WAVES IN A PRESTRESSED ELASTIC LAYER IN COMPRESSION INTERACTING WITH AN IDEAL FLUID

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The laws established for the propagation of waves in laminated media are widely used in seismology, seismic exploration, acoustics, and other disciplines [2-6, 16-20]. There are also the studies [6, 16, 17], which examined wave processes in such systems as part of a larger investigation of the transmission of signals in large bodies of water covered with ice. These studies are of considerable interest and are being used in the solution of several important practical problems in underwater acoustics. At the same time, most of the results in [2-6, 16-20] were obtained using the classical theory of elasticity. This approach makes it impossible to describe many of the properties of actual solids or to examine and explain the observed effects. Among the models that more fully reflect the behavior of actual elastic bodies is the model of a prestressed body in [7-12]. This model makes it possible to consider the initial stresses present in real materials and obtain information on their effect on the characteristics of the wave process.

In connection with this, we use this model here to study the effect of preliminary deformation on the phase velocities of waves in a system comprised of an elastic layer and a fluid. We will formulate hydroelastic problems for bodies with initial stresses and solve them in accordance with [8-12] in order to analyze the propagation of small perturbations in a compressible layer interacting with an ideal fluid. We will conduct our investigation in the coordinates of the uniform initial state z_i within the framework of linearized three-dimensional equation [8, 10-12]. We then restrict ourselves to consideration of the plane problem by assuming that the external forces acting on the elastic and liquid media are distributed uniformly along the axis oz_3. In this case, the system of equations describing joint motions will have the form

$$\left(\tilde{\omega}_{ij\alpha\beta}\frac{\partial^2}{\partial z_i \partial z_\beta} - \delta_{j\alpha}\tilde{\rho}\frac{\partial^2}{\partial t^2}\right)u_\alpha = 0; \tag{1}$$

$$\tilde{Q}_j \equiv N_i^0\tilde{\omega}_{ij\alpha\beta}\frac{\partial u_\alpha}{\partial z_\beta}; \tag{2}$$

$$\frac{\partial \vec{v}}{\partial t} + \frac{1}{\rho_0}\vec{\nabla}p = 0; \tag{3}$$

$$\frac{1}{\rho_0}\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{v} = 0; \tag{4}$$

$$(4)\frac{\partial p}{\partial \rho} = a_0^2, \quad a_0 = \text{const}; \tag{5}$$

$$p_{ij} = -p\delta_{ij}, \quad P_j = p_{ij}N_i^0; \tag{6}$$

$$\tilde{Q}_1\Big|_{z_2=h} = 0, \quad \tilde{Q}_2\Big|_{z_2=h} = 0; \tag{7}$$

$$\tilde{Q}_1\Big|_{z_2=0} = 0, \quad \tilde{Q}_2\Big|_{z_2=0} = P_2\Big|_{z_2=0}, \quad \frac{\partial u_2}{\partial t}\Big|_{z_2=0} = v_2\Big|_{z_2=0} \tag{8}$$

Here, Eq. (1) describes the motion of an elastic body subjected to uniform initial strains; Eqs. (2) are used to determine the components of the stress vector on the surface of the solid; Eq. (3) describes small vibrations of a quiescent ideal compressible

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fluid; Eq. (4) is the continuity condition; Eq. (5) is the linearized equation of state of the liquid medium; Eqs. (6) determine the stresses in the liquid; the absence of stresses on the free surface of the elastic layer of thickness h is characterized by Eqs. (7); Eqs. (8) are nothing more than the dynamic and kinematic conditions on the interface of the ideal liquid half-space and the compressible elastic layer. We will use the representations of general solutions proposed in [7-12] to simplify the solution. In the case of plane strain being considered here, they will have the form

$$u_1 = -\frac{\partial^2 \chi_1}{\partial z_1 \partial z_2};$$

$$u_2 = \frac{(\lambda_1^2 a_{11} + S_{11}^0)}{\lambda_2^2(a_{12} + \mu_{12})}\left[\frac{\partial^2}{\partial z_1^2} + \lambda_1^{-2}\lambda_2^2\frac{(\lambda_1^2\mu_{12} + S_{22}^0)}{(\lambda_1^2 a_{11} + S_{11}^0)}\frac{\partial^2}{\partial z^2} - \frac{\rho\lambda_1^{-2}}{(\lambda_1^2 a_{11} + S_{11}^0)}\frac{\partial^2}{\partial t^2}\right]\chi_1; \tag{9}$$

