# THE RADIATION WAVE ZONE

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The criterion for the formation of a radiation wave zone of a charge moving in an arbitrary way is discussed. The appearance of a wave zone is investigated in detail for synchrotron radiation and radiation from a charge moving along a hyperbola. Special attention is given to the ultrarelativistic case.

The concept of the wave zone plays a fundamental role in the theory of radiation from a charge. Meanwhile, no rigorous criterion for the formation of a wave zone was available for a long time. In the special case of radiation originating when a charge moves in a circle the problem has been considered in [1] (see also [2]). Below we consider the formation of a wave zone in the case of radiation from a particle moving in an arbitrary way.

First of all we recall the generally accepted condition for the appearance of the radiation wave zone.

It is well known (see [3]) that the field of a charge moving in an arbitrary way is described by the expressions

$$\begin{aligned} \boldsymbol{E} &= \frac{e}{R^2[1-(\boldsymbol{n\beta})]^3}\left\{(1-\beta^2)(\boldsymbol{n}-\boldsymbol{\beta})+\frac{R}{c^2}[\boldsymbol{n}[(\boldsymbol{n}-\boldsymbol{\beta})\boldsymbol{a}]]\right\}, \\ \boldsymbol{H} &= -\frac{e}{R^2[1-(\boldsymbol{n\beta})]^3}\left\{(1-\beta^2)[\boldsymbol{n\beta}]-\frac{R}{c^2}[\boldsymbol{n}[\boldsymbol{n}[(\boldsymbol{n}-\boldsymbol{\beta})\boldsymbol{a}]]]\right\}. \end{aligned} \tag{1}$$

Here v is the velocity; $a$ is the acceleration; e is the magnitude of the charge; c is the velocity of light; $\beta = v/c$; R is the distance from the charge to the point of observation; and $\mathbf{n} = \mathbf{R}/R$.

The field intensities **E** and **H** are taken at the moment $\tilde{t}$ of observation, and the quantities v, $a$, and R on the right-hand sides of Eqs. (1) are referred to a moment of time t which differs from $\tilde{t}$ by the retardation time of the radiation $\Delta t = \tilde{t} - t = R/c$.

The fields **E** and H are orthogonal: $\mathbf{H} = [\mathbf{nE}]$, and, as is evident, they split into two parts, one of which depends only on the velocity of the charge (velocity field) and another which depends only on the acceleration (acceleration field). The velocity field corresponds to the so-called convective field of a uniformly moving charge, and at large distances from the charge it decays according to the law $1/R^2$. The acceleration field has the transversality property, characteristic of the field of a plane electromagnetic wave:

$$\tilde{\boldsymbol{H}} = [\boldsymbol{n}\tilde{\boldsymbol{E}}], \quad (\boldsymbol{n}\tilde{\boldsymbol{E}}) = (\boldsymbol{n}\tilde{\boldsymbol{H}}) = 0,$$

and decays like $1/R$. This field is usually associated with the radiation field.

It should be remarked, however, that such an approach is not sufficiently correct, and it is advisable to understand the statement "the charge radiates" in a broader sense, taking into account both the acceleration field and the velocity field, as has been explained in [4].

When acceleration is present, then at large distances from the charge

$$R \gg \frac{1-\beta^2}{a} c^2 \tag{2}$$

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the radiation is represented by a purely plane-wave field. Relation (2) is customarily taken as the condition for the appearance of a wave zone of radiation.

In our view such a criterion is rather crude; actually the radiation field can also appear as a plane-wave field at smaller distances from the charge in small spatial regions (this has also been hinted at in [4]).

In order to make a detailed investigation of a radiation wave zone at one distance or another from the charge we consider the instantaneous angular distribution of the power radiated from the charge.

