ENERGY RADIATED BY AN ARBITRARY MATERIAL SYSTEM

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The solution of the Einstein field equations for the case of an arbitrary material system is analyzed on the basis of the asymptotic solutions found by Sachs for a radiative system devoid of symmetry. The tetrad formalism is used to find the energy and momentum of the system.

The tetrad description of the gravitational field has been used by several investigators [1, 2] to find a satisfactory solution of the energy problem: at each spatial point the vector h_i characterizing the curvilinear coordinates is supplemented by an orthogonal basis having tetrad components $h(a)$. From the mutual projections (Lamé coefficients) a metric tensor is constructed:

$$g_{ij} = h_i(a)\, h_j^{(a)}. \tag{1}$$

Since the tetrad components are given ambiguously by this relation, it is important to introduce auxiliary conditions; the Rodichev gauge conditions

$$\nabla_l h^{l(a)} = \frac{1}{\Lambda} \frac{\partial}{\partial x^l} \Lambda h^{l(a)} = 0, \tag{2}$$

where $\Lambda = 1$, have been used for this purpose.

The ambiguity in the choice of auxiliary conditions spurred Möller [2] to use the tetrads as special auxiliary quantities, which impart a meaning only to integral expressions. In this manner the auxiliary conditions, i.e., tetrad gauge conditions, can be avoided.

Rodichev, on the other hand [1], uses tetrads as locally Lorentzian systems, which move in a gravitational field along geodesic lines at an acceleration equivalent to the field intensity. The tetrad distribution therefore characterizes the gravitational field.

Since there are two bodies of opinion on this question, we will compare the corresponding results for this case. Sachs [4] found the form of the metric for an arbitrary material system devoid of symmetry and radiating gravitational waves. Here the concept of the "mass" of the system was introduced (including the mass of the sources and the gravitational field), and this mass was said to decrease as time elapsed as a result of the radiation.

Applying the theory which he developed, and using a special class of coordinate systems which asymptotically become Lorentzian systems, Möller found expressions for the energy and momentum consistent with those found by Sachs.

Let us examine the solution from the Rodichev point of view [1]. In the coordinate system

$$x^i = \{x, y, z, t\}, \quad r = \sqrt{x^2 + y^2 + z^2} \tag{3}$$

the asymptotic Sachs solution can be written as [3]

$$g_{ik} = \eta_{ik} + y_{ik} + z_{ik} + O_3, \tag{4}$$

where

$$y_{ik} = \alpha_{ik} r^{-1}, \quad z_{ik} = \beta_{ik} r^{-2}, \tag{5}$$

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and α_{ik} and β_{ik} are functions only of Θ and φ and of the retarded time $u = t - r$; and O_n is a term which falls off as r^{-n} as $r \to \infty$. If, following [3], we introduce

$$\begin{gathered} n_i = \{n_i, 0\}, \\ m_i = \{\cos\theta\cos\varphi,\ \cos\theta\sin\varphi,\ -\sin\theta,\ 0\}, \\ l_i = \{-\sin\varphi,\ \cos\varphi,\ 0,\ 0\}, \\ \mu_i = \eta_i^4 - n_i,\ \ t_i = m_i - il_i, \end{gathered} \tag{6}$$

we find

$$\alpha_{i\kappa} = 2Re\{ct_i t_\kappa\} + Re\{\psi(t_i\mu_\kappa - \mu_i t_\kappa)\} + 2M\mu_i\mu_\kappa; \tag{7}$$

$$\beta_{i\kappa} = 2|c|^2(m_i m_\kappa + l_i l_\kappa) + \frac{1}{2}|c|^2(n_i\mu_\kappa + \mu_i n_\kappa) - Re\left\{\left(2N + \frac{1}{2}\Delta|c|^2\right)(t_i\mu_\kappa + \mu_i t_\kappa)\right\} + Re\{\Delta^*(N\sin\theta)\}\mu_i\mu_\kappa, \tag{8}$$

where

$$\Delta = \frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}, \tag{9}$$

$$\psi = \frac{1}{\sin^2\theta}\Delta^*(c\sin^2\theta), \tag{10}$$

N and c are complex functions of the variables u, θ, φ; and M is a real function of these variables. These functions are also related by

$$M_0 = -|c_0|^2 + \frac{1}{2}A_0, \quad 3N_0 = -\Delta M - (4c\cos\theta + \Delta c + 3c\Delta)c_0^*, \tag{11}$$

where

$$A = \frac{1}{\sin\theta}Re\{\Delta^*(\psi\sin\theta)\}. \tag{12}$$

Here the asterisk denotes complex conjugation, and the "0" denotes partial differentiation with respect to u.

