CORRECTION FOR RADIATION IN THE SCATTERING OF DIRAC PARTICLES IN AN EXTERNAL CENTRAL FIELD

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This scattering is considered for the relativistic case with allowance for radiation corrections by a method previously described [1-3] for solving radial equations for the nonrelativistic case, the basis being an approximate method [4] employing trial wave functions whose parameters are found without resort to variational methods. Detailed formulas are deduced. The scattering of Klein-Gordon particles is also considered.

The relativistic quantum mechanics of particle collision involves the same difficulties as the relativistic theory of line spectra, which arise, in part, because no rigorous and relativistically invariant theory has been available even for a system consisting of only two particles. However, the motion of a particle in an external field can be described with adequate rigor in certain instances, in which case it is usual to neglect processes such as pair production and emission or absorption of quanta. Quantum field theory must be used from the start in any more rigorous discussion of problems involving (say) change in the number of particles, and this gives rise to additional difficulties [5], which it is not my object to discuss here.

It is true that only relatively few problems can be handled on the basis of particle plus external field, but this approximation is of great theoretical and practical importance, being one of the basic methods of current relativistic quantum theory. But even then there have so far been no exact analytic methods for collision problems, apart from some instances of coulomb scattering, scattering at a potential well, and certain other fields of purely trial interest.

Perturbation theory is used in relativistic studies of continuous spectra, but this is restricted to the region of weak interactions, which is a severe restriction. Comparison with numerical integration [6] shows that the Born approximation [7, 8] becomes unsuitable for $Z \geq 10$ even in the scattering of electrons of energy 100 MeV. The essential difference between the relativistic scattering of particles described by Dirac's equation and nonrelativistic scattering described by Schrödinger's equation is that in the second the interaction with the field may be treated as a small perturbation at high energies, because $\varkappa \to \infty$ implies that the phase $\eta_{lj} \to 0$, whereas the latter is in general not so for the scattering of Dirac particles [9], the phase remaining substantial even for $\varkappa \to \infty$, so perturbation theory cannot be applied [10, 11].

Here I extend my recent treatment for nonrelativistic problems [1-3] to give methods of solving the radial equations for the scattering of particles of spin 1/2 in a static external scalar central field.

The treatment may be made much more general and precise by using some results from the quantum theory of fields instead of employing simply the one-particle relativistic Dirac equation: use is made (for particles of spin 1/2) of a modified Dirac wave equation (Dirac's equation incorporating radiation corrections), which was first derived by Schwinger [12]. The additional terms in the hamiltonian arise because the external field polarizes the vacuum, and this may interact (via its virtual field) with the particle and with the zero-point fluctuations of the vacuum. Pauli [13] pointed out that Dirac's equation could be generalized somewhat without loss of relativistic invariance, although he did not consider polarization of the vacuum.

## Exact Solution

The equation to be used is [14] the modified relativistically invariant Dirac equation for the scattering of a particle of mass m and spin 1/2 (the system of units is such that $\hbar = c = 1$):

$$\sum_{\mu} (\pi_\mu \gamma_\mu - im) \Psi = \sum_{\mu\nu} \left[ \frac{g_1 q}{4m} \gamma_\mu \gamma_\nu F_{\mu\nu} - \frac{g_2 q}{m^2} \gamma_\mu \Box^2 A_\mu \right] \Psi; \tag{1}$$

$\Psi$ is the bispinor describing the particle and $\gamma_\mu$ is the usual 4-dimensional Dirac matrix ($\gamma_4$ is diagonal). The operator

$$\pi_\mu = - i\partial_\mu + q \cdot A_\mu, \qquad A_\mu (\bar{A}, i\varphi), \tag{2}$$

in which $\partial_\mu$ denotes the derivative with respect to $x_\mu$; $\bar{A}$ and $\varphi$ are the vector and scalar parts of the four-dimensional potential; $\Box^2$ is the D'Alembert operator; $F_{\mu\nu}$ is the antisymmetric tensor of second rank that describes the external field:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu; \qquad x_\mu (\bar{r}, i, t), \tag{3}$$

and q is the constant for the interaction between the particle and the field.

