NONAXISYMMETRIC TEMPERATURE AND THERMOSTRESS IN ISOTROPIC AND CURVED ORTHOTROPIC LAYERED SHELLS OF REVOLUTION

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The use of composite materials in layered structural elements requires effective numerical methods of studying the temperature field and thermostress in these materials under thermal and mechanical loads. Methods of calculating the thermostress in bodies of revolution have been worked out quite extensively for inelastically deformable isotropic materials [1, 7, 10] for both axisymmetric and nonaxisymmetric loads. Methods of studying the thermostress in curved orthotropic bodies of revolution have been considered in [6, 8, 9] for axisymmetric loads. In the present paper, we propose a method of calculating the temperature field and thermostress in isotropic and curved orthotropic bodies of revolution in the case of nonaxisymmetric thermal and mechanical loads. The method is based on the so-called semi-analytical finite element method [5, 10] in which the displacement, temperature, and external loads are written as Fourier series in the azimuthal coordinate and the finite-element approximation is used to obtain the unknown displacement and temperature amplitudes in the radial section of the body.

1. Statement of the Problem. We consider a layered (compound) body of revolution of arbitrary shape consisting of inelastically deformable isotropic and elastic curved orthotropic materials. We assume ideal thermal and mechanical contact on the surfaces between the layers. We use cylindrical coordinates (z, r, φ).

The body is assumed to be in a natural state of stress when the temperature field is $T_0(z, r)$. The body is subjected to nonaxisymmetric time-dependent heating by a medium with temperature $\theta(z, r, \varphi)$ and bulk $\vec{K}(K_z, K_r, K_\varphi)$ and surface $\vec{t}_n(t_{nz}, t_{nr}, t_{n\varphi})$ loads act over part Σ_t of its surface. The displacement is given as $\vec{u}_0(u_z^0, u_r^0, u_\varphi^0)$ over the remaining part of the surface.

The thermal and mechanical properties of the body are assumed to be temperature dependent. In addition, one of the principal directions of the elastic and thermal conductivity tensors of the anisotropic layers is in the direction of the azimuthal coordinate φ, while the other two mutually perpendicular directions lie in a radial section of the body. The angle between these directions, and the coordinate axes z and r is different for each layer and depends on its structure.

The above problem reduces to solving the heat-conduction problem for the temperature of the body at a set of times and then calculating the resulting stress and deformation of the body.

2. Determination of the Temperature Fields. The determination of the time-dependent temperature field in an anisotropic body of revolution under nonaxisymmetric heating reduces to the integration of the differential equation

$$c\rho\frac{\partial T}{\partial t} = -\operatorname{div}\vec{q} \tag{2.1}$$

subject to the following initial and boundary conditions:

$$T = T_0(z,r) \quad \text{at} \quad t = 0, \tag{2.2}$$

$$\vec{n}\vec{q} = \alpha(T - \theta) \quad \text{on} \quad \Sigma, \tag{2.3}$$

where $\vec{q}(q_z, q_r, q_\varphi) = -\Lambda \cdot \operatorname{grad} T$ is the heat flux density vector in the body, $\Lambda(\lambda_{ij})$ is the thermal conductivity tensor, ρ is the density of the material, c is the specific heat, $\alpha(z, r)$ is the heat transfer coefficient, and $\vec{n}$ is the outward normal to the surface of the body Σ.

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For an isotropic material $\lambda_{zz} = \lambda_{rr} = \lambda_{\varphi\varphi} = \lambda$, $\lambda_{zr} = \lambda_{z\varphi} = \lambda_{r\varphi} = 0$. For an orthotropic body the thermal conductivities are determined by the principal values $\lambda_{\alpha\alpha}$ and $\lambda_{\beta\beta}$ of the thermal conductivity tensor with the help of the transformation equations between the coordinate system α, β and z, r:

$$\begin{gathered}\lambda_{zz} = \lambda_{\alpha\alpha}\cos^2 v + \lambda_{\beta\beta}\sin^2 v, \\ \lambda_{rr} = \lambda_{\alpha\alpha}\sin^2 v + \lambda_{\beta\beta}\cos^2 v, \\ \lambda_{zr} = (\lambda_{\alpha\alpha} - \lambda_{\beta\beta})\sin v\cos v, \quad \lambda_{z\varphi} = \lambda_{r\varphi} = 0,\end{gathered} \tag{2.4}$$

where v(z, r) is the angle between the principal axis of anistropy α and the axis z of rotation symmetry of the body.