It can be shown by a standard method that in a coordinate system moving with the charge the angular distribution of radiated power in the situation under discussion has the form

$$dW = \frac{c}{4\pi}\frac{e^2\beta^4}{\rho^2}\left\{\frac{\sin^2\varphi}{(1-\beta\cos\vartheta)^3} + \frac{[(\beta-\cos\vartheta)\cos\varphi + \varkappa\sin\vartheta]^2}{(1-\beta\cos\vartheta)^5}\right\}d\Omega, \tag{3}$$

where $\rho$ is the radius of curvature of the trajectory, and

$$\varkappa = \operatorname{ctg}\alpha + \frac{1-\beta^2}{\beta}\frac{\rho}{R}, \quad \operatorname{ctg}\alpha = \frac{a_\parallel}{a_\perp},$$

$$a_\parallel = \frac{(\beta a)}{\beta}, \quad a_\perp = \frac{|[\beta a]|}{\beta} = \frac{c^2\beta^2}{\rho}.$$

The coordinate system is chosen in such a way that the charge is at the origin, the velocity vector $\mathbf{v}$ of the charge is oriented along the z-axis, and the radius-of-curvature vector $\boldsymbol{\rho}$ of the trajectory is directed antiparallel to the x-axis, i.e., $(\mathbf{n}\beta) = \beta\cos\vartheta$, $(\boldsymbol{\rho}\mathbf{n}) = -\rho\sin\vartheta\cos\varphi$. The element of solid angle is $d\Omega = \sin\vartheta d\vartheta d\varphi$.

We recall that on the right-hand side of (3) all quantities are taken at the moment of time t, which differs from the moment of observation $\tilde{t}$ by the retardation time $\Delta t$ of the radiation.

From an analysis of expression (3) it follows that in the most interesting cases (with $\alpha \neq 0$) the criterion for the formation of the radiation wave zone can be represented in the form

$$R \gg \frac{1-\beta^2}{\beta} f(\vartheta, \varphi; \alpha; \beta)\rho, \tag{4}$$

where f is some function of its arguments.

A simple criterion is obtained in the case of synchrotron radiation ($\alpha = \pi/2$) in the osculating plane of the trajectory of the charge ($\varphi = \pi/2, 3\pi/2$),

$$R \gg \frac{1-\beta^2}{\beta}\frac{\sin\vartheta}{1-\beta\cos\vartheta}\rho \tag{5a}$$

and in the rectifying plane ($\varphi = 0, \pi$)

$$R \gg \frac{1-\beta^2}{\beta}\frac{\sin\vartheta}{|\beta-\cos\vartheta|}\rho. \tag{5b}$$

From this it follows that when a charge is moving in a circle, for small angles $\vartheta \approx 0$, the radiation wave zone can begin in the immediate neighborhood of the charge, no matter what the velocity is with which the charge is moving. In the ultrarelativistic case $\beta \to 1$ the radiation of the charge has a number of distinctive characteristics, and we shall therefore consider this case in some detail below.

The distinctive features become pronounced for hyperbolic motion of the charge, when $\alpha = 0$.

In fact, if we put $\alpha = 0$ in expression (3), the surviving terms will have identical axial symmetry, so that the criterion for the formation of the radiation wave zone loses its dependence on the angles $\vartheta$, $\varphi$, and assumes the form

$$R \gg (1-\beta^2)\beta\frac{c^2}{a_\parallel}. \tag{6}$$

The time dependence of $\beta$ and $a_\parallel$ can be found in explicit form. If the velocity of the charge was $v_1$ at the orgin of time t = 0, then

$$\beta(t) = \frac{B}{(1+B^2)^{1/2}}, \quad a_{\parallel}(t) = \frac{a_0}{(1+B^2)^{3/2}}, \tag{7}$$

where

$$B = \frac{a_0}{c} t + \frac{\beta_1}{\sqrt{1-\beta_1^2}},$$

and $a_0$ is the acceleration of the charge in its proper reference frame. From this the criterion for the appearance of a wave zone must be

$$R \gg ct + \frac{\beta_1}{\sqrt{1-\beta_1^2}} \frac{c^2}{a_0}. \tag{6'}$$

Using criterion (6'), one can clear up the situation with Pauli's well-known paradox [5] concerning the absence of a wave zone when a charge is moving along a hyperbola (see also [4]).