The terms on the right in (1) represent radiation corrections, $g_1$ and $g_2$ being dimensionless constants given by the quantum theory of fields.

Consider an external static scalar central field; here $\overline{A} = 0$ and $\varphi = \varphi(r)$. The permutation relations satisfied by the Dirac matrix, taken with (2) and (3), enable us to put the right side of (1) in the form

$$iq\left[\frac{g_1}{2m}\sum_{j=1}^{3}\gamma_j\partial_j\varphi - \frac{g_2}{m^2}\Delta\varphi\right]\gamma\Psi, \tag{4}$$

in which $\Delta$ is the Laplace operator.

The variables are separated in spherical coordinates. The problem is a stationary one, so the time dependence of $\Psi$ takes the form $\Psi(\overline{r}, t) = \psi(\overline{r}) \exp - iEt$, in which E is the total energy of the particle. We represent the bispinor $\psi$ as a set of two spinors:

$$\psi(\overline{r}) = \begin{vmatrix} \Sigma_1(\overline{r}) \\ \Sigma_2(\overline{r}) \end{vmatrix}.$$

We expand $\Sigma_i(\overline{r})$ with respect to spherical spinors and represent these in the form

$$\begin{gathered}\Sigma_1(\overline{r}) = \frac{i}{r}G_{Ejl}(r)\,\Omega_{jlM}(\Theta, \varphi),\\ \Sigma_2(\overline{r}) = \frac{1}{r}F_{Ejl}(r)\,\Omega_{jl'M}(\Theta, \varphi), l' = 2_j - j,\end{gathered} \tag{5}$$

in which $\Omega$ is a spherical spinor, j and M being the quantum numbers for the total angular momentum and for the projection of this on the Z axis; $l$ is the first parameter of the spherical function. The subscripts will in future be omitted when this can lead to no misunderstanding. The $\Omega$ are known [15], so the problem reduces to that of finding the radial functions. $\Sigma_1$ and $\Sigma_2$ are respectively the large and small components of $\psi$, so $\Sigma_2$ vanishes in the nonrelativistic approximation.

We substitute (5) into (1) and use (2)-(4) together with the known properties of spherical spinors [15]; tedious manipulations give us a system of four radial equations:

$$G'_{\varkappa l}(r) + D_{\varkappa l}(r)\,G_{\varkappa l}(r) - (E + m + q\varphi^+(r))\,F_{\varkappa l}(r) = 0, \tag{6a}$$

$$F'_{\varkappa l}(r) - D_{\varkappa l}(r)\,F_{\varkappa l}(r) + (E - m + q\,\varphi^+(r))\,G_{\varkappa l}(r) = 0, \tag{6b}$$

$$\varkappa = l \quad \text{for} \quad j = l - 1/2 \text{ and } \varkappa = -l - 1 \quad \text{for} \quad j = l + 1/2 \tag{7}$$

(as usual in Dirac's theory, $\varkappa$ takes only the value $-1$ for $l = 0$), with

$$D_\varkappa(r) = \frac{\varkappa}{r} + g_1\frac{\varphi'(r)\,q}{2m}\ ;\ \varphi^+(r) = \varphi(r) + \frac{g_2}{m^2}\left[\varphi''(r) - \frac{2}{r}\varphi'(r)\right]. \tag{8}$$

The primes denote differentiation with respect to r. The two values for $\varkappa$ (for a given momentum) correspond to the two linearly independent components of the spinor wave, which represent the two possible states of polarization. We eliminate F from (6) to get a second-order differential equation for G:

$$G'' - \frac{T'_-}{T_-}\cdot G' + (D' - D\frac{T'_-}{T_-} - D^2 + T_+T_-)\,G = 0, \tag{9}$$

in which

$$T_\pm(r) = E \mp m + q\varphi^+(r), \tag{10}$$

F being expressed in terms of G from (6) and (10) as

$$F_\varkappa(r) = T_-^{-1}(r)\,[G'_\varkappa(r) + D_\varkappa(r)\,G_\varkappa(r)]. \tag{11}$$

