APPROACH TO THE NUMERICAL SOLUTION OF BOUNDARY-VALUE PROBLEMS IN THE THEORY OF SHELLS IN COORDINATES OF GENERAL FORM*

Ya. M. Grigorenko and A. M. Timonin UDC 539.3

In the general case, one is concerned with nonlinear boundary-value problems of the theory of compliant thin shells having a complex three-dimensional shape with various cutouts and boundary contours with a nontrivial configuration, whose centroidal or coordinate surface is referred to a nonorthogonal, nonconjugate coordinate system [1, 3, 10]. Certain approaches to the calculation of shells of complex geometry are presented in [2, 7, 9].

In this article we construct a governing system of nonlinear partial differential equations from general tensor relations of the geometrically nonlinear theory of thin shells for the case of nonorthogonal parametrization of the centroidal surface, and we propose an approach to the solution of a nonlinear boundary-value problem, based on linearization, reduction of the two-dimensional problem to a one-dimensional problem, and numerical solution of the latter by the expansion of certain functions in discrete Fourier series. The approach has been used for the solution of certain problems in [5, 6].

We consider in invariant tensor form the stress–strain state of compliant thin shells, whose centroidal surface is parametrized by two curvilinear (Gaussian) coordinates α^1 and α^2, where the coordinate lines α^i = const coincide with the contours of the open shells at their boundary.

We write the complete system of equations:

geometrical relations

$$2\varepsilon_{ij} = e_{ij} + e_{ji} + v_i v_j,$$
$$2\kappa_{ij} = -\nabla_i v_j - \nabla_j v_i + b_i^{\alpha} e_{j\alpha} + b_j^{\alpha} e_{i\alpha}, \tag{1}$$
$$e_{ij} = \nabla_i u_j - b_{ij} w, \quad v_i = -\left(\frac{\partial w}{\partial \alpha^i} + b_i^{\alpha} u_{\alpha}\right);$$

equilibrium equations

$$\nabla_{\alpha} T^{\alpha j} - b_{\alpha}^{j} Q^{\alpha} + q^i = 0,$$
$$\nabla_{\alpha} Q^{\alpha} + b_{\alpha\beta} T^{\alpha\beta} + q^3 = 0, \tag{2}$$
$$\nabla_{\alpha} M^{\alpha j} - Q^j - T^{j\alpha} v_{\alpha} = 0;$$

elasticity relations

$$S^{ij} = \frac{Eh}{1-\mu^2}\left[\mu a^{ij} a^{\alpha\beta} + (1-\mu) a^{i\alpha} a^{j\beta}\right] \varepsilon_{\alpha\beta},$$
$$M^{ij} = -\frac{Eh^3}{12(1-\mu^2)}\left[\mu a^{ij} a^{\alpha\beta} + (1-\mu) a^{i\alpha} a^{j\beta}\right] \kappa_{\alpha\beta}, \tag{3}$$
$$T^{ij} = S^{ij} - b_{\gamma}^{j} M^{\gamma i}.$$

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S. P. Timoshenko Institute of Mechanics, Academy of Sciences of Ukraine, Kiev. Translated from Prikladnaya Mekhanika, Vol. 30, No. 4, pp. 14-20, April, 1994. Original article submitted February 2, 1993.

