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Compact Interval Spaces in which All Closed Subsets Are Homeomorphic to Clopen Ones, II

To the memory of Ernest Corominas (1913–1992)

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Abstract. A topological space X whose topology is the order topology of some linear ordering on X, is called an *interval space.* A space in which every closed subspace is homeomorphic to a clopen subspace, is called a *CO space* and a space is *scattered* if every non-empty subspace has an isolated point. We regard linear orderings as topological spaces, by equipping them with their order topology. If L and K are linear orderings, then L^*, $L + K$, $L \cdot K$ denote respectively the reverse ordering of L, the ordered sum of L and K and the lexicographic order on $L \times K$ (so $\omega \cdot 2 = \omega + \omega$ and $2 \cdot \omega = \omega$). Ordinals are considered as linear orderings, and cardinals are initial ordinals. For cardinals $\kappa, \lambda \geqslant 0$, let $L(\kappa, \lambda) = \kappa + 1 + \lambda^*$. **Theorem**: *Let X be a compact interval scattered space. Then X is a CO space if and only if X is homeomorphic to a space of the form $\alpha + 1 + \Sigma_{i<n} L(\kappa_i, \lambda_i)$, where α is any ordinal, $n \in \omega$, for every $i < n$, κ_i, λ_i are regular cardinals and $\kappa_i \geqslant \lambda_i$, and if $n > 0$, then $\alpha \geqslant \max(\{\kappa_i : i < n\}) \cdot \omega$.* By Part I of this work, the hypothesis "scattered" is unnecessary.

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1. Introduction

DEFINITION 1.1. Let X be a topological space. X is a *CO space*, if every closed subspace of X is homeomorphic to a clopen subspace of X.

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A non-empty subset A of a topological space X is perfect, if for every open subset U of X, $|U \cap A| \neq 1$. A space X is *scattered*, if it does not have a non-empty perfect subset.

In this work, we characterize all compact scattered CO interval spaces, and show that they are very close to being ordinals.

Let $\langle L, < \rangle$ be a linear ordering. For $a, b \in L \cup \{-\infty, +\infty\}$, let (a, b) denote $\{c \in L : a < c < b\}$. The set $\{(a, b) : a, b \in L \cup \{-\infty, +\infty\}\}$ is a basis for a topology. This topology is called the *order topology*, or the *interval topology* of $\langle L, < \rangle$, and is denoted by $\tau_L^<$. A topological space $\langle X, \tau \rangle$ is called an *interval space*, if there is a linear ordering $<$ on X such that $\tau = \tau_L^<$.

We regard linear orderings as topological spaces by equipping them with their interval topology.

If $<$ is a linear ordering on a set L, then $<^*$ denotes the reverse ordering of $<$. That is, for every $a, b \in L$, $a <^* b$ if $b < a$. Let $M_i = \langle L_i, <_i \rangle$ $(i = 0, 1)$ be linear orderings. Then $(M_0)^*$ denotes $\langle L_0, (<_0)^* \rangle$; $M_0 + M_1$ denotes $\langle \bigcup_{i<2} \{i\} \times L_i, < \rangle$, where $\langle i, a \rangle < \langle j, b \rangle$ if $i < j$, or $i = j$ and $a <_i b$; and $M_0 \cdot M_1$ denotes $\langle L_0 \times L_1, < \rangle$, where $\langle a, b \rangle < \langle c, d \rangle$, if $b <_1 d$, or $b =_1 d$ and $a <_0 b$. Clearly $(M_0)^*$, $M_0 + M_1$ and $M_0 \cdot M_1$ are linear orderings.

Ordinals are considered to be linear ordered sets, and cardinals are considered to be initial ordinals. An infinite cardinal κ is *regular*, if every unbounded subset of κ has cardinality κ.

The goal of this work is to prove the following theorem:

THEOREM 1.2. *Let X be a scattered compact interval space. Then the following are equivalent.*

(*i*) *X is a CO-space.*

(*ii*) *X is homeomorphic to $\alpha + 1 + \Sigma_{i<n} (\kappa_i + 1 + \lambda_i^*)$, where α is an ordinal, $n < \omega$, and for every $i < n$, κ_i and λ_i are regular cardinals such that $\kappa_i \geqslant \lambda_i \geqslant \omega$, and if $n > 0$, then $\alpha \geqslant \max\{\kappa_i : i < n\} \cdot \omega$.*

The class $\mathscr{S}_{\mathrm{CO}}$ of Hausdorff compact scattered CO spaces seems to be interesting. There is a space $X \in \mathscr{S}_{\mathrm{CO}}$ which is not of the form described in Theorem 1.2. Let Y be the one point compactification of a discrete space of cardinality $\aleph_1$, and X be the disjoint union of the space Y and $\omega^2 + 1$, where both Y and $\omega^2 + 1$ are clopen in X. $X \in \mathscr{S}_{\mathrm{CO}}$, but does not have the form described in Theorem 1.2. We do not know however, whether X is a sporadic example, or whether it leads to a new class of compact scattered CO spaces.