The tetrad components (for large r) satisfying (1) and (2) are written in the form

$$\begin{gathered} h^{i(a)} = \eta^{ia} - \frac{1}{2}y^{ia} - \frac{1}{2}z^{ia} + \frac{3}{8}y^{ar}y_r^i - q^{ia}, \\ h_i(a) = \eta_{ia} + \frac{1}{2}y_{ia} + \frac{1}{2}z_{ia} - \frac{1}{8}y_{ar}y_i^r - q_{ia}, \end{gathered} \tag{13}$$

where

$$\begin{aligned} q^{ia} = \frac{1}{r^2}[&Re\{c_2 - \psi - ic_3/\sin\theta + c\operatorname{ctg}\theta\}(n^i t^a - t^i n^a) + \\ &+ \frac{1}{2}Re\left\{\psi_2 - i\psi_3/\sin\theta + \psi\operatorname{ctg}\theta - \frac{1}{2}|cc_0|\right\}(n^i\mu^a - \mu^i n^a) \\ &+ Re\left\{\frac{1}{2}i\psi - ic\operatorname{ctg}\theta\right\}(n^i l^a - l^i n^a) + Re\left\{c\operatorname{ctg}\theta - \frac{1}{2}\psi\right\}(n^i m^a - m^i n^a)]. \end{aligned} \tag{14}$$

Using Eq. (13), we can construct the energy–momentum tensor [1]:

$$t^\mu_{(a)} = \frac{1}{\varkappa}\left\{2\Delta(li\kappa)C(ial) - \frac{1}{2}\delta(a\kappa)\Delta(lpi)C(pil)\right\}h^\mu(\kappa) = \frac{1}{\varkappa}\nabla_i\Delta^{\mu i}_{(a)}, \tag{15}$$

where

$$\Delta_{lpi} = C_{lip} + C_{pil} + C_{pli}. \tag{16}$$

We then find

$$t^\kappa_{(a)} = \frac{4|c_0|^2}{\varkappa r^2}\mu_a\mu^\kappa + O_3. \tag{17}$$

Our expression differs from that of Möller [3] by a numerical factor. The total energy radiated per unit time by a sphere is

$$-\frac{dH}{dt}=\frac{1}{2}\int_{0}^{\pi}|c_0|^2\sin\theta\, d\theta, \tag{18}$$

in agreement with [3].

For the 4-momentum enclosed in a sphere of sufficiently large radius r at time t we find

$$P_{(a)}=\frac{1}{\varkappa}\int\nabla_i\Delta_{(a)}{}^{\mu i}_{\cdot\cdot}\,dV_\mu\approx\frac{1}{\varkappa}\int\Delta^{4\lambda}_{(a)}\,dS_\lambda. \tag{19}$$

Following [3], we have

$$\Delta^{4\lambda}_{(a)}\,dS_\lambda=-\Delta^{4\lambda}_{(a)}\,\mu_\lambda r^2 d\theta\sin\theta\, d\varphi.$$

Using (6), (13), and (17), we can convert $\Delta^{4\lambda}_{(a)}\mu_\lambda$ to

$$-\Delta^{4\lambda}_{(a)}\,\mu_\lambda=\frac{1}{2\varkappa r^2}[(-4\mathrm{M}+2Re\{A\})\,\mu_a-Re\{\psi\}\,t_a]+O_3. \tag{20}$$

If for sufficiently large r we set O_3 equal to zero and integrate by parts, we find

$$\int A\mu_a d\Omega=\int\psi t_a\, d\Omega=0. \tag{21}$$

We thus find

$$P_{(a)}=-\frac{2}{\varkappa}\int\mathrm{M}\mu_a d\Omega=\{P_i,\,-H\}. \tag{22}$$

Writing relations for the total energy and momentum, we find results precisely equal to the analogous expressions given in [3, 4]:

$$P_i=\frac{2}{\varkappa}\int\mathrm{M}(u,\,\theta,\,\varphi)\,n_i d\Omega,\quad H=m(u)=\frac{1}{2}\int_{0}^{\pi}\mathrm{M}(u,\,\theta,\,\varphi)\,d\Omega. \tag{23}$$

CONCLUSIONS

1. Applying the Rodichev theory [1] to the Sachs system [4], we find a total energy and momentum which coincide in integral form with the analogous results of [3] (the results do not coincide in the differential form).

2. The ambiguity in the determination of total energy and momentum is completely eliminated by their invariance with respect to gauge-conserving tetrad rotation.

In the general case, on the other hand, they are noninvariant with respect to tetrad rotations.

3. The fact that Möller's results are the same as those of [4] and the same as our results can be attributed to the choice of a convenient coordinate system. In an evaluation of integral quantities at sufficently large r this coordinate system asymptotically converts into a Lorentzian system which is the same as ours, so the results are the same.

In our case the total energy and momentum are generally covariant tensors, so any changes in the coordinate grid leave these quantities unchanged. Möller's expressions do not have this property. Furthermore, Möller himself [5] pointed out that even the use of the Einstein pseudotensor in an asymptotic Lorentzian coordinate system gives the same value for the 4-momentum as his complex.

However, the results will not coincide in the case of the most general coordinate transformations, so the Einstein and Möller expressions may in this case take on any prespecified value.

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