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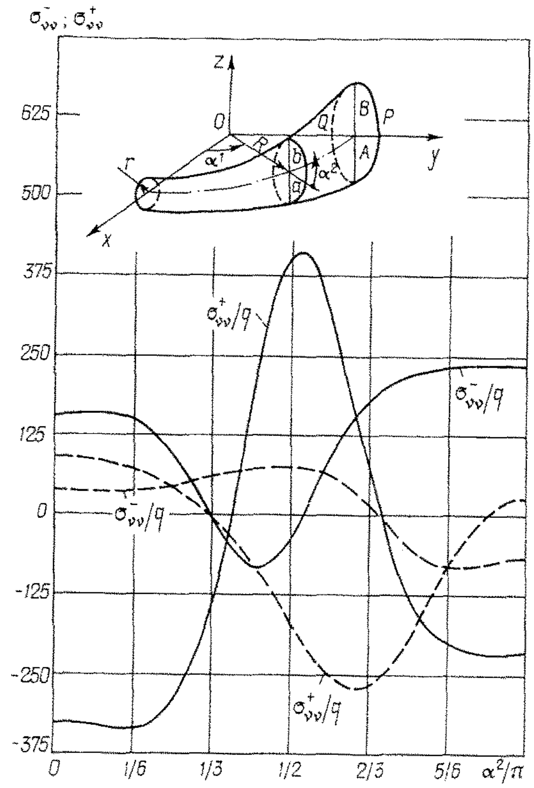

Fig. 1

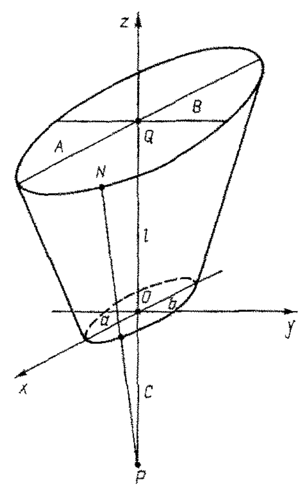

Fig. 2

In Eqs. (1)-(3) i, j = 1, 2; summation from 1 to 2 is implied by twice-repeated indices α, β; a^{ij} denotes the contravariant components of the first metric tensor a_{ij} of the centroidal surface; b_{ij} and b_i^j are the covariant and mixed components of the second metric tensor; ∇_i is the symbol of covariant differentiation in metric a_{ij}; u_i are the covariant components of the tangential displacement vector; w is the bending deflection of the centroidal surface of the shell; T^{ij} and M^{ij} are the contravariant components of the force and torque tensors, S_{ij} are the components of the symmetric force tensor, Q^i are the shearing forces; q^j and q^3 are the components of the external surface load; E, μ, and $h(\alpha^1, \alpha^2)$ are the elastic modulus, Poisson ratio, and thickness of the shell.

At each point of the coordinate line α^1 = const we consider the right orthogonal coordinate system formed by the unit vectors along the tangential normal $\bar{\nu}$, the tangent $\bar{\tau}$, and the normal $\bar{n}$ to the centroidal surface.

To evaluate the arbitrary constants in the system of equations (1)-(3), we specify four boundary conditions, one from each pair, on each contour of the shell α^1 = const:

$$(Q_{\nu\nu}, u_\nu);\ (Q_{\nu\tau}, u_\tau);\ (Q_{\nu n}, w);\ (M_{\nu\nu}, v_\nu). \tag{4}$$

Here $u_\nu = \nu^i u_i$, $u_\tau = \tau^i u_i$, $v_\nu = \nu^i v_i$,

$$\begin{gathered} Q_{\nu\nu} = T_{\nu\nu} + k_{\nu\tau} M_{\nu\tau},\quad Q_{\nu\tau} = T_{\nu\tau} + k_\tau M_{\nu\tau}, \\ Q_{\nu n} = T_{\nu n} + \frac{1}{\sqrt{a_{22}}}\frac{\partial M_{\nu\tau}}{\partial \alpha^2}, \\ T_{\nu\nu} = \nu_i \nu_j T^{ij},\quad T_{\nu\tau} = \nu_i \tau_j T^{ij},\quad T_{\nu n} = \nu_i Q^i, \\ M_{\nu\nu} = \nu_i \nu_j M^{ij},\quad M_{\nu\tau} = \nu_i \tau_j M^{ij}, \\ \nu_1 = \sqrt{\frac{a}{a_{22}}},\quad \nu_2 = 0,\quad \tau_1 = \frac{a_{12}}{\sqrt{a_{22}}},\quad \tau_2 = \sqrt{a_{22}}, \\ \nu^1 = \sqrt{\frac{a_{22}}{a}},\quad \nu^2 = -\frac{a_{12}}{\sqrt{a a_{22}}},\quad \tau^1 = 0,\quad \tau^2 = \frac{1}{\sqrt{a_{22}}}; \end{gathered} \tag{5}$$