Writing the thermal conductivities λ_{ij} in the form $\lambda_{ij} = \lambda_{ij}^0(1 - \omega_{ij}^T)$, where λ_{ij}^0 are the temperature-dependent parts of the thermal conductivities, the components of the heat flux density are

$$\begin{gathered}q_z = -\left(\lambda_{zz}^0 \frac{\partial T}{\partial z} + \lambda_{zr}^0 \frac{\partial T}{\partial r}\right) + q_z^{\partial}, \\ q_r = -\left(\lambda_{zr}^0 \frac{\partial T}{\partial z} + \lambda_{rr}^0 \frac{\partial T}{\partial r}\right) + q_r^{\partial}, \\ q_\varphi = -\lambda_{\varphi\varphi}^0 \frac{1}{r}\frac{\partial T}{\partial \varphi} + q_\varphi^{\partial},\end{gathered} \tag{2.5}$$

where

$$\begin{gathered}q_z^{\partial} = \lambda_{zz}^0 \omega_{zz}^T \frac{\partial T}{\partial z} + \lambda_{zr}^0 \omega_{zr}^T \frac{\partial T}{\partial r}, \\ q_r^{\partial} = \lambda_{zr}^0 \omega_{zr}^T \frac{\partial T}{\partial z} + \lambda_{rr}^0 \omega_{rr}^T \frac{\partial T}{\partial r}, \\ q_\varphi^{\partial} = \lambda_{\varphi\varphi}^0 \omega_{\varphi\varphi}^T \frac{1}{r}\frac{\partial T}{\partial \varphi}.\end{gathered} \tag{2.6}$$

Assuming that the nonlinear terms q_z^{∂}, q_r^{∂}, and q_φ^{∂} are known for the preceding instant of time or from the preceding order of approximation, the heat-conduction equation (2.1) can be solved by iteration.

We multiply the heat-conduction equation (2.1) and the boundary conditions (2.3) by the temperature variation and integrate the first expression over the volume of the body V and the second over the surface Σ. Adding the two results, using the Ostrogradskii–Gauss formula and (2.5), and assuming that a certain fixed instant of time the quantities c, ρ, α, θ, q_z^{∂}, q_r^{∂}, and q_φ^{∂} are known functions of the coordinates, we obtain a variational equation for the three-dimensional heat-conduction problem in the case of a curved orthotropic body of revolution.

The dimensionality of the heat-conduction problem is lowered by assuming a solution in the form of a trigonometric series in the azimuthal coordinate

$$T(z,r,\varphi,t) = \sum_{m=0}^{\infty} T_m(z,r,t)\cos m\varphi \sum_{m=1}^{\infty} \overline{T}_m(z,r,t)\sin m\varphi. \tag{2.7}$$

The temperature θ of the surrounding medium and the additional terms q_z^{∂}, q_r^{∂}, and q_φ^{∂} are also written as trigonometric series

$$\begin{gathered}\{\theta, q_z^{\partial}, q_r^{\partial}\} = \sum_{m=0}^{\infty} \{\theta_m, q_z^{\partial(m)}, q_r^{\partial(m)}\}\cos m\varphi + \sum_{m=1}^{\infty} \{\overline{\theta}_m, \overline{q}_z^{\partial(m)}, \overline{q}_r^{\partial(m)}\}\sin m\varphi, \\ q_\varphi^{\partial} = \sum_{m=1}^{\infty} q_\varphi^{\partial(m)}\sin m\varphi + \sum_{m=0}^{\infty} \overline{q}_\varphi^{\partial(m)}\cos m\varphi,\end{gathered} \tag{2.8}$$

whose coefficients can be found from the known relations of functional analysis [10].

Substituting (2.7) and (2.8) into the variational equation, assuming that the function $\partial T/\partial t$ is held fixed and that the heat transfer coefficient α does not depend on the azimuthal coordinate, and proceeding as in [9], we obtain the following equations for the temperature amplitudes:

$$\begin{gathered}(1 + \delta_{0m})\pi\delta I_m = 0 \quad (m = 0, 1, \dots), \\ \pi\delta\overline{I}_m = 0 \quad (m = 1, 2, \dots),\end{gathered} \tag{2.9}$$

where

$$I_m = \int\limits_F \Big[\frac{1}{2}\lambda^0_{zz}\Big(\frac{\partial T_m}{\partial z}\Big)^2 + \frac{1}{2}\lambda^0_{rr}\Big(\frac{\partial T_m}{\partial r}\Big)^2 + \frac{1}{2}\lambda^0_{\varphi\varphi}\frac{m^2}{r^2}T_m^2 + + \lambda^0_{zr}\frac{\partial T_m}{\partial z}\frac{\partial T_m}{\partial r} + c\rho\frac{\partial T_m}{\partial t}T_m - q_z^{\partial(m)}\frac{\partial T_m}{\partial z} - - q_r^{\partial(m)}\frac{\partial T_m}{\partial r} + \frac{m}{r}q_\varphi^{\partial(m)}T_m\Big]r\,dz\,dr + \int\limits_S \frac{1}{2}\alpha T_m(T_m - 2\theta_m)r\,ds, \tag{2.10}$$

and δ_{0m} is the Kronecker delta. The integration in (2.10) goes over the area F of a radial half section of the body and along its contour S.

