3/2} \frac{\partial E}{\partial K_3}, \quad l_1 = -2\rho_0 K_3^{1/2}\left(\frac{\partial E}{\partial K_1} + K_1 \frac{\partial E}{\partial K_2}\right),$$
$$l_2 = 2\rho_0 K_3^{1/2} \frac{\partial E}{\partial K_2} \tag{11}$$

It is expedient to consider the invariants tr G, tr G^2, tr G^3 side by side with the invariants K_1, K_2, K_3 which are related by

$$K_1 = \mathrm{tr}\, G, \quad K_2 = {}^1/_2\, [(\mathrm{tr}\, G)^2 - \mathrm{tr}\, G^2], \quad K_3 = {}^1/_6\, [(\mathrm{tr}\, G)^3 -$$
$$- 3\mathrm{tr}\, G\, \mathrm{tr}\, G^2 + 2\mathrm{tr}\, G^3]$$

Equations (8) then yield

$$d\, \mathrm{tr}\, G/dt = -2\mathrm{tr}\,(GW) + \mathrm{tr}\,\Phi, \quad d\, \mathrm{tr}\, G^2/dt = -4\, \mathrm{tr}\,(G^2 W) + 2\mathrm{tr}\,(G\Phi)$$
$$d\, \mathrm{tr}\, G^3/dt = -6K_3\, \mathrm{tr}\, W + 6K_2\, \mathrm{tr}\,(GW) - 6K_1\, \mathrm{tr}\,(G^2W) + 3\mathrm{tr}\,(G^2\Phi), \quad dK_1/dt = -2\mathrm{tr}\,(GW) + \mathrm{tr}\,\Phi$$
$$dK_2/dt = -2K_1\, \mathrm{tr}\,(GW) + 2\mathrm{tr}\,(G^2W) + \mathrm{tr}\,[(K_1 I - G)\,\Phi],$$
$$dK_3/dt = -2K_3\, \mathrm{tr}\, W + \mathrm{tr}\,(G^{-1}\Phi) \tag{12}$$

It is noted that since the equation K_3 implies the continuity equation then the identity

$$\mathrm{tr}\,(G^{-1}\Phi) = 0$$

must be valid; the latter was considered in [2] as a constraint on the manner of introducing the relaxation terms φ_{ij}.

The equation for the time-change of the stress matrix is now obtained. It follows from (10) that

$$\frac{d\Sigma}{dt} = l_1 \frac{dG}{dt} + l_2 \frac{dG}{dt} G + l_2 G \frac{dG}{dt} + \left(\frac{\partial l_0}{\partial K_i} I + \frac{\partial l_1}{\partial K_i} G + \frac{\partial l_2}{\partial K_i} G^2\right) \frac{dK_i}{dt} + \left(\frac{dl_0}{\partial S} I + \frac{\partial l_1}{\partial S} G + \frac{\partial l_2}{\partial S} G^2\right) \frac{dS}{dt}$$

Using (6), (8), (12) one obtains

$$\frac{d\Sigma}{dt} = -\Sigma U - U^*\Sigma + 4l_0 W - 2l_2 GWG + Q_0\, \mathrm{tr}\, W + Q_1\, \mathrm{tr}\,(GW) + Q_2\, \mathrm{tr}\,(G^2W) + \Psi \tag{13}$$

$$Q_i = q_{ij} G^j \quad (i, j = 0, 1, 2)$$
$$q_{00} = -2K_3 \frac{\partial l_0}{\partial K_3}, \quad q_{01} = -2K_3 \frac{\partial l_1}{\partial K_3}, \quad q_{02} = -2K_3 \frac{\partial l_2}{\partial K_3}$$
$$q_{10} = -2\left(\frac{\partial l_0}{\partial K_1} + K_1 \frac{\partial l_0}{\partial K_2}\right), \quad q_{11} = -2\left(\frac{\partial l_1}{\partial K_1} + K_1 \frac{\partial l_1}{\partial K_2}\right),$$
$$q_{12} = -2\left(\frac{\partial l_2}{\partial K_1} + K_1 \frac{\partial l_2}{\partial K_2}\right) \tag{14}$$
$$q_{20} = 2\frac{\partial l_0}{\partial K_2}, \quad q_{21} = 2\frac{\partial l_1}{\partial K_2}, \quad q_{22} = 2\frac{\partial l_2}{\partial K_2}$$
$$\Psi = l_1\Phi + l_2(G\Phi + \Phi G) - \frac{1}{2} Q_1\, \mathrm{tr}\,\Phi - \frac{1}{2} Q_2\, \mathrm{tr}\,(G\Phi)$$