$$\vec{v} = \frac{\partial}{\partial t}\vec{\nabla}\chi_2; \quad p = -\rho_0\frac{\partial^2 \chi_2}{\partial t^2}, \tag{10}$$

where the potentials χ_i satisfy the equations

$$\left[\left(\frac{\partial^2}{\partial z_1^2} + \frac{\lambda_2^2(\lambda_1^2\mu_{12} + S_{22}^0)}{\lambda_1^2(\lambda_1^2 a_{11} + S_{11}^0)}\frac{\partial^2}{\partial z^2} - \frac{\rho}{\lambda_1^2(\lambda_1^2 a_{11} + S_{11}^0)}\frac{\partial^2}{\partial t^2}\right)\right.$$

$$\times\left(\frac{\partial^2}{\partial z_1^2} + \frac{\lambda_2^2(\lambda_2^2 a_{22} + S_{22}^0)}{\lambda_1^2(\lambda_2^2\mu_{12} + S_{11}^0)}\frac{\partial^2}{\partial z_2^2} - \frac{\rho}{\lambda_1^2(\lambda_2^2\mu_{12} + S_{11}^0)}\frac{\partial^2}{\partial t^2}\right) \tag{11}$$

$$\left.-\frac{\lambda_2^4(a_{12} + \mu_{12})^2}{(\lambda_1^2 a_{11} + S_{11}^0)(\lambda_2^2\mu_{12} + S_{11}^0)}\frac{\partial^4}{\partial z_1^2 \partial z_2^2}\right]\chi_1 = 0;$$

$$\left(\frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2} - \frac{1}{a_0^2}\frac{\partial^2}{\partial t^2}\right)\chi_2 = 0. \tag{12}$$

Here and below, we use the following notation: v_i represents components of the fluid-velocity perturbation vector $\vec{v}$; ρ and p are perturbations of density and pressure in the liquid; ρ_0 and a_0 are density and sonic velocity for the liquid in the quiescent state; u_i are components of the vector describing the displacement of the solid $\vec{u}$; $\tilde{\rho}$ is the density of the material of the elastic layer; λ_i are the initial extensions; N_i^0 are components of a unit normal to the surface of the body in the initial deformed state; S_{ii}^0 are the initial stresses; a_{ij} and μ_{ij} are coefficients of the equations of state of the solid, dependent on the form of the elastic potential (expressions for them were given in [9]); Q_j are components of the stress vector on the surface of the solid; P_j are components of the stress vector in the liquid.

Study of the dispersion properties of the given system is of particular interest in the analysis of wave processes in laminated media. In connection with this, we will henceforth focus on deriving the characteristic equation. We will obtain it in the most general case, for prestressed compressible bodies whose elastic properties are described by a potential of arbitrary form. To do this, we will seek the parameters characterizing the wave process within the class of harmonic travelling waves. It should be noted that that the functions chosen here

$$\chi_j = X_j(z_1)\exp[i(kz_2 - \omega t)], \quad (j=1,2), \tag{13}$$

are the simplest and most convenient for theoretical studies and do not significantly limit the generality of the results that are obtained. Inserting (13) into (11) and (12), we obtain two uncoupled ordinary differential equations. Based on physical considerations, the following functions will be solutions of these equations:

$$X_1(z_2) = A_1 e^{\alpha_1 z_2} + B_1 e^{-\alpha_1 z_2} + A_2 e^{\alpha_2 z_2} + B_2 e^{-\alpha_2 z_2}, \tag{14}$$

$$X_2(z_2) = C_1 e^{\alpha_3 z_2}, \tag{15}$$

where

$$\alpha_{1,2}^2 = \frac{-\sum_{i=1}^{3} a_i \pm \sqrt{(\sum_{i=1}^{3} a_i)^2 - 4a_1a_2}}{2}; \quad \alpha_3^2 = k^2 - \frac{\omega^2}{a_0^2};$$

$$a_1 = \frac{\rho\omega^2 - \lambda_1^2 k^2(\lambda_1^2 a_{11} + S_{11}^0)}{\lambda_2^2(\lambda_1^2\mu_{12} + S_{22}^0)}; \quad a_2 = \frac{\rho\omega^2 - \lambda_1^2 k^2(\lambda_2^2\mu_{12} + S_{11}^0)}{\lambda_2^2(\lambda_2^2 a_{22} + S_{22}^0)};$$

$$a_3 = \frac{k^2\lambda_1^4(a_{12} + \mu_{12})^2}{(\lambda_1^2\mu_{12} + S_{22}^0)(\lambda_2^2 a_{22} + S_{22}^0)}. \tag{16}$$