An expression for $\bar{I}_m$ can be obtained from (2.10) by replacing T_m, θ_m, $q_z^{\partial(m)}$, $q_r^{\partial(m)}$, and $q_\varphi^{\partial(m)}$ by the corresponding quantities with overbars and by replacing m by $-m$.

The three-dimensional heat-conduction problem therefore reduces to a series of two-dimensional problems for each harmonic of the temperature field. In each approximation and in each time step, the problem reduces to a heat-conduction problem for a body of revolution whose thermal characteristics are independent of the azimuthal coordinate and which is subjected to an additional heat source whose intensity is determined by the functions $\omega_{ij}^T\lambda_{ij}^0$ and by the temperature gradients obtained in the preceding approximation (or in the preceding time step).

The finite element method is used to determine the unknown temperature amplitudes in (2.7). The finite elements are chosen to be triangles in the radial section of the body. The radial half section is split up into M triangular elements by N nodes. We assume that the expansion coefficients of the temperature in (2.7) vary linearly within each element.

Replacing the integration over the area F and along the contour S of the radial half section by the sum of integrals over the triangular elements and along their sides, writing the derivative $\partial T_m/\partial t$ in terms of finite differences, and proceeding as in [4], we obtain relations for the unknown temperature amplitudes at the corners i, j, and k of a triangular element. Then solution of the problem by the explicit difference scheme gives recursion relations for the coefficients T_m in (2.7) at time $t + \Delta t$ in terms of the coefficient at time t if we assume that in each element only the side ij lies along the contour of the body:

$$T_{mi}(t+\Delta t) = T_{mi}(t) - \frac{\Delta t}{\sum\limits_{q=1}^{M} \langle c\rho\rangle_q H_i^{(q)}} \sum_{q=1}^{M}\Big\{A_{ij}^{(q)}\theta_{mi}^{(q)} + B_{ij}^{(q)}\theta_{mj}^{(q)} - - [D_{ii}^{(q)} + m^2N_{ii}^{(q)} + A_{ij}^{(q)}]T_{mi}(t) - [D_{ij}^{(q)} + m^2N_{ij}^{(q)} + B_{ij}^{(q)}]T_{mj}(t) - - [D_{ik}^{(q)} + m^2N_{ik}^{(q)}]T_{mk}(t) + L_i^{(q)}\langle q_z^\partial\rangle_q + + P_i^{(q)}\langle q_r^\partial\rangle_q - mR_i^{(q)}\langle q_\varphi^\partial\rangle_q \quad (i = 1, 2, \ldots, N). \tag{2.11}$$

The calculation of the temperature field will be stable if the following condition holds on each node:

$$\Delta t \le \min\Big\{\frac{\sum\limits_{q=1}^{M}\langle c\rho\rangle_\theta H_i^{(q)}}{\sum\limits_{q=1}^{M}[D_{ii}^{(q)} + m^2N_{ii}^{(q)} + A_{ij}^{(q)}]}\Big\} \quad (i = 1, 2, \ldots, N)(m = 0, 1, \ldots). \tag{2.12}$$

Because the integration step size Δt of the heat-conduction differential equation (2.1) is small, the additional terms q_z^∂, q_r^∂, and q_φ^∂ averaged over each element can be calculated using the temperatures found in the preceding time step.

Relations similar to (2.11) can be found for the coefficients $\bar{T}_m$.

Once T_m and $\bar{T}_m$ have been found for all harmonics, the temperature at $t + \Delta t$ can be calculated using (2.7).

3. Determination of the Stress and Deformation. The nonaxisymmetric stress and deformation of layered bodies of revolution will be calculated using the Lagrange variational equation

$$\int\limits_V \sigma_{ij}\delta\varepsilon_{ij}dV - \int\limits_V \vec{R}\delta\vec{u}dV - \int\limits_{\Sigma_t} \vec{t}_n\delta\vec{u}d\Sigma = 0. \tag{3.1}$$

The equation of state for an inelastically deformable isotropic material will be described in terms of deformations of elements of the body along small-curvature trajectories [10]. The essence of this method is the assumption that the relations between the stresses and deformations in the l-th loading step are in the form of the generalized Hooke's law for an isotropic and homogeneous body with additional terms taking into account the departure of the material from an elastic body.