In the general case, when the charge is moving in an arbitrary way, the function f in (4) has a simpler form for radiation in the rectifying plane ($\varphi = 0, \pi$):

$$R \gg \frac{1-\beta^2}{\beta} \frac{\sin\vartheta}{|\beta - \cos\vartheta \pm \operatorname{ctg}\alpha \sin\vartheta|} \rho. \tag{8}$$

From this it follows that there are always two directions for which the wave zone is at infinity. It is easily seen that in this case

$$\cos\vartheta = \beta\sin^2\alpha \pm \cos\alpha\sqrt{1-\beta^2\sin^2\alpha}. \tag{9}$$

This condition determines the two directions along which radiation due to the acceleration of the charge is absent [3].

According to (8) the wave zone tends to approach the charge when it moves at higher velocities ($\alpha \neq 0$!).

In the ultrarelativistic case, when practically all the radiation is concentrated in a narrow cone $\vartheta \approx \sqrt{1-\beta^2}$, it is convenient to obtain an integral criterion for the formation of a wave zone. To this end we integrate expression (3) over all angles and obtain

$$W = \frac{2}{3} \frac{ce^2\beta^4}{\rho^2} \left\{ \frac{1}{(1-\beta^2)^2} + \frac{\kappa^2}{(1-\beta^2)^3} \right\}. \tag{10}$$

It follows from this that in the special case of synchrotron radiation, the criterion for the formation of a wave zone, when $\beta \to 1$ has the form

$$R \gg \frac{\sqrt{1-\beta^2}}{\beta} \rho. \tag{11}$$

A similar result follows from the angular criterion (5) if the passage to the limit $\vartheta \to \sqrt{1-\beta^2}$ is made.

Matters become complicated when the charge moves in an arbitrary way. It would appear that at large velocities the principal contribution comes from the second term in (10). However, this is in fact not the case. Actually the quantity cot $\alpha$, appearing in $\kappa$, is a function of the velocity of the charge and the intensities of the external electromagnetic fields, whose action is responsible for the motion of the charge:

$$\operatorname{ctg}^2\alpha = (1-\beta^2)^2 \frac{\left(\beta \frac{dP}{dt}\right)^2}{\left[\beta \frac{dP}{dt}\right]^2} = (1-\beta^2)^2 \frac{(\beta E)^2}{\beta^2 (E + [\beta H])^2 - (\beta E)^2}; \tag{12}$$

$$P = m_0 v / \sqrt{1-\beta^2}.$$

Thus expression (10) can be rewritten in the form

$$W = \frac{2}{3} \frac{e^4}{m^2c^3(1-\beta^2)} \left\{ ([\nu E] + [\nu[\beta H]])^2 + \left[ (\nu E)\sqrt{1-\beta^2} + \frac{m\beta c^2}{eR} \right]^2 \right\},$$

where the notation $\boldsymbol{\nu} = \mathbf{v}/v$ is used.

From this it follows that in the ultrarelativistic case when $H \neq 0$ and $\boldsymbol{\nu}$ is not parallel to $\mathbf{E}$, the wave zone criterion is the condition $R \gg (\sqrt{1-\beta^2/\beta})\rho$, which is identical to (11).

Thus, in the general case of a charge moving ultrarelativistically in an arbitrary way the criterion for the formation of a wave zone does not differ from the corresponding criterion for synchrotron radiation. This fact is not unexpected and conforms with the conclusions of a number of authors concerning general regularities of radiation from ultrarelativistic charges [2, 6, 7]. In the particular case of a charge moving along a hyperbola the integral criterion coincides exactly with the angular criterion for any velocity of the motion.

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