u_ν, u_τ, and w are the physical components of the displacement vector along the $\bar{\nu}$, $\bar{\tau}$, and $\bar{n}$ axes, respectively; v_ν is the angle of rotation of the normal to the centroidal surface of the shell about the $\bar{\tau}$ axis; $Q_{\nu\nu}$, $Q_{\nu\tau}$, $Q_{\nu n}$, and $M_{\nu\nu}$ are generalized forces corresponding to the generalized displacements u_ν, u_τ, w, and v_ν, respectively; k_τ and $k_{\nu\tau}$ are the normal curvature and geodesic torsion of the centroidal surface in the direction $\bar{\tau}$; ν_i, τ_i, ν^i, and τ^i are the covariant and contravariant components of the unit vectors $\bar{\nu}$ and $\bar{\tau}$. A series of cumbersome transformations reduces the basic system of equations (1)-(3) to a governing system of partial differential equations in the functions (4), which has the form

$$\frac{\partial \bar{Z}}{\partial \alpha^1} = \bar{G}\left(\alpha^1, \alpha^2, \frac{\partial^k \bar{Z}}{\partial (\alpha^2)^k}\right)\quad (k = \overline{0,4}), \tag{6}$$

where $\bar{Z}^T = \{Q_{\nu\nu}, Q_{\nu\tau}, Q_{\nu n}, M_{\nu\nu}, u_\nu, u_\tau, w, v_\nu\}$, $\bar{Z} = \{Z_i\}$ ($i = 1,\ldots, 8$) is a vector of resolvent kernels, and $\bar{G} = \{g_i\}$ is the vector right-hand side, which is a nonlinear vector function of $\bar{Z}$. Expressions for the right-hand side in the linear problem are given in [5].

Linearization reduces the nonlinear boundary-value problem for the system of equations (6) with the boundary conditions (4) to a sequence of linear two-dimensional boundary-value problems for the system of equations

$$\begin{gathered} \frac{\partial \bar{Z}^{(s+1)}}{\partial \alpha^1} = \bar{F}\left(\alpha^1, \alpha^2, \frac{\partial^k \bar{Z}^{(s)}}{\partial (\alpha^2)^k}, \frac{\partial^k \bar{Z}^{(s+1)}}{\partial (\alpha^2)^k}\right)\quad (k = \overline{0,4};\ s = 0,1,\ldots), \\ \bar{F} = \bar{G}\left(\alpha^1, \alpha^2, \frac{\partial^k \bar{Z}^{(s)}}{\partial (\alpha^2)^k}\right) + J\left(\alpha^1, \alpha^2, \frac{\partial^k \bar{Z}^{(s)}}{\partial (\alpha^2)^k}\right)(\bar{Z}^{(s+1)} - \bar{Z}^{(s)}); \\ \bar{F} = \{f_i\}\,(i = \overline{1,8});\ J\left(\alpha^1, \alpha^2, \frac{\partial^k \bar{Z}}{\partial (\alpha^2)^k}\right) \end{gathered} \tag{7}$$

is the Jacobian matrix of the right-hand side of the system (7). The boundary conditions are linearized analogously. The solution of the linear problem for some approximate solution can be adopted as the initial approximation in a number of cases. To solve the resulting linear boundary-value problem in each approximation, we use a method based on the discrete Fourier series expansion of the functions on the right-hand side of the system (7) [4]. We then seek a solution of the boundary-value problem for the system of equations (7) with the corresponding boundary conditions by means of the expansions

$$X(\alpha^1, \alpha^2) = \sum_{n=0}^{N} X_n(\alpha^1)\cos n\alpha^2,\quad Y(\alpha^1, \alpha^2) = \sum_{n=1}^{N} Y_n(\alpha^1)\sin n\alpha^2, \tag{8}$$

