NUMERICAL MODELING OF THE DYNAMIC BEHAVIOR OF REINFORCED SHELLS OF REVOLUTION UNDER NONSTEADY LOADS

P. Z. Lugovoi and V. F. Meish

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The broad use of composite materials in high technology demands the further development of theoretical models of nonuniform structures. In accordance with the surveys [2, 7], an important direction in this regard is the study of transients in reinforced shells with allowance for the discrete placement of the ribs. Such studies are conducted using refined models.

In the present study, within the framework of the geometrically nonlinear theory of shells and the Timoshenko rod theory, we examined the equations of motion of reinforced shells with allowance for the discrete placement of the ribs. The shells that we will discuss are subjected to unsteady axisymmetric loading. We construct a numerical algorithm which allows us to effectively examine unsteady wave processes in the given structures. A numerical example is presented.

1. Formulation of the Problem. The reinforced shell is examined as a system consisting of the shell proper (the skin) and ribs (rings) rigidly attached to the skin along contact lines. The mathematical model describing the unsteady deformation of the structure is a hyperbolic system of nonlinear differential equations from the Timoshenko theory of shells and rods. In constructing the mathematical model of the equations of motion, we used the simplified variant of the nonlinear shell and rod theory proposed by Novozhilov [8]. It was assumed that the extensions, shears, and angles of rotation were small compared to unity. Here, the extensions and shears were of a higher order of smallness than the angles of rotation. The strain state of the skin can be determined by three components of the generalized displacement vector u, w, ϕ. To describe the strain state of the j-th rib, we will use a generalized vector giving the displacement of the center of gravity of the rib cross section with the components u_j, w_j, and ϕ_j. The contact conditions linking the middle surface of the skin and the centers of gravity of the cross sections of the reinforcing elements have the form

$$\begin{gathered} u_j = u(\alpha_j) + h_j \varphi(\alpha_j), \\ w_j = w(\alpha_j), \quad \varphi_j = \varphi(\alpha_j), \end{gathered} \tag{1.1}$$

where α_j is the coordinate of the line of contact of the j-th rib with the shell; $h_j = 0.5 \times h + H_j$; h is the thickness of the shell; and H_j is the distance from the axis of the j-th rib to the surface of the shell.

We will use the Hamilton–Ostrogradskii variational principle to derive the equations of motion of the reinforced shell. With allowance for the integral representation of contact conditions (1.1) in [1], we write the variational equation as follows:

$$\int_0^L \left\{ T_{11}\left[\frac{\partial}{A_1\partial\alpha_1}(\delta u) + \theta_1\frac{\partial}{A_1\partial\alpha_1}(\delta w) - \theta_1 k_1 \delta u\right] + \frac{T_{22}}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\delta u - \right.$$
$$- Q_1 k_1 \delta u + Q_{13}\frac{\partial}{A_1\partial\alpha_1}(\delta u) + T_{11}k_1\delta w + T_{22}k_2\delta w + M_{11}\frac{\partial}{A_1\partial\alpha_1}(\delta\varphi) +$$
$$+ \frac{M_{22}}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\delta\varphi + Q_{13}\delta\varphi + \sum_{j=1}^{l}(T_{22j}k_{2j}\delta w + M_{22j}k_{2j}\delta\varphi)\,\delta(\alpha_1 - \alpha_{1j}) +$$

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$$+\rho h\left(\frac{\partial^2 u}{\partial t^2}\delta u+\frac{\partial^2 w}{\partial t^2}\delta w+\frac{h^2}{12}\frac{\partial^2\varphi}{\partial t^2}\delta\varphi\right)+$$

$$+\sum_{j=1}^{l}\rho_j F_j\left[\left(\frac{\partial^2 u}{\partial t^2}\pm h_j\frac{\partial^2\varphi}{\partial t^2}\right)\delta u+\frac{\partial^2 w}{\partial t^2}\delta w+\right.$$

$$\left.\pm h_j\left(\frac{\partial^2 u}{\partial t^2}\pm h_j\frac{\partial^2\varphi}{\partial t^2}\right)\delta\varphi\right]\delta(\alpha_1-\alpha_{1j})-P_1\delta u-P_2\delta w-m_1\delta\varphi\Big\}A_1A_2 d\alpha_1- \tag{1.2}$$

$$-A_1A_2[T_{11}\delta u+(Q_{13}+\theta_1 T_{11})\delta w+M_{11}\delta\varphi]|_{s=0,L}=0,$$

where A_1 and A_2 are coefficients of the first quadratic form; k_1 and k_2 are the principle curvatures of the coordinate surface; $s = \alpha_1 A_1$; t represents the space and time coordinates; P_1, P_2, and m_1 are components of the generalized load vector; ρ and ρ_j are the densities of the materials of the skin and the j-th ring; F_j is the cross-sectional area of the j-th ring; and $\delta(\alpha_1 - \alpha_{1j})$ is the Dirac delta function.