The right-hand side of (13) is now transformed using a generalization of Cayley–Hamilton formula (see, for example, [3]),

$$GWG = -G^2W - WG^2 + K_1(GW + WG) - K_2 W +$$
$$+ (G^2 - K_1 G + K_2 I)\, \mathrm{tr}\, W + (G - K_1 I)\, \mathrm{tr}\,(GW) + I\, \mathrm{tr}\,(G^2W)$$

Then (13) can be rewritten as

$$d\Sigma/dt = -\Sigma U - U^*\Sigma + 2\,(2l_0 + K_2 l_2)\, W - 2K_1 l_2\,(GW +$$
$$+ WG) + 2l_2\,(G^2W + WG^2) + R_0\, \mathrm{tr}\, W + R_1\, \mathrm{tr}\,(GW) + R_2\, \mathrm{tr}\,(G^2W) + \Psi$$
$$R_i = r_{ij} G^j\ (i, j = 0, 1, 2) \tag{15}$$

$$r_{00} = q_{00} - 2K_2l_2, \quad r_{01} = q_{01} + 2K_1l_2, \quad r_{02} = q_{02} - 2l_2$$
$$r_{10} = q_{10} + 2K_1l_2, \quad r_{11} = q_{11} - 2l_2, \quad r_{12} = q_{12}$$
$$r_{20} = q_{20} - 2l_2, \quad r_{21} = q_{21}, \quad r_{22} = q_{22}$$

The above can be written in a form similar to (7),

$$\begin{aligned} d\Sigma/dt + \Sigma U + U^*\Sigma = \beta_0 I \operatorname{tr} W + \beta_1 W + \beta_2 G \operatorname{tr} W + \\ + \beta_3 I \operatorname{tr}(GW) + {}^1/_2\beta_4 (GW + WG) + \beta_5 G^2 \operatorname{tr} W + \beta_6 G \operatorname{tr}(GW) + \\ + \beta_7 I \operatorname{tr}(G^2W) + {}^1/_2\beta_8(G^2W + WG^2) + \beta_9 G^2 \operatorname{tr}(GW) + \beta_{10} G \operatorname{tr}(G^2W) + \beta_{11} G^2 \operatorname{tr}(G^2W) + \Psi \end{aligned} \tag{16}$$

$$\begin{aligned} &\beta_0 = r_{00}, \ \beta_1 = 2(2l_0 + K_2l_2), \ \beta_2 = r_{01}, \ \beta_3 = r_{10}, \ \beta_4 = -4K_1l_2, \\ &\beta_5 = r_{02}, \quad \beta_6 = r_{11}, \quad \beta_7 = r_{20}, \quad \beta_8 = 4l_2, \quad \beta_9 = r_{12}, \quad \beta_{10} = r_{21}, \\ &\beta_{11} = r_{22} \end{aligned} \tag{17}$$

Thus a relation close to the hypoelastic one has been obtained which differs from (7) in that the right-hand side of (16) depends on G and not on Σ. The coefficients $\beta_0, \beta_1, \ldots, \beta_{11}$ as well as the derivatives of the energy are uniquely determined by the invariants K_1, K_2, K_3.

The matrix Ψ describes the attenuation of tangential stresses; only in the case of elastic strains one sets $\Psi = 0$.

It is usual when constructing a complete system of equations to proceed from the relations (7) to the continuity equation and to the equations for the deviator of the stress tensor, $\Sigma' = \Sigma - 1/3 \ I \operatorname{tr} \Sigma$ which can be obtained from (7),

$$\begin{aligned} d\Sigma'/dt + {}^1/_2\Sigma'(U - U^*) - {}^1/_2(U - U^*)\Sigma' = a_1(W - {}^1/_3 I \operatorname{tr} W) + \\ + a_2\Sigma' \operatorname{tr} W + a_3(\Sigma'W + W\Sigma' - {}^2/_3 I \operatorname{tr}(\Sigma'W)) + \\ + a_4(\Sigma'^2 - {}^1/_3 I \operatorname{tr} \Sigma'^2) \operatorname{tr} W + a_5\Sigma' \operatorname{tr}(\Sigma'W) + a_6(\Sigma'^2 W + \\ + W\Sigma'^2 - {}^2/_3 I \operatorname{tr}(\Sigma'^2)) + a_7(\Sigma'^2 - {}^1/_3 I \operatorname{tr} \Sigma'^2) \operatorname{tr}(\Sigma'W) + \\ + a_8 \Sigma' \operatorname{tr}(\Sigma'^2 W) + a_9(\Sigma'^2 - {}^1/_3 I \operatorname{tr} \Sigma'^2) \operatorname{tr}(\Sigma'^2 W) \end{aligned} \tag{18}$$

where the coefficients a_i are determined by the coefficients α_i in Eq. (7).