I consider at a later point the solution of (9) with exact treatment of the radiation corrections; at this point I consider only the terms in $g_2$ in (1). Here $D_\varkappa = \frac{\varkappa}{r}$ from (8). We put

$$N_\varkappa(r) = T_-^{-1/2}(r)\, G_\varkappa(r), \tag{12}$$

and transform (9) to

$$N''_\varkappa(r) + \left[\tilde{\kappa}^2 - \frac{\varkappa(\varkappa+1)}{r^2} - V_\varkappa(r)\right] N_\varkappa(r) = 0, \tag{13}$$

in which

$$\tilde{\kappa}^2 = (E^2 - m^2), \tag{14}$$

$$V_\varkappa(r) = -2qE\varphi^+ - (q\varphi^+)^2 + \frac{T'_-}{T_-}\left[\ln\frac{r^\varkappa T_-^{3/4}}{T_-'^{1/2}}\right]' \cdot q \tag{15}$$

and $\varkappa$ is given by (7). G(r) satisfies the usual boundary conditions

$$G_{jl}(0) = 0; \tag{16}$$

$$G_{jl}(r) \backsim A_{jl}\left(\tilde{\kappa} - l\frac{\pi}{2} - \eta_{jl}\right). \tag{17}$$

The boundary conditions for N(r) are implied by (12), (16), and (17).

Equation (13) has been solved [1] in the nonrelativistic quantum theory of collisions by means of generalized power series; the relativistic (13) differs from the equations considered in [1] only in the value of $\bar{\kappa}$ and in the precise form of $V_\varkappa(r)$, so I shall not repeat the derivation of N(r) and merely give the final results, which follow simply and directly from [1]. The following are some general comments on this.

It is clear from (17) that we have so far considered only the case in which $V_\varkappa(r)$ decreases more rapidly than 1/r for $r \to \infty$; as in [1], (13) has, in general, only two singular points (at 0 and $\infty$). I assume that $\varphi(r)$ has no divergence higher than a coulomb one at $r \to 0$; but $V_\varkappa(r)$ here differs from V(r) in being for the relativistic scattering of a particle of spin 1/2 and having a pole of second (not first) order for $r \to 0$. At first sight it seems that the term in $\varphi^+(r)$, which describes the radiation corrections, would give a pole of third order (which usually corresponds to tensor forces); but this is not so, and it is readily shown that all third-order poles mutually cancel in combinations of radiation terms. This is of major importance here, for the general analytic theory of differential equations [16] shows that only the absence of third-order poles allows us to apply directly the method developed in [1, 2] for the solution of (13).

It merely remains to find the root of the characteristic equation for $N_\varkappa(r)$ and to derive the recurrence formulas.

We have $\varphi^+(r) = \nu/r$ for $r \to 0$, with $\nu = \varphi_{-1} + 2g_2 m^{-2}\varphi_1$, in which the $\varphi_i$ are the coefficients in the expansion $\varphi(r) = \sum_{i=-1}^{\infty} \varphi_i r^i$. Then the only characteristic root $\lambda_\varkappa$ that satisfies (12) and (16) is of the form

$$\lambda_\varkappa = 1/2 + [\varkappa^2 - q^2\nu^2]^{1/2}. \tag{18}$$

By analogy with [1], we obtain N as a generalized power series:

$$N_{\varkappa l}(r) = C_{\varkappa l0} r^{\lambda_\varkappa} \sum_{n=0}^{\infty} \alpha_{\varkappa ln} r^n;\ C_{\varkappa ln} = \alpha_{\varkappa ln} C_{\varkappa l0}, \tag{19}$$

in which $C_{\varkappa l0}$ is some constant (generally complex); the $\alpha_{\varkappa ln}$ are given by the recurrence relation

$$\alpha_{\varkappa ln} = \frac{\sum_{i=1}' b_{\varkappa l-2}\, \alpha_{\varkappa ln-i}}{n^2 + 2\lambda_\varkappa n - n}\quad \alpha_{\varkappa l0} = 1;\quad n = 1,2,3\ldots \tag{20}$$

The prime to the sum indicates that the $b_{\varkappa 0}$ in (20) must be replaced by $b_{\varkappa 0} - \tilde{K}^2$; the $b_{\varkappa i}$ are the coefficients of

$$V_\varkappa(r) = \sum_{i=-2}^{\infty} b_{\varkappa i} r^i. \tag{21}$$

The general theory [16] shows that series (19) for $N_{\varkappa l}(r)$ converges to the exact solution, the radius of convergence extending to the nearest singularity. The nearest and only singularity lies at $\infty$, so the series converges to the exact solution (at least for r finite), which serves to solve the problem, on account of the behavior of $V_{\varkappa}(r)$ for r large.