To derive the dispersion equation, we take solutions (14) and (15) and insert them in succession into Eqs. (13) and boundary conditions (7) and (8). After completing the obvious transformations, we obtain a system of five linear homogeneous algebraic equations. We then change these equations to dimensionless form, introducing the dimensionless quantities as follows:

$$\bar{\mu}_{ij} = \frac{\mu_{ij}}{\mu}; \quad \bar{a}_{ij} = \frac{a_{ij}}{\mu}, \quad \bar{S}_{ij}^0 = \frac{S_{ij}^0}{\mu}, \quad \bar{a}_0 = \frac{a_0}{c_S},$$
$$c_S^2 = \frac{\mu}{\rho}, \quad \bar{\rho}_0 = \frac{\rho_0}{\rho}, \quad \bar{c} = \frac{c}{c_S}, \quad \bar{h} = k_S h. \tag{17}$$

After we equate the determinant of this system to zero, we obtain the sought dispersion equation. In dimensionless quantities, it will have the form

$$\begin{vmatrix} f_1 & f_2 & f_1 & f_2 & 0 \\ g_1 & g_2 & -g_1 & -g_2 & \rho_0 c^3 \\ l_1 & l_2 & l_1 & l_2 & \alpha_3 c \\ f_1 m_1 & f_2 m_2 & f_1 n_1 & f_2 n_2 & 0 \\ g_1 m_1 & g_2 m_2 & -g_1 n_1 & -g_2 n_2 & 0 \end{vmatrix} = 0, \tag{18}$$

where

$$f_j = -\frac{\lambda_2 a_{12}(\lambda_1^2\mu_{12} + S_{22}^0)\alpha_j^2}{\lambda_1\lambda_3(a_{12} + \mu_{12})} - \frac{\lambda_1\mu_{12}(\lambda_1^2 a_{11} + S_{11}^0)}{\lambda_2\lambda_3(a_{12} + \mu_{12})} + \frac{\mu_{12}c^2}{\lambda_1\lambda_2\lambda_3(a_{12} + \mu_{12})};$$

$$g_j = \frac{\lambda_1\lambda_2 a_{12}\alpha_j}{\lambda_3} - \frac{(\lambda_2^2 a_{22} + S_{22}^0)(\lambda_1^2 a_{11} + S_{11}^0)\alpha_j}{\lambda_1\lambda_2\lambda_3(a_{12} + \mu_{12})}$$
$$+ \frac{\lambda_2(\lambda_1^2\mu_{12} + S_{22}^0)(\lambda_2^2 a_{22} + S_{22}^0)\alpha_j^3}{\lambda_1^3\lambda_3(a_{12} + \mu_{12})} + \frac{(\lambda_2^2 a_{22} + S_{22}^0)\alpha_j c^2}{\lambda_1^3\lambda_2\lambda_3(a_{12} + \mu_{12})};$$

$$l_j = \frac{(\lambda_1^2 a_{11} + S_{11}^0)}{\lambda_2^2(a_{12} + \mu_{12})} - \frac{(\lambda_1^2\mu_{12} + S_{22}^0)\alpha_j^2}{\lambda_1^2(a_{12} + \mu_{12})} - \frac{c^2}{\lambda_1^2\lambda_2^2(a_{12} + \mu_{12})};$$

$$m_j = e^{\frac{\alpha_j h}{c}}; \quad n_j = e^{-\frac{\alpha_j h}{c}}; \quad j = 1, 2.$$

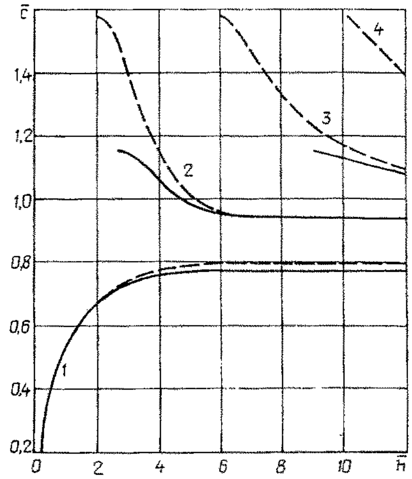

Fig. 1

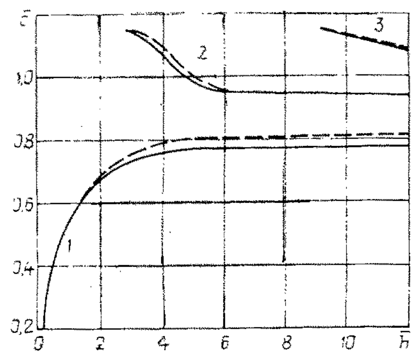

Fig. 2

We should note that we omitted the bars above the dimensionless quantities in the above expressions to simplify the notation.