In the case of a curved orthotropic material in which one of the principal axes of anisotropy is along the coordinate φ, and the other two axes lie in the z, r plane, the relations between the deformations and the stresses can be written in the form [2]

$$\begin{gathered}
\varepsilon_{zz}=a'_{11}\sigma_{zz}+a'_{12}\sigma_{rr}+a'_{13}\sigma_{\varphi\varphi}+a'_{14}\sigma_{zr}+\varepsilon^{T}_{zz},\\
\dots\quad\dots\quad\dots\quad\dots\\
2\varepsilon_{zr}=a'_{14}\sigma_{zz}+a'_{24}\sigma_{rr}+a'_{34}\sigma_{\varphi\varphi}+a'_{44}\sigma_{zr}+2\varepsilon^{T}_{zr},\\
\dots\quad\dots\quad\dots\quad\dots\\
2\varepsilon_{r\varphi}=a'_{56}\sigma_{z\varphi}+a'_{66}\sigma_{r\varphi},
\end{gathered}\tag{3.2}$$

where $\varepsilon_{ij}{}^{T} = \alpha_{ij}{}^{T}(T - T_0)$ is the thermal deformation, and the coefficients, a_{ij}', $\alpha_{ij}{}^{T}$ are expressed in terms of the mechanical characteristics of the material and the coefficients of thermal expansion along the principal directions of anisotropy α, β, φ, and the angle v.

Solving (3.2) for the components of the stress and writing the coefficients in the form $A_{ij} = A_{ij}{}^{0}(1 - \omega_{ij})$, where $A_{ij}{}^{0}$ is the average value of the coefficient and $A_{ij}{}^{0}\omega_{ij}$ determines the position dependence, the relations between the stresses and deformations can be written as

$$\begin{gathered}
\sigma_{zz}=A^{0}_{11}\varepsilon_{zz}+A^{0}_{12}\varepsilon_{rr}+A^{0}_{13}\varepsilon_{\varphi\varphi}+2A^{0}_{14}\varepsilon_{zr}-\sigma^{**}_{zz},\\
\sigma_{rr}=A^{0}_{12}\varepsilon_{zz}+A^{0}_{22}\varepsilon_{rr}+A^{0}_{23}\varepsilon_{\varphi\varphi}+2A^{0}_{24}\varepsilon_{zr}-\sigma^{**}_{rr},\\
\sigma_{\varphi\varphi}=A^{0}_{13}\varepsilon_{zz}+A^{0}_{23}\varepsilon_{rr}+A^{0}_{33}\varepsilon_{\varphi\varphi}+2A^{0}_{34}\varepsilon_{zr}-\sigma^{**}_{\varphi\varphi},\\
\sigma_{zr}=A^{0}_{14}\varepsilon_{zz}+A^{0}_{24}\varepsilon_{rr}+A^{0}_{34}\varepsilon_{\varphi\varphi}+2A^{0}_{44}\varepsilon_{zr}-\sigma^{**}_{zr},\\
\sigma_{z\varphi}=2A^{0}_{55}\varepsilon_{z\varphi}+2A^{0}_{56}\varepsilon_{z\varphi}-\sigma^{**}_{z\varphi},\\
\sigma_{r\varphi}=2A^{0}_{56}\varepsilon_{z\varphi}+2A^{0}_{66}\varepsilon_{z\varphi}-\sigma^{**}_{r\varphi},
\end{gathered}\tag{3.3}$$

where

$$\begin{gathered}
\sigma^{**}_{zz}=A^{0}_{11}\varepsilon^{T}_{zz}+A^{0}_{12}\varepsilon^{T}_{rr}+A^{0}_{13}\varepsilon^{T}_{\varphi\varphi}+2A^{0}_{14}\varepsilon^{T}_{zr}+A^{0}_{11}\omega_{11}(\varepsilon_{zz}-\varepsilon^{T}_{zz})+\\
+A^{0}_{12}\omega_{12}(\varepsilon_{rr}-\varepsilon^{T}_{rr})+A^{0}_{13}\omega_{13}(\varepsilon_{\varphi\varphi}-\varepsilon^{T}_{\varphi\varphi})+2A^{0}_{14}\omega_{14}(\varepsilon_{zr}-\varepsilon^{T}_{zr}),\\
\dots\quad\dots\quad\dots\quad\dots\\
\sigma^{**}_{r\varphi}=2A^{0}_{56}\omega_{56}\varepsilon_{z\varphi}+2A^{0}_{66}\omega_{66}\varepsilon_{r\varphi}.
\end{gathered}\tag{3.4}$$