where

$$X = \{Z_i, f_i\},\ Y = \{Z_j, f_j\} \quad (i=1,3,4,5,7,8;\ j=2,6).$$

Substituting the expansions (8) into the system of equations (7) and doing likewise for the boundary conditions, after appropriate transformations we arrive at a coupled system of ordinary differential equations of order 6 + 8N, which has the form

$$\frac{dZ_{i,0}^{(s+1)}}{d\alpha^1} = f_{i,0}(\alpha^1; Z_{j,0}^{(s)}; Z_{j,0}^{(s+1)}; Z_{l,m}^{(s)}; Z_{l,m}^{(s+1)});$$

$$\frac{dZ_{k,n}^{(s+1)}}{d\alpha^1} = f_{k,n}(\alpha^1; Z_{j,0}^{(s)}; Z_{j,0}^{(s+1)}; Z_{l,m}^{(s)}; Z_{l,m}^{(s+1)}); \tag{9}$$

$$(i,j=1,3,4,5,7,8;\ k,l=\overline{1,8};\ n,m=\overline{1,N};\ s=0,1,\ldots).$$

The linear boundary-value problem is solved in each iteration for the system (7) by a stable numerical method of discrete orthogonalization.

We give the results of solving problems for certain shell elements on the basis of the proposed approach.

We first consider the linearly formulated problem of the stressed state of a toroidal shell element having a variable elliptical cross section (Fig. 1). The shell is subjected to an internal pressure q = const, and the edges of the shell $\alpha^1 = 0$ and $\alpha^1 = \pi/2$ are rigidly fixed. The expressions for the Cartesian coordinates of points of the centroidal surface have the form

$$x = [R + a(\alpha^1)\cos\alpha^2]\cos\alpha^1,$$

$$y = [R + a(\alpha^1)\cos\alpha^2]\sin\alpha^1, \quad z = b(\alpha^1)\sin\alpha^2,$$

$$a(\alpha^1) = r + 2(A-r)\alpha^1/\pi, \quad b(\alpha^1) = r + 2(B-r)\alpha^1/\pi.$$

Figure 1 shows the distributions of the stresses $\sigma_{\nu\nu}^-$ and $\sigma_{\nu\nu}^+$ at the edges of the element. The solid curves are associated with the edge $\alpha^1 = \pi/2$, and the dashed curves with the edge $\alpha^1 = 0$. The problem is solved for R = 500, r = 100, A = 100, B = 200, h = 5 (mm), and N = 8.

The results of solving two problems are given below in the geometrically nonlinear formulation. In one of them we consider the deformation of a conical shell of elliptical cross section, whose centroidal surface is formed by moving a radial line segment PN passing through a point P on the z axis and through a point N around an elliptical path with semiaxes A and B (Fig. 2). The small base of the cone is also an ellipse with semiaxes a = λA and b = λB, where λ = c/(l + c), c = PO, and l = OQ. The expressions for the Cartesian coordinates of points of the centroidal surface of the shell have the form

$$x = A\frac{\alpha^1 + c}{l + c}\sin\alpha^2;\ y = B\frac{\alpha^1 + c}{l + c}\cos\alpha^2;\ z = \alpha^1; \quad (0 \leqslant \alpha^1 \leqslant l;\ 0 \leqslant \alpha^2 \leqslant 2\pi).$$

The shell is loaded by an internal pressure q = 1.5 MPa. The edges of the shell are rigidly fixed. The problem is solved for E = 70 GPa, μ = 0.3, A = 450, B = 275, c = 500, l = 600, h = 2.5 (mm), and N = 6.

Table 1 gives the values of the normal stresses $\sigma_{\nu\nu}$ for three approximations s for $\alpha^2 = 0$ and $\alpha^2 = \pi/2$. The values of the stresses on the outer surface are indicated in the numerator position of each entry in the table, and their values on the inner surface are shown in the denominator position. It is evident from the table how the nonlinear solution (s = 3) differs from the linear solution (s = 1).