Dispersion equation (18) is a general equation and can be used to obtain relations characterizing wave processes for a number of special cases. For example, if we make the thickness of the elastic layer h approach infinity, we obtain two half-spaces. The propagation of surface waves along their interface was examined in [13, 15]. If we also set ρ_0 equal to zero in this case, then (18) becomes an equation that describes the parameters of surface waves in an elastic half-space. This topic was studied in [1]. Given these simplifications, the characteristic equations we obtain within the framework of linearized theory are also valid for arbitrary prestressed compressible bodies whose form is independent of the elastic potential.

The results of studies conducted earlier using the classical theory of elasticity [2-5] can also be obtained from (18). This requires only that λ_i approach unity and that S_{ii}^0 approach zero in (18). With the additional simplifications mentioned above, (18) becomes the well-known and widely studied equations of Rayleigh [5, 22] and Strouhal [5, 23].

We subsequently solved dispersion equation (18) numerically on a computer, examining the case when the layer is loaded only along the oz_1 axis, i.e., $S_{11}^0 \neq 0$, $S_{22}^0 = 0$. As is known [11, 12], there is no analogy between linear and linearized problems for such an initial stress state, and results for predeformed bodies cannot be obtained from solutions from the classical theory of elasticity. It should be noted that we used the theory of large (finite) initial strains [7, 12] in deriving dispersion equation (18). The latter is thus a general equation and, with additional simplifications [9, 12], can be used to obtain (as special cases) equations that are valid for variants of the theory of small initial strains. We chose organic glass as the material for the elastic layer for our numerical calculations. As is known, organic glass is a brittle material and can be subjected only to small initial strains before fracturing. We therefore examined uniform initial states with relatively small strains ($\sigma_{11}^0 = 0.004$). As a result, the coefficients of the equations of state of the elastic layer were determined within the framework of the theory of acoustoelasticity [14]. Also, as was already noted, dispersion equation (18) was derived for elastic potentials described by arbitrary functions that we assumed could be continuously differentiated only twice. No other restrictions were imposed on the functions. At the same time, the form chosen for the elastic potential is important in the numerical realization of the problem and can have a significant effect on the results of theoretical calculations. As was noted in [14], to have the theoretical results agree satisfactorily with experimental observations for compressible prestressed materials characterized by a high degree of stiffness (organic glass, steel), it is necessary to use elastic potentials that depend on three invariants. In light of this, we will use the simplest three-invariant potential — Murnaghan's potential [21] — to describe the elastic properties of a solid layer made of organic glass.

Figures 1-4 show the results of numerical solution of dispersion equation (18). Figures 1 and 2 show the dependence of dimensionless phase velocity $\bar{c}(\bar{c} = c/c_s,\ c_s^2 = \mu/\rho)$ on the thickness of the elastic layer $\bar{h}$, ($\bar{h} = k_s h$, k_s is the wave number of the shear waves in the body without initial stresses) in the absence of initial strains ($\bar{\sigma}_{11}^0 = 0$).

The character of the effect of pretensioning ($\bar{\sigma}_{11}^0 = 0.004$) is illustrated in Figs. 3 and 4 by graphs showing the dependence of the relative change in phase velocity $c_\varepsilon[c_\varepsilon = (c_0 - c)/c]$, c is phase velocity for the modes in the body with-out initial stresses and c_0 is the phase velocity in the prestrained body) on the thickness of the elastic layer $\bar{h}$. The