It follows from (12) that the asymptotes of N and G coincide apart from a constant of no significance here; the asymptote of N(r) therefore also has the form of (17). The formulas for the phases $\eta$ in terms of an asymptote of the form of (17) have been derived and expressed in terms of the $\alpha$ of (20) [2], so the scattering phases for (17) are known. Also, formulas are known [15] for the effective relativistic scattering cross-section in terms of the asymptotic phases $\eta_{jl}(\tilde{K})$ and $\eta_{j-l-1}(\tilde{K})$ from (17) for both directions of spin; these are not given here.

Now I turn to the limits on the applicability of this method arising from the force of interaction between the particle and the external field.

We discarded the second root of the characteristic equation and used only the first root on the basis that it did not satisfy (16); but (18) shows that this applies only to real $\lambda_{\varkappa}$, i. e., for $\varkappa^2 - q^2 \cdot \nu^2 > 0$. It is clear that, roughly speaking, $q^2\nu^2 = \alpha^2$, in which $\alpha$ is the fine-structure constant; $\alpha^2 \approx 5 \cdot 10^{-5}$ for electromagnetic interactions of elementary particles, and so (18) is correct (and the method is applicable) even if the interaction force exceeds the force of electromagnetic interaction by not less than two orders of magnitude. The second root must be taken into account for fields stronger than this; then (19) will be replaced by a combination of two series. The case requires special examination if this modification is impracticable.

Approximate Solution

The above solution gives rise to considerable labor when numerical results are required, except for the relative values of the wave functions for r small. This makes it of interest to have also a simple approximate analytic solution, for which purpose it is convenient to use trial wave functions whose parameters are determined without resort to variational methods; this method has been used [4] to solve analogous nonrelativistic problems and is readily extended to the present case. Moreover, both radiation corrections can be incorporated.

First I consider the changes in the basic nonrelativistic forms of the method of trial wave functions [4] that have been considered in [17-19].

In the generalized method of determining the parameters [17] we have to replace the Schrödinger-type radial operator by a relativistic Dirac one, as (9) shows; the radiation corrections are also incorporated in this:

$$L_{\varkappa l} = \frac{\alpha^2}{dr^2} - \frac{T'_-}{T_-} \cdot \frac{d}{dr} + \left(D' - D\,\frac{T'_-}{T_-} - D^2 + T_+\,T_-\right), \qquad (22)$$

in which T and D and given by (10) and (8).

We introduce the n-parameter trial wave function $\overline{G}_{\varkappa l}(r)$. We apply operator $L_{\varkappa l}$ to $\overline{G}_{\varkappa l}$, multiplying from the left by the $A_{j\varkappa l}(r)$ and integrating from 0 to $\infty$; this gives a system of equations that may be solved via (17) give the scattering phases:

$$\int\limits_0^\infty A_{j\varkappa l}(r)\,L_{\varkappa l}\,\overline{G}_{\varkappa l}(r)\,dr = 0;\; I = 1,2\ldots n. \qquad (23)$$

The $A_{j\varkappa l}(r)$ must ensure that the integrals converge; (7) shows that $\varkappa$ has two values.

It can be shown that asymptotic self-consistency [18] may be used, as in the nonrelativistic case [17], to obtain an appropriate set of $A_{j\varkappa l}(r)$ from the generalized method given above. We may also substitute $\overline{G}$ directly into Parzen's [9] integral identities for the phases.