We assume relations between the stresses and deformations of the form (3.3) for both isotropic and curved orthotropic materials. For an elastically deformable isotropic material we then have [10]

$$\begin{gathered}
A^{0}_{11}=A^{0}_{22}=A^{0}_{33}=2G_0+\lambda_0,\quad A^{0}_{12}=A^{0}_{13}=A^{0}_{23}=\lambda_0,\\
A^{0}_{44}=A^{0}_{55}=A^{0}_{66}=G_0,\quad A^{0}_{14}=A^{0}_{24}=A^{0}_{34}=A^{0}_{56}=0,\\
\sigma^{**}_{ij}=2G\sum_{k=1}^{l}\Delta_k\varepsilon^{(n)}_{ij}+2G_0\omega_1\varepsilon_{ij}+(K\varepsilon_T+3\lambda_0\omega_1\varepsilon_0)\delta_{ij}.
\end{gathered}\tag{3.5}$$

Hence a single algorithm can be constructed to determine the thermostress in isotropic and curved orthotropic layered bodies of revolution.

Substituting (3.3) into the variational equation (3.1) and holding that the additional stresses $\sigma_{ij}{}^{**}$ are constant, we obtain the following variational equation for the stress and deformation of a compound body of revolution:

$$\begin{gathered}
\delta E=\delta\Big\{\int_V\Big[\frac{1}{2}(A^{0}_{11}\varepsilon^{2}_{zz}+A^{0}_{22}\varepsilon^{2}_{rr}+A^{0}_{33}\varepsilon^{2}_{\varphi\varphi})+2(A^{0}_{44}\varepsilon^{2}_{zr}+A^{0}_{55}\varepsilon^{2}_{z\varphi}+A^{0}_{66}\varepsilon^{2}_{r\varphi})+\\
+A^{0}_{12}\varepsilon_{zz}\varepsilon_{rr}+A^{0}_{13}\varepsilon_{zz}\varepsilon_{\varphi\varphi}+A^{0}_{23}\varepsilon_{rr}\varepsilon_{\varphi\varphi}+
\end{gathered}$$

$$+2(A_{14}^0\varepsilon_{zz}+A_{24}^0\varepsilon_{rr}+A_{34}^0\varepsilon_{\varphi\varphi})\varepsilon_{zr}+4A_{56}^0\varepsilon_{z\varphi}\varepsilon_{r\varphi}-$$
$$-\sigma_{zz}^{**}\varepsilon_{zz}-\sigma_{rr}\varepsilon_{rr}-\sigma_{\varphi\varphi}^{**}\varepsilon_{\varphi\varphi}-2\sigma_{zr}^{**}\varepsilon_{zr}-2\sigma_{z\varphi}^{**}\varepsilon_{z\varphi}-$$
$$-2\sigma_{r\varphi}^{**}\varepsilon_{r\varphi}]rdzdrd\varphi-\int_V(K_zu_z+K_ru_r+K_\varphi u_\varphi)rdzdrd\varphi-\tag{3.6}$$
$$-\int_{\Sigma_t}(t_{nz}u_z+t_{nr}u_r+t_{n\varphi}u_\varphi)rdsd\varphi\}=0.$$

This variational equation is usually solved by the method of successive approximations, in which the additional terms σ_{ij}^{**} are calculated using the results of the preceding approximation or preceding load step.

As in the case of the heat-conduction problem, the solution of (3.6) will be constructed in terms of trigonometric Fourier series. The unknowns are taken to be the displacements and they are written as trigonometric series in the azimuthal coordinate

$$u_z(z,r,\varphi,t)=\sum_{m=0}^{\infty}u_z^{(m)}(z,r,t)\cos m\varphi+\sum_{m=1}^{\infty}\bar{u}_z^{(m)}(z,r,t)\sin m\varphi,\quad (z,r),$$
$$u_\varphi(z,r,\varphi,t)=\sum_{m=1}^{\infty}u_\varphi^{(m)}(z,r,t)\sin m\varphi+\sum_{m=0}^{\infty}\bar{u}_\varphi^{(m)}(z,r,t)\cos m\varphi,\tag{3.7}$$

whose coefficients will be determined using the finite element method.