In Eqs. (1.2), we introduced the following relations connecting components of the force-moment tensor of the middle surface of the shell and the components of the strain tensor:

$$T_{11}=\frac{E_1 h}{1-\nu_1\nu_2}(\varepsilon_{11}+\nu_2\varepsilon_{22}),\qquad T_{22}=\frac{E_2 h}{1-\nu_1\nu_2}(\varepsilon_{22}+\nu_1\varepsilon_{11}),$$

$$M_{11}=\frac{E_1 h^3}{12(1-\nu_1\nu_2)}(\varkappa_{11}+\nu_2\varkappa_{22}),\qquad M_{22}=\frac{E_2 h^3}{12(1-\nu_1\nu_2)}(\varkappa_{22}+\nu_1\varkappa_{11}), \tag{1.3}$$

$$Q_{13}=G_{13}k^2 h\varepsilon_{13},\qquad T_{22j}=E_jF_j\varepsilon_{22j},\qquad M_{22j}=E_jI_j\varkappa_{22j}.$$

The below expressions connect the components of the strain tensor in terms of the components of the displacement vector for the skin and the j-th ring

$$\varepsilon_{11}=\frac{1}{A_1}\frac{\partial u}{\partial\alpha_1}+\frac{1}{2}\theta_1^2+k_1 w,\qquad \varepsilon_{22}=\frac{u}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}+k_2 w,$$

$$\varepsilon_{13}=\varphi+\theta_1,\quad \theta_1=\frac{1}{A_1}\frac{\partial w}{\partial\alpha}-k_1 u,\qquad \varkappa_{11}=\frac{1}{A_1}\frac{\partial\varphi}{\partial\alpha_1}, \tag{1.4}$$

$$\varkappa_{22}=\frac{\varphi}{A_1A_2}\frac{\partial A_2}{\partial\alpha},\qquad \varepsilon_{22j}=wk_{2j},\qquad \varkappa_{22j}=\varphi k_{2j}.$$

We introduced the following notation in Eqs. (1.3-1.4): E_1, E_2, G_{13}, ν_1, ν_2, E_j are physico-mechanical parameters of the skin and the j-th ring; F_j and I_j are geometric parameters of the j-th ring.

By virtue of the independence of the variations δu, δw, and $\delta\phi$, we can use Eq. (1.2) to obtain the equations of motion of the reinforced shell in differential form

$$\frac{1}{A_1A_2}\left[\frac{\partial}{\partial\alpha_1}(A_2T_{11})-\frac{\partial A_2}{\partial\alpha_1}T_{22}\right]+(T_{11}\theta_1+Q_{13})k_1+P_1=$$

$$=\rho h\frac{\partial^2 u}{\partial t^2}+\sum_{j=1}^{l}\rho_jF_j\left(\frac{\partial^2 u}{\partial t^2}\pm h_j\frac{\partial^2\varphi}{\partial t^2}\right)\delta(\alpha_1-\alpha_{1j}),$$

$$\frac{1}{A_1A_2}\left\{\frac{\partial}{\partial\alpha_1}[A_2(Q_{13}+\theta_1T_{11})]\right\}-T_{11}k_1-T_{22}k_2-$$

$$-\sum_{j=1}^{l}T_{22j}k_{2j}\delta(\alpha_1-\alpha_{1j})+P_2=\rho h\frac{\partial^2 w}{\partial t^2}+\sum_{j=1}^{l}\rho_jF_j\frac{\partial^2 w}{\partial t^2}\delta(\alpha_1-\alpha_{1j}),$$

$$\frac{1}{A_1A_2}\left[\frac{\partial}{\partial\alpha_1}(A_2M_{11})-\frac{\partial A_2}{\partial\alpha_1}M_{22}\right]-Q_{13}-\sum_{j=1}^{l}M_{22j}k_{2j}\delta(\alpha_1-\alpha_{1j})+m_1=$$

$$= \rho \frac{h^3}{12} \frac{\partial^2 \varphi}{\partial t^2} \pm \sum_{j=1}^{i} \rho_j F_j \left[h_i \left(\frac{\partial^2 u}{\partial t^2} \pm h_j \frac{\partial^2 \varphi}{\partial t^2} \right) \right] \delta(\alpha_1 - \alpha_{1j}). \tag{1.5}$$

Equations (1.5), (1.3), and (1.4), are supplemented by the corresponding boundary conditions and zero initial conditions.

2. Numerical Algorithm for Solution of the Problem. The numerical algorithm that will be used to solve the boundary-value problem is based on finite-difference discretization of variational equation (1.2) and an explicit finite-difference scheme of integration over time [3]. When allowance is made for the discrete placement of the ribs of reinforced shells, the main difficulty encountered in solving boundary-value problems for such shells is the presence of discontinuous coefficients in the equations of motion. Following [6], we will seek a solution on the smooth part and "join" the different solutions at the lines of discontinuity. In the problem being examined here, the line of discontinuity is the projection of the center of gravity of the cross section of the j-th rib on the middle surface of the skin.