The virial theorem [17] must be applied with a relativistic relation first derived by Novozhilov [20] for the Dirac equation applied to a continuous spectrum:

$$\int\limits_0^\infty [\varphi(r) + r\varphi'(r)](\overline{G}^2 + \overline{F}^2)\,dr = \frac{A^2}{qm}(\tilde{\kappa}\partial_{\tilde{\kappa}} + m\partial_m)\eta. \qquad (24)$$

This differs from the nonrelativistic one in containing the derivative of the phase with respect to the mass, which hinders [20] its use in calculations. We can use (24) in a good approximation only for high energies such that $\tilde{K}\partial_{\tilde{K}}\eta \gg\gg m\partial_m\eta$. No difficulty arises over the convergence of the integrals for $l > 1$.

Finally, the expansion method [19] involves simply expansion with respect to small r in $L_{\varkappa l}G_{\varkappa l}(r) = 0$. The parameters of the trial functions are determined as in the nonrelativistic case; in all cases they must be such as to satisfy (16) for $r \rightarrow 0$.

## Conclusions

The same formulas for the $\alpha_{\varkappa l}$ are reached for the case $\varphi(r) \propto 1/r$ for $r \rightarrow \infty$, which has not been considered here; (17) is then replaced by the asymptote of the coulomb field. The $\alpha_{\varkappa l}$ might be related to the phases from this asymptote by the method of [2] for an asymptote of the form of (17); but I consider that it would be of interest to make a detailed study of the convergence of (19) for this case. The approximate method for this case involves only a certain change in the virial relation [20] (together with the change in the asymptote), where $\varphi(r)$ loses its coulomb part, as it were.

It has been claimed [14] that the radiation corrections (in the form used here) do not always give a complete description of the interaction of the particles with the vacuum in the ultrarelativistic case.

The method is applicable, generally speaking, to the scattering of any particle of spin 1/2 in any field; but special practical interest attaches to cases in which perturbation theory is inapplicable on account of the strong interactions between the particle and the external field.

The method may also be applied to the scattering of particles of Klein-Gordon type (spin 0); here the wave function for the scattered particle has one component, whose angular part is a spherical function, the equation for the radial part being simply (9). The simplification of $T_{\pm}$ and D is trivial.

## REFERENCES

1. V. D. Kamenetskii, Izv. VUZ. Fizika, no. 4, 86, 1963.
2. V. D. Kamenetskii, Izv. VUZ. Fizika, no. 1, 107, 1964.
3. V D. Kamenetskii, Izv. VUZ. Fizika, no. 4, 82, 1963; no. 2, 60, 1964.
4. V. D. Kamenetskii, Optika i Spektroskopiya, 9, 111, 1960; Dissertation, Leningrad, 1961.
5. N. N. Bogolyubov and D. V. Shirkov, Introduction to the Theory of Quantized Fields [in Russian], GTTI, Moscow, 1957.
6. D. Yennie, D. Ravenhall, and R. Wilson, Phys. Rev., 92, 13251, 1953; 95, 500, 1954.
7. J. Smith, Phys. Rev., 95, 271, 1954.
8. L. Schiff, Phys. Rev., 92, 988, 1953.
9. G. Parzen, Phys. Rev., 80, 261, 1950.
10. J. Bartlett and R. Watson, Proc. Am. Acad. Arts. Sci., 74, 53, 1940.
11. W. McKinley and H. Feshbach, Phys. Rev., 74, 1759, 1948.
12. J. Schwinger, Phys. Rev., 76, 790, 1949.
13. W. Pauli, The General Principles of Wave Mechanics [Russian translation], GTTI, Moscow, 1947.
14. H. A. Bethe and E. E. Salpeter, Quantum Mechanics of One- and Two-Electron Atoms [Russian translation], GIFML, 1960.
15. A. I. Akhiezer and V. B. Berestetskii, Quantum Electrodynamics [in Russian], GTTI, Moscow, 1963.
16. P. M. Morse and H. Feshbach, Methods of Theoretical Physics [Russian translation], vol. 1, IL, Moscow, 1958.
17. V. D. Kamenetskii, Izv. VUZ. Fizika, no. 5, 134, 1961.
18. V. D. Kamenetskii, Izv. VUZ. Fizika, no. 3, 42, 1961.
19. V. D. Kamenetskii and B. M. Yavorskii, Izv. VUZ. Fizika, no. 4, 66, 1960.
20. Yu. V. Novozhilov, Vestnik Leningrad Univ., no. 4, 5, 1957.

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