To obtain the solution of the variational equation (3.6) in the series form (3.7), the components of the surface t_{ni} and the volume K_i forces, and also the function σ_{ij}^{**} are represented in series form analogous to (2.8). Then, substituting these series, (3.7), and the expressions for the components of the deformation obtained from (3.7) using the Cauchy relations into the variational equation (3.6), we obtain the following equations for the displacement amplitudes:

$$(1+\delta_{0m})\pi\delta\,\mathrm{E}_m=0,$$
$$(1+\delta_{0m})\pi\delta\bar{\mathrm{E}}_m=0\quad(m=0,1,\ldots),\tag{3.8}$$

where

$$\mathrm{E}_m=\int_F[\frac{1}{2}(A_{11}^0\varepsilon_{zz}^{(m)2}+A_{22}^0\varepsilon_{rr}^{(m)2}+A_{33}^0\varepsilon_{\varphi\varphi}^{(m)2})+$$
$$+2(A_{44}^0\varepsilon_{zr}^{(m)2}+A_{55}^0\varepsilon_{z\varphi}^{(m)2}+A_{66}^0\varepsilon_{r\varphi}^{(m)2})+$$
$$+A_{12}^0\varepsilon_{zz}^{(m)}\varepsilon_{rr}^{(m)}+A_{13}^0\varepsilon_{zz}^{(m)}\varepsilon_{\varphi\varphi}^{(m)}+A_{23}^0\varepsilon_{rr}^{(m)}\varepsilon_{\varphi\varphi}^{(m)}+$$
$$+2(A_{14}^0\varepsilon_{zz}^{(m)}+A_{24}^0\varepsilon_{rr}^{(m)}+A_{34}^0\varepsilon_{\varphi\varphi}^{(m)})\varepsilon_{zr}^{(m)}+4A_{56}^0\varepsilon_{z\varphi}^{(m)}\varepsilon_{r\varphi}^{(m)}-$$
$$-(\sigma_{zz}^{**(m)}\varepsilon_{zz}^{(m)}+\sigma_{zz}^{**(m)}\varepsilon_{rr}^{(m)}+\sigma_{\varphi\varphi}^{**(m)}\varepsilon_{\varphi\varphi}^{(m)})+2\sigma_{zr}^{**(m)}\varepsilon_{zr}^{(m)}+\tag{3.9}$$
$$+2\sigma_{z\varphi}^{**(m)}\varepsilon_{z\varphi}^{(m)}+2\sigma_{r\varphi}^{**(m)}\varepsilon_{r\varphi}^{(m)})]rdzdr-$$
$$-\int_F(K_z^{(m)}u_z^{(m)}+K_r^{(m)}u_r^{(m)}+K_\varphi^{(m)}u_\varphi^{(m)})rdzdr-$$
$$-\int_S(t_{nz}^{(m)}u_z^{(m)}+t_{nr}^{(m)}u_r^{(m)}+t_{n\varphi}^{(m)}u_\varphi^{(m)})rds;$$

$$\varepsilon_{zz}^{(m)}=\frac{\partial u_z^{(m)}}{\partial z},\quad \varepsilon_{rr}^{(m)}=\frac{\partial u_r^{(m)}}{\partial z},\quad \varepsilon_{\varphi\varphi}^{(m)}=\frac{1}{r}(u_r^{(m)}+mu_\varphi^{(m)}),$$
$$\varepsilon_{zr}^{(m)}=\frac{1}{2}\left(\frac{\partial u_z^{(m)}}{\partial r}+\frac{\partial u_r^{(m)}}{\partial z}\right),\quad \varepsilon_{z\varphi}^{(m)}=\frac{1}{2}\left(\frac{\partial u_\varphi^{(m)}}{\partial z}-\frac{m}{r}u_z^{(m)}\right),\tag{3.10}$$
$$\varepsilon_{r\varphi}^{(m)}=\frac{1}{2}\left(\frac{\partial u_\varphi^{(m)}}{\partial r}-\frac{m}{r}u_r^{(m)}+\frac{1}{2}u_\varphi^{(m)}\right)\quad(m=0,1,\ldots).$$

The $\vec{\mathrm{E}}_m$ can be obtained from (3.9) and (3.10) by replacing the quantities without overbars by quantities with overbars, and by replacing m by $-$m.

Therefore, linearizing the constitutive relations of thermoplasticity for an isotropic material and representing the relations between the stresses and deformations in the form (3.3), the solution of the nonaxisymmetric problem for the stress and deformation reduces to the calculation in each approximation of the extremum values of the functional (3.9) with respect to

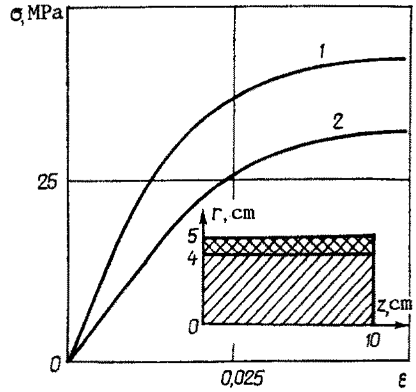

Fig. 1

coefficients of the series (3.7), which vary only in the radial section of the body of revolution. The variational equation for E_0 describes the axisymmetric stress without torsion, while the equation for $\bar{E}_0$ describes the stress with torsion.