Equation (16) can be written in the form (18). To this end a hypothesis is required that the strain tensor can be computed from the stress tensor with the aid of the equation of state (for the entropy remaining constant). The hypothesis is formulated as follows:

1) for a fixed deviator of the stress tensor the density can be calculated from the known pressure;

2) for a fixed density the deviator of the strain tensor can be calculated from the known deviator of the stress tensor.

An example of an equation of state is now given which satisfies these requirements. In the principal axes of the tensor h_{ik} one has

$$\sigma_i = \rho E_{h_i}(h_1, h_2, h_3, S), \quad \rho = \rho_0 \exp[-(h_1 + h_2 + h_3)]$$

Let us consider the state equation

$$E = E^{(0)}(\rho, S) + 2f(\rho, S)D$$
$$D = {}^1/_2(d_1^2 + d_2^2 + d_3^2), \quad d_i = h_i - {}^1/_3(h_1 + h_2 + h_3)$$

One finds

$$\sigma_i = -p(\rho, D, S) + 2f(\rho, S)d_i$$
$$p(\rho, D, S) = -{}^1/_3(\sigma_1 + \sigma_2 + \sigma_3) = \rho^2 E_\rho^{(0)} + 2\rho^2 f_\rho D$$

Hence

$$d_i = \frac{1}{2f(\rho, S)}\left(\sigma_i - \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}\right)$$

which is an illustration of part 2) of the hypothesis.

Moreover,

$$p = \rho^2 E_\rho^{(0)}(\rho, S) + \frac{\rho^2 f_\rho(\rho, S)}{4f^2(\rho, S)} \sum_{i=1}^{3} \left(\sigma_i - \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}\right)^2$$

and for part 1) of the hypothesis to be valid one needs

$$\partial p / \partial \rho \neq 0$$

It is usual when using the hypoelastic relations on the right-hand side of Eq. (18) to neglect the terms containing Σ' and higher powers of Σ'. The complete system of equations is then (see, for example, [4])

$$\frac{d\rho}{dt} + \rho \frac{\partial u_i}{\partial x_j} \delta_{ij} = 0, \quad \rho \frac{du_i}{dt} + \frac{\partial p}{\partial x_i} - \frac{\partial \sigma_{ij}'}{\partial x_j} = 0$$

$$\rho \frac{dE}{dt} + p \frac{\partial u_i}{\partial x_j} \delta_{ij} - \sigma_{ij}' \frac{\partial u_i}{\partial x_j} = 0 \tag{19}$$

$$\frac{d\sigma_{ij}'}{dt} + {}^1/_2 \sigma_{i\alpha}' \left(\frac{\partial u_\alpha}{\partial x_j} - \frac{\partial u_j}{\partial x_\alpha}\right) - {}^1/_2 \sigma_{\alpha j}' \left(\frac{\partial u_i}{\partial x_\alpha} - \frac{\partial u_\alpha}{\partial x_i}\right) = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_\alpha}{\partial x_\beta} \delta_{\alpha\beta} \delta_{ij}\right)$$

where μ is identical with a_1 in (18). In the model (19) one usually assumes that $\mu = \mu(\rho)$. The thermodynamic properties of the equations are infringed as a result of neglecting the terms with Σ'. Whereas for the system (3)-(5) the conservation law holds for the entropy (6), if the viscous terms are neglected, for the system (19) with the equation of state $E = E(\rho, \sigma_{ij}', S)$ one has $p = \rho^2 E_\rho$

$$E_s \frac{dS}{dt} = \frac{dE}{dt} - E_\rho \frac{d\rho}{dt} - \frac{\partial E}{\partial \sigma_{ij}'} \frac{d\sigma_{ij}'}{dt} = \frac{1}{\rho} \sigma_{ij}' \frac{\partial u_i}{\partial x_j} + \sigma_{ik}' \frac{\partial E}{\partial \sigma_{ij}'} \left(\frac{\partial u_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k}\right) -$$
$$- \frac{\mu}{\rho} \frac{\partial E}{\partial \sigma_{ij}'} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{2\mu}{3\rho} \left(\frac{\partial E}{\partial \sigma_{11}'} + \frac{\partial E}{\partial \sigma_{22}'} + \frac{\partial E}{\partial \sigma_{33}'}\right) \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)$$

In view of the fact that the terms with Σ' are neglected, the right-hand side of the above equation is not equal to $\varkappa$ on the right-hand side of (16). In particular, for purely elastic processes ($\varphi_{ij} = 0$) the law of conservation of entropy does not hold for the solutions (19).

Thus the system (3)-(5) can be reduced to a hypoelastic form if the strain tensor can be calculated from the stress tensor. The coefficients appearing in the hypoelastic form can be uniquely evaluated from the state equation. Arbitrariness in the choice of coefficients may result in an infringement of thermodynamics. The system (3)-(5) has a more suitable form for computations than its hypoelastic form since strains can be expressed by stresses only for simple forms of the equations of state.

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