We will discretize the domain of the variable α_1 in such a way that the coordinates of the center of gravity of the cross section of the j-th rib on the midline coincide with an integral point of the grid region. We will isolate a transitional element joining the skin and the j-th rib — such as for $\alpha_{1j-1/2} \le \alpha_1 \le \alpha_{1j+1/2}$ — and we will write the joining conditions for this element

$$u^+ = u^-, \quad w^+ = w^-, \quad \varphi^+ = \varphi^-; \tag{2.1}$$

$$T_{11}^+ - T_{11}^- + \Delta s P_1 = \rho h \Delta s \frac{\partial^2 u}{\partial t^2} + \rho_j F_j \left(\frac{\partial^2 u}{\partial t^2} \pm h_j \frac{\partial^2 \varphi}{\partial t^2} \right); \tag{2.2}$$

$$(Q_{13} + \theta T_{11})^+ - (Q_{13} + \theta_1 T_{11})^- - T_{22c} k_2 - T_{22j} k_{2j} + P_2 \Delta s =$$
$$= \rho h \Delta s \frac{\partial^2 w}{\partial t^2} + \rho_j F_j \frac{\partial^2 w}{\partial t^2};$$

$$M_{11}^+ - M_{11}^- - Q_{13c} - M_{22j} k_{2j} + m_1 \Delta s = \rho \frac{h^3 \Delta s}{12} \frac{\partial^2 \varphi}{\partial t^2} \pm$$
$$\pm \rho_j F_j h_j \left(\frac{\partial^2 u}{\partial t^2} \pm h_j \frac{\partial^2 \varphi}{\partial t^2} \right),$$

where $T_{22c} = E_2 h \Delta s w \cdot k_2$, $Q_{13c} = G_{13} k^2 h \Delta s \phi$, $\Delta s = \Delta\alpha_1 A_1$, $\Delta\alpha_1$ is the interval of the space coordinate.

The difference scheme for Eqs. (1.5), (1.3), and (1.4) on the smooth part of the solution is written in the form

$$\frac{1}{A_{1i} A_{2i}} [(A_{2i-1/2} T_{11i-12}^n)_{\alpha_1} - (A_{2i-1/2})_{\alpha_1} T_{22i}^n] + \tag{2.3}$$

$$+ (Q_{13}^n + \theta_1^n T_{11}^n)_i k_{1i} + P_{1i}^n = \rho h (u_i^n)_{\bar{t}t}, \tag{2.4}$$

. . . .

$$T_{11i-1/2}^n = \frac{E_1 h}{1 - \nu_1 \nu_2} (\varepsilon_{11i-1/2}^n + \nu_2 \varepsilon_{22i-1/2}^n),$$

. . . .

$$\varepsilon_{11i-1/2}^n = \frac{1}{A_{1i-1/2}} (u_{i-1}^n)_{\alpha_1} + \frac{1}{2} (\theta_{1i-1/2}^n)^2 + k_{1i-1/2} w_{i-1/2}^n, \tag{2.5}$$

. . . .

We similarly approximate the equations of motion of the joint element (2.2). The above algorithm makes it possible to solve the initial equations of motion with second-order accuracy for both the space and time coordinates in problems of the theory of reinforced shells in which allowance is made for discrete rib placement.

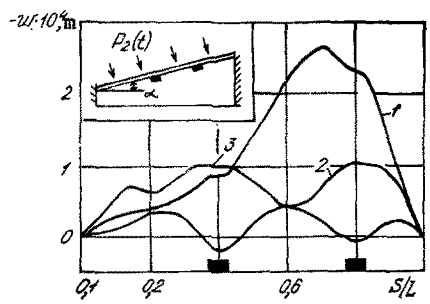

Fig. 1

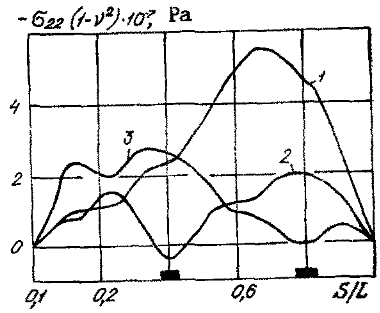

Fig. 2

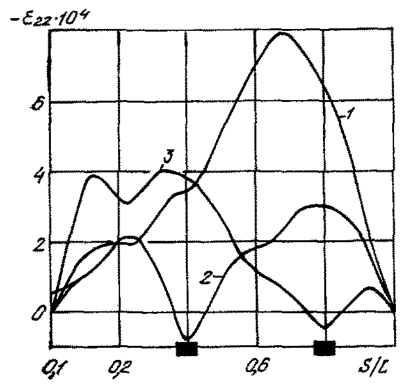

Fig. 3

We make use of the necessary condition of stability from [5] to study the stability of linearized difference equations (2.3-2.5). In accordance with this condition

$$\Delta t \leqslant 2/\omega, \tag{2.6}$$

where $\omega = \max(\omega_0, \omega_j)$, $j = \overline{1, \ell}$ are the highest natural frequencies of the discrete systems for the skin and j-th transitional element; and Δt is the interval of the time coordinate.