The finite element method was used to find the unknown deformation amplitudes. As in the solution of the heat-conduction problem, the finite element in the (z, r) plane is chosen to be a triangle in which the coefficients of the series (3.7) vary linearly.

Application of the finite element reduces the variational equations (3.8) and (3.9) in each approximation to a system of 3N linear algebraic equations for the coefficients of the series (3.7) at the corners of the triangular elements:

$$
\begin{gathered}
\sum_{q=1}^{M} B_{\beta c}^{zi(q)} u_{\beta c} = D_{zi}, \\
\sum_{q=1}^{M} B_{\beta c}^{ri(q)} u_{\beta c} = D_{ri}, \\
\sum_{q=1}^{M} B_{\beta c}^{\varphi i(q)} u_{\beta c} = D_{\varphi i} \\
(i=1,2,\ldots,N;\ \ \beta=z,r,\varphi;\ \ c=i,j,k),
\end{gathered}
\tag{3.11}
$$

where the series are summed over the repeating indices β and c over the values indicated in the parentheses.

The coefficients of (3.11) are not written out explicitly because of their complexity. They can be obtained in analogy with [5, 10].

Solving the system of equations (3.11) for the coefficients $u_{\beta c}$, $\bar{u}_{\beta c}$ (β = z, r, φ; c = i, j, k) for each harmonic m, (3.7) and (3.10) are used to calculate the deformations and displacements at all nodes of different radical sections of the body of revolution. Knowing the components of the deformation and the functions σ_{ij}^{**} obtained in the preceding approximation, the stresses (3.3) can be calculated.

4. Example. As an example, we determine the time-dependent temperature field and stress in a bilayered cylinder whose radial half-section is shown in Fig. 1. We assume convective heat exchange with the surrounding medium. The inner layer of the cylinder is isotropic and the outer layer is orthotropic in cylindrical coordinates. At t = 0, the cylinder has temperature T_0 = 20°C. It is then subjected to convective heat exchange with a medium of temperature $\theta = 500(1 + 0.1 \cos\varphi)$°C through the cylindrical surface and with a medium of constant temperature θ = 300°C through the end of the cylinder at z = 10 cm. The end z = 0 is thermally insulated. The heat transfer coefficient α between the surrounding medium and the material of the cylinder is assumed constant and equal to 0.1 W/cm^2·K.

The thermal characteristics of the isotropic material are $\lambda_{zz} = \lambda_{rr} = \lambda_{\varphi\varphi}$ = 0.02 W/cm·K, $c\rho$ = 2 J/(cm^3·K), $\lambda_{zz}^T = \lambda_{rr}^T = \lambda_{\varphi\varphi}^T = 1.1 \cdot 10^{-5}$°C^{-1}. Stress–strain diagrams are shown in Fig. 1 for two values of the temperature (T = 20° for curve 1 and 500°C for curve 2). The Poisson coefficient is assumed to be independent of temperature and equal to 0.32. The thermal and mechanical characteristics of the orthotropic layer are

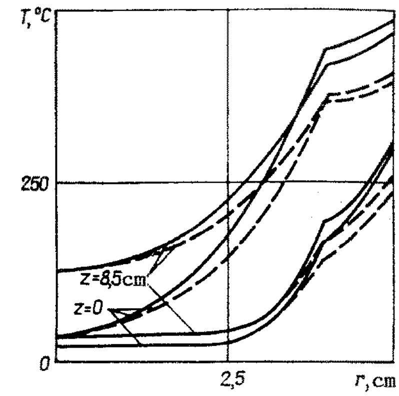

Fig. 2

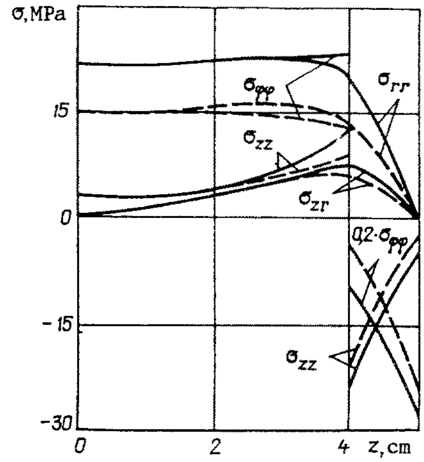

Fig. 3

$$\lambda_{zz} = 0{,}233 \ \text{W/cm} \cdot K, \ \lambda_{rr} = 0{,}133 \ \text{W/cm} \cdot K, \ \lambda_{\varphi\varphi} = 0{,}333 \ \text{W/cm} \cdot K,$$