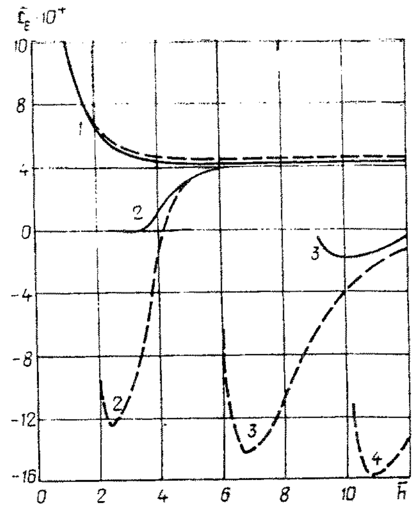

Fig. 3

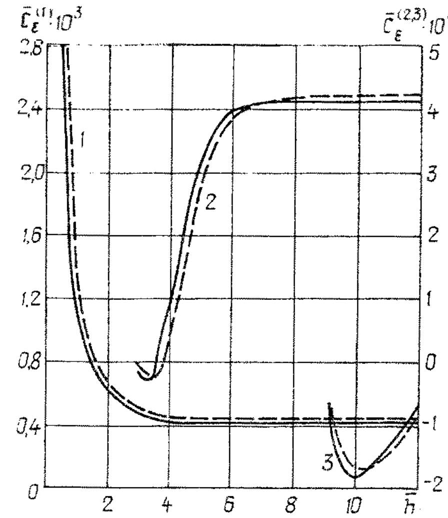

Fig. 4

calculations were performed for organic glass and liquids characterized by the following parameters: organic glass $-a = -3.91 \cdot 10^9$ Pa; $b = -7.02 \cdot 10^9$ Pa; $c = -14.1 \cdot 10^9$ Pa; $\rho = 1160$ kg/m^3; $\lambda = 3.96 \cdot 10^9$ Pa; $\mu = 1.86 \cdot 10^9$ Pa [12, 14], liquid $-\rho_0 = 700$ kg/m^3, $\rho_0 = 1000$ kg/m^3; $a_0 = 1459$ m/sec; $a_0 = 2000$ m/sec.

Figure 1 illustrates the effect of sonic velocity in the liquid on the velocities associated with the modes of the system. The dispersion curves were obtained for liquid media characterized by the following parameters: $\rho_0 = 1000$ kg/m^3 and $a_0 = 2000$ m/sec (dashed lines); $\rho_0 = 1000$ kg/m^3 and $a_0 = 1459$ m/sec (solid lines). It is not hard to see that the compressibility of the fluid significantly alters the wave properties of the system. The effect is particularly noticeable in regard to the frequencies at which the second and subsequent modes are excited. For the first mode, phase velocity typically increases only with an increase in the thickness of the elastic layer. Our analysis shows that use of the approximate model of an incompressible liquid in calculations leads to greatly exaggerated values of phase velocity for the higher modes and gross errors in the determination of the frequencies at which these modes are excited.

Figure 2 shows features of the effect of liquid density. It shows dispersion curves obtained for liquids characterized by the same sonic velocity ($a_0 = 1459$ m/sec) but different densities. Here, the solid lines correspond to the case when the elastic layers interact with a fluid having the density 1000 kg/m^3. The dashed curves were obtained for a fluid with $\rho_0 = 700$ kg/m^3. For the relations in this figure as well, density affects phase velocity in the first mode only with an increase in the thickness of the elastic layer. For the higher modes, this effect is seen mainly in the number of the exciting frequencies.

Figures 3 and 4 reflect the effect of initial stresses on phase velocity. Figure 3 shows $\bar{c} = f(\bar{h})$ for fluid half-spaces characterized by different sonic velocities. The dashed lines were obtained with $a_0 = 2000$ m/sec, while the solid curves were obtained with $a_0 = 1459$ m/sec. An analysis of the graphs in this figure shows that the effect of initial stresses ($\bar{\sigma}_{11}{}^0 = 0.004$) is closely allied with the compressibility of the fluid. For the less compressible fluids ($a_0 = 2000$ m/sec), the presence of initial stresses leads to a decrease in phase velocity in the higher modes near the thicknesses at which these modes are excited. With a further increase in the thickness of the elastic layer, preliminary deformation increases phase velocity. Figure 4 illustrates the effect of initial tension ($\bar{\sigma}_{11}{}^0 = 0.004$) on wave velocity for fluids with different densities. This figure shows the function $\bar{c}_\varepsilon = f(\bar{h})$. The solid lines correspond to the case in which $\rho_0 = 1000$ kg/m^3, while the dashed lines were obtained for a hypothetical fluid with a density equal to 700 kg/m^3. Sonic velocity was assumed to be the same in both cases ($a_0 = 1459$ m/sec).

Thus, it follows from the relations shown in Figs. 3 and 4 that there are certain layer thicknesses for which initial stresses will have no effect on the phase velocities of the higher modes excited by the interaction of the layers with a fluid.

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