As the studies showed, shear vibrations of the skin had the highest natural frequency. For shell thicknesses less than the space-coordinate interval, condition (2.6) results in a decrease in the efficiency of explicit difference scheme (2.3-2.5). The interval of integration can be made larger by using the method proposed in [4] to regularize difference schemes for uniform shells and plates of the Timoshenko type.

3. Numerical Results. As a numerical example, we will examine the problem of the nonsteady behavior of a reinforced conical shell. Let the shell, of length L, be subjected to a instantaneous load $P_2(t)$. The load is applied in the direction normal to the shell surface. We assume that the ends of the shell are rigidly fastened, i.e., the boundary conditions at the points $S = S_0$, $S = S_N$ have the form $u = w = \phi = 0$. The shell is reinforced by two ribs at the points $S = S_1$, $S = S_2$.

In calculating the nonsteady behavior of conical shells, we will use a coordinate system in which the S coordinate is reckoned from the edge of the shell. The coefficients of the first quadratic form and the curvature of the coordinate surface in this system are written in the form

$$A_1 = 1, \quad A_2 = R_S, \quad k_1 = 0, \quad k_2 = R_S^{-1} \cos\alpha, \quad R_S = R_0 + S \sin\alpha, \tag{3.1}$$

where α is the angle of taper; and R_0 is the radius of the shell in the section $S = S_0$.

Using Eqs. (2.3-2.5) and allowing for (3.1), we find that the maximum natural frequency of vibration of the discrete system is as follows for conditions (2.6)

$$\omega_{\max} = C_{11}\frac{2}{\Delta S}\left\{1 + \frac{\Delta S \nu_2 \sin\alpha}{2R_0} + \frac{C_{22}^2}{C_{11}^2}\left[\left(\frac{\Delta S}{R_0}\right)^2 \sin\alpha + \frac{\nu_1 \Delta S}{2R_0}\right] + \right.$$
$$\left. + \frac{C_{13}^2}{C_{11}^2}\left[\frac{\Delta S}{4h} + \left(\frac{\Delta S}{h}\right)^2\right]\right\}^{1/2},$$

$$C_{11}^2 = \frac{E_1}{\rho(1-\nu_1\nu_2)}, \qquad C_{22}^2 = \frac{E_2}{\rho(1-\nu_1\nu_2)}, \qquad C_{13}^2 = \frac{12G_{13}}{\rho}.$$

The problem was solved with the following geometric and physicomechanical parameters for the skin and reinforcing elements:

$$E_1 = E_2 = E_j = 7\cdot 10^{10}\ \text{Pa};\quad \nu_1 = \nu_2 = 0{,}3;\quad \rho = \rho_j = 2{,}7\cdot 10^3\ \text{kg/m}^3;$$
$$L = 0{,}5\ \text{m};\quad R_0 = 5\cdot 10^{-2}\ \text{m};\quad S_0 = 0{,}1\ \text{m};\quad \alpha = \pi/6;\quad h = 1\cdot 10^{-2}\ \text{m};$$
$$H_j = 1{,}5\cdot 10^{-2}\ \text{m};\quad F_j = 3\cdot 10^{-4}\ \text{m}^2;\quad P_2(t) = A_m H(t),\ \text{where}\ A_m = 10^6\ \text{Pa}$$
$$H(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}.$$

We assumed that the geometric parameters of the ribs were identical. The centers of gravity of the ribs are projected on the middle plane of the skin at the points $S_1 = 0.4$ L; $S_2 = 0.8 \times$ L.

Figures 1-3 show the results of calculations performed in the given problem for the deflection w, strain ε_{22}, and stress σ_{22}. Curves 1-3 correspond to these quantities at the dimensionless times $\bar{t}_1 = 2$, $\bar{t}_2 = 3$, $\bar{t}_3 = 4$ ($\bar{t} = t\cdot c_{11}/L$) in the space coordinate. The nonsteady behavior of the reinforced structure was calculated for the time interval $\bar{t} = 15$. As shown by the numerical results, the maximum values of w, ε_{22}, and σ_{22} are seen at the time $\bar{t} = 1$-2. These results make it possible to evaluate the effect of discrete reinforcement on the kinematic and force parameters of the given structure.

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