In the second problem we consider the geometrically nonlinear deformation of a cylindrical tank with an ellipsoidal bottom fitted with an off-axis (nonpolar) branch pipe (Fig. 3).

The structure consists of four shell elements:

1) an axisymmetric cylindrical shell

$$x = A\sin\alpha^2;\quad y = A\cos\alpha^2;$$

$$z = \alpha^1 - L;$$

$$0 \leqslant \alpha^1 \leqslant L;\ 0 \leqslant \alpha^2 \leqslant 2\pi;$$

TABLE 1

α^2	Small base of cone $\sigma_{\nu\nu}\cdot 10^{-2}$, MPa			Large base of cone $\sigma_{\nu\nu}\cdot 10^{-2}$, MPa		
	$s = 1$	$s = 2$	$s = 3$	$s = 1$	$s = 2$	$s = 3$
0	−10,404 6,120	−7,494 4,946	−7,407 4,898	−9,293 9,124	−7,989 8,797	−7,828 8,706
$\frac{\pi}{2}$	6,623 −0,372	8,614 −1,221	8,544 −1,183	4,474 2,385	5,079 1,987	5,048 1,995

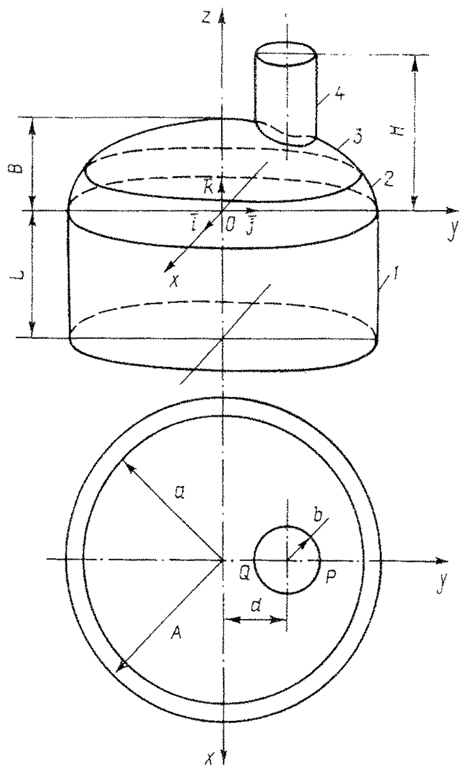

Fig. 3

2) an ellipsoidal shell of revolution

$$x = A\cos\alpha^1 \sin\alpha^2;$$
$$y = A\cos\alpha^1 \cos\alpha^2;\quad z = B\sin\alpha^1;$$

$$0 \leqslant \alpha^1 \leqslant \arccos(a/A);$$
$$0 \leqslant \alpha^2 \leqslant 2\pi;$$

3) an ellipsoidal shell with an off-axis opening, where the projection of this element onto the plane perpendicular to the axis of the structure forms a region bounded by two nonconcentric circles,

$$x = [a + \alpha^1(b - a)]\sin\alpha^2;\ y = [a + \alpha^1(b - a)]\cos\alpha^2 + \alpha^1 d;$$
$$z = \frac{B}{A}\sqrt{A^2 - x^2 - y^2};\quad 0 \leqslant \alpha^1 \leqslant 1;\quad 0 \leqslant \alpha^2 \leqslant 2\pi;$$

4) a cylindrical shell, one edge of which is the line of intersection of an ellipsoid with a cylinder

$$x = b\sin\alpha^2;\quad y = d + b\cos\alpha^2;\quad z = (1 - \alpha^1)\frac{B}{A}\sqrt{A^2 - x^2 - y^2} + \alpha^1 H;$$
$$0 \leqslant \alpha^1 \leqslant 1;\quad 0 \leqslant \alpha^2 \leqslant 2\pi;$$