$$c\rho = 4{,}41 \ \text{J/(cm}^3 \cdot K), \ \nu_{zr} = 0{,}22, \ \nu_{z\varphi} = \nu_{r\varphi} = 0{,}022,$$

$$E_z = E_r = 1{,}4 \cdot 10^4 \ \text{MPa}, \ E_\varphi = 7 \cdot 10^4 \ \text{MPa}, \ G_{zr} = 0{,}3 \cdot 10^4 \ \text{MPa},$$

$$G_{z\varphi} = G_{r\varphi} = 1{,}5 \cdot 10^4 \ \text{MPa}, \ \alpha^T_{zz} = \alpha^T_{rr} = 1{,}1 \cdot 10^{-5} \ 1/^\circ\text{C}, \ \alpha^T_{\varphi\varphi} = 2{,}2 \cdot 10^{-5} \ 1/^\circ\text{C}.$$

The calculated temperature and stress fields are shown in Figs. 2 and 3.

Figure 2 shows the radial dependence of the temperature near the insulated end of the cylinder and in the section z = 8.5 cm for 30 and 150 seconds of heating and for two values of the azimuthal coordinate: $\varphi = 0$ (solid curves) and $\varphi = \pi$ (dashed curves). To test the method of solving the heat-conduction problem, two additional calculations were carried out: first, the thermal conductivities λ_{ij}^0 of the anisotropic layer were assumed to be the same in all directions and equal to the average value, while the direction dependence was taken into account by the functions $\omega_{ij}^T\lambda_{ij}^0$; second, the λ_{ij}^0 were assumed to be equal to the largest of the λ_{ij}^0. In all three cases, the calculated temperatures were the same to within 1-2 degrees.

The radial dependence of the stress for 150 sec of heating is shown in Fig. 3 for $\varphi = 0$ and $z = 8.5$ cm for the elastic solution (solid curves) and the solution taking into account the inelastic deformation of the inner layer of the cylinder (dashed curves). The calculations show that the nonaxisymmetric temperature distribution leads to stress variations of 5-7% in the outer layer. The maximum values occur at $\varphi = 0$. Calculations using the average values of the thermal and mechanical characteristics of the outer layer predict stresses that are much too low.

REFERENCES

1. A. I. Gulyar, "Method of calculating three-dimensional structures using an analytical finite element method for closed noncircular finite elements," Soprotivlenie Materialov Teor. Sooruzhen., No. 44, 44-46 (1984).
2. S. G. Lekhnitskii, Theory of Elasticity of an Anisotropic Body [in Russian], Nauka, Moscow (1977).
3. V. I. Lipovskii and A. G. Makarenkov, "Time-dependent temperature fields in curved orthotropic bodies of revolution," Mathematical Methods of Heat and Mass Transfer [in Russian], Dnepropetrovsk (1985), pp. 22-27.
4. V. G. Savchenko, "Time-dependent temperature fields in bodies of revolution under nonaxisymmetric heating," Probl. Prochn., No. 1, 33-36 (1982).
5. V. G. Savchenko, "On a method of solving three-dimensional nonaxisymmetric problems of thermoplasticity," Thermal Stresses in Structural Elements, No. 18, 24-29 (1978).
6. A. S. Sakharov, A. V. Gondlyakh, S. L. Mel'nikov, and A. N. Snipko, "Numerical modeling of breakdown processes in layered composite shells," Mekh. Kompozit. Mater., No. 3, 459-465 (1989).
7. A. S. Sakharov, A. I. Gulyar, and A. G. Topor, "Numerical solution of thermoelastic equilibrium problems for nonaxisymmetrically loaded bodies of revolution," Prikl. Mekh., **22**, No. 6, 7-13 (1986).

8. Yu. N. Shevchenko, V. V. Piskun, and V. A. Kovalenko, "Thermostress in a curved orthotropic inhomogeneous body of revolution," Prikl. Mekh., **19**, No. 7, 3-11 (1983).
9. Yu. N. Shevchenko, V. V. Piskun, and V. A. Kovalenko, "Elastoplastic state of axisymmetrically loaded isotropic and orthotropic layered bodies of revolution," Prikl. Mekh., **28**, No. 1, 31-39 (1992).
10. Yu. N. Shevchenko and V. G. Savchenko, Thermoviscoplasticity: Mechanics of Coupled Fields in Structural Elements [in Russian], Vol. 2, Naukova Dumka, Kiev (1987)