TABLE 2

Point	$\hat{\sigma}_{\nu\nu}$			$\hat{\sigma}_{\tau\tau}$			$\hat{w}$		
	$s = 1$	$s = 2$	$s = 3$	$s = 1$	$s = 2$	$s = 3$	$s = 1$	$s = 2$	$s = 3$
P	4,141 / −3,491	3,680 / −3,055	3,658 / −3,036	3,540 / −1,248	3,101 / −1,069	3,094 / −1,076	4,207	3,269	3,196
Q	1,633 / −1,366	1,221 / −0,977	1,185 / −0,945	2,250 / 1,533	1,878 / 1,383	1,856 / 1,382	3,418	2,674	2,614

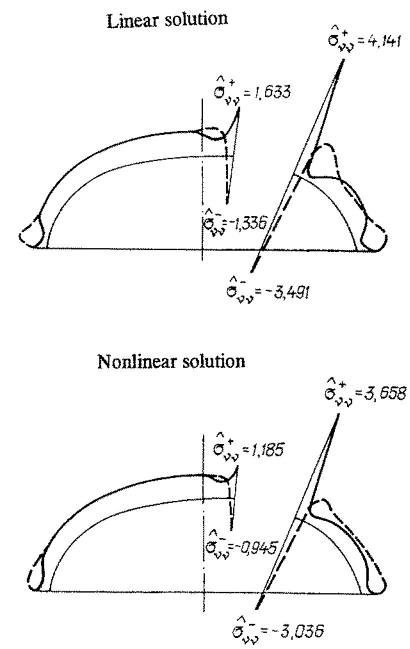

Fig. 4

Here A and B are the semiaxes of the ellipsoid of revolution, L is the length of the cylindrical part, a is the radius of the dividing parallel between regions 2 and 3, b is the radius of the cylindrical branch pipe, d is the eccentricity, and H is the distance from the edge of the branch pipe.

The tank is loaded by an internal pressure q, and the edge of the cylindrical part is rigidly fixed: $u_\nu = u_\tau = w = v_\nu = 0$. Applied to the edge of the branch pipe is a uniformly distributed axial force of intensity

$$Q_0 = qb/2 - Q_{\nu\nu} = Q_0;\ u_\tau = w = v_\nu = 0.$$

Calculations are carried out for the initial data A = 500, B = 300, L = 400, a = 450, b = 100, d = 200, H = 500, h = 2.5 (mm); E = 100 GPa, μ = 0.3, q = 2.5 MPa, and N = 4.

Table 2 shows how the values of the dimensionless stresses $\hat{\sigma}_{\nu\nu}$ and $\hat{\sigma}_{\tau\tau}$ normal to the respective lines α^1 = const and α^2 = const and the dimensionless bending deflection $\hat{w}$ change with the order of approximation s. Here

$$\hat{\sigma}_{\nu\nu} = \sigma_{\nu\nu}/\sigma_0,\quad \hat{\sigma}_{\tau\tau} = \sigma_{\tau\tau}/\sigma_0,\quad \hat{w} = w/w_0,$$

$$\sigma_0 = \frac{qA}{h},\quad w_0 = \frac{qA^2}{Eh}\left(1 - \frac{\mu}{2}\right).$$

The values in the table are given for bottom points P and Q situated in the symmetry plane of the structure (see Fig. 3). The stresses on the outer surface are given in the numerator position of each entry, and the stresses on the inner surface are given in the denominator position.

Figure 4 shows the distribution, along the bottom generatrices situated in the symmetry plane, of the dimensionless stresses $\sigma_{\nu\nu}^{\pm}$ obtained from the solution of the problem in the linear and geometrically nonlinear formulations. The superscripted minus sign corresponds to the inner surface of the bottom (dashed curves), and the superscripted plus sign corresponds to the outer surface (solid curves).

The results demonstrate the efficiency of the proposed approach for the calculation of shells of complex geometry in linear and geometrically nonlinear settings.

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