Accordingly we have the following conjecture.

QUESTION 1.3. *Describe the scattered compact Hausdorff CO spaces.*

Before turning to the proof of Theorem 1.2, we need some notations, preliminaries and results that we can find in Bekkali, Bonnet and Rubin ([1992], Part I). The first definition is concerned with general topological spaces.

DEFINITION 1.4. (*a*) Let X, Y and Z be topological spaces and $y \in Y$. τ_Y will denote the topology of Y. If $B \subseteq Y$, then τ_B^Y will denote the topology on B inherited from τ_Y. Also, for any $B \subseteq Y$, we denote by $\mathrm{cl}_Y(B)$ (resp. $\mathrm{int}_Y(B)$) the closure of B (resp. the interior of B) in Y. If Y is understood from the context, we omit the underscript $_Y$.

(*b*) We define $\sim$ on the class of topological spaces as follows: $Y \sim Z$ if and only if there is a homeomorphism from Y onto Z. Moreover $f\colon Y \sim Z$ means that f is an homeomorphism from Y onto Z.

A *t-embedding* from Y into Z is a homeomorphism from Y onto $\mathrm{rng}(f) \subseteq Z$; it will be denoted by $Y \leqslant^t Z$. Note that $Y <^t Z$ means $Y \leqslant^t Z$ and $Z \nleqslant^t Y$.

(*c*) $Y \times Z$ denotes the topological product of Y and Z. $Y \sqcup Z$ denotes the topological sum of Y and Z. That is, $Y \sqcup Z = Y' \cup Z'$ where $Y' \sim Y$, $Z' \sim Z$, $Y' \cap Z' = \emptyset$, and Y', Z' are open in $Y \sqcup Z$. The choice of Y' and Z' is done in some canonical way, e.g., $Y' = \{0\} \times Y$ and $Z' = \{1\} \times Z$.

(*d*) Let X, Y be Hausdorff spaces. We put:

$$\mathrm{Op}(Y) = \{O \subseteq Y : O \text{ is open in } Y\},$$

$$\mathrm{Cl}(Y) = \{F \subseteq Y : F \text{ is closed in } Y\},$$

$$\mathrm{Clop}(Y) = \{U \subseteq Y : U \text{ is clopen in } Y\},$$

$$\mathrm{Op}_Y(y) = \{O \in \mathrm{Op}(Y) : y \in O\},$$

$$\mathrm{Cl}_Y(y) = \{F \in \mathrm{Cl}(Y) : y \in F\},$$

$$\mathrm{Clopen}_Y(y) = \{U \in \mathrm{Clop}(Y) : y \in U\},$$

$$\mathrm{Clcopies}_Y(X) = \{A \in \mathrm{Cl}(Y) : A \sim X\},$$

$$\mathrm{Clopcopies}_Y(X) = \{A \in \mathrm{Clop}(Y) : A \sim X\}.$$

(*e*) Let Y be a Hausdorff space.

$$\mathrm{Iso}(Y) \stackrel{\mathrm{def}}{=} \{y \in Y : y \text{ is isolated in } Y\}.$$

We define $D_\alpha(Y)$, for α ordinal, by the following rules:

$$D(Y) = Y - \mathrm{Iso}(Y),$$

$$D_0(Y) = Y,$$

$$D_{\alpha+1}(Y) = D(D_\alpha(Y)), \text{ and}$$

$$D_\alpha(Y) = \bigcap_{\nu < \alpha} D_\nu(Y), \text{ for } \alpha \text{ limit.}$$

$$\mathrm{rk}(Y) \stackrel{\mathrm{def}}{=} \sup\{\alpha \mid D_{\alpha+1}(Y) \neq D_\alpha(Y)\} \text{ is called the } \textit{rank of } Y.$$

Thus Y is scattered if and only if there is an α such that $D_\alpha(Y) = \emptyset$.

Also, for a scattered space Y, if $u \in Y$ then we put $\mathrm{rk}_Y(u) \stackrel{\mathrm{def}}{=} \max\{\alpha : y \in D_\alpha(Y)\}$.

(*f*) If α is an ordinal, it is regarded as a topological space, by equipping it with its interval topology. We recall that if α is an ordinal, then $\mathrm{cf}(\alpha)$ denotes its *cofinal type*: if $\alpha = 0$, then $\mathrm{cf}(\alpha) = 0$; if α is a successor (i.e., $\alpha = \beta + 1$), then $\mathrm{cf}(\alpha) = 1$; and if α is limit, then $\mathrm{cf}(\alpha)$ is the first ordinal γ such that there is a strictly increasing sequence $\langle \alpha_\mu \rangle_{\mu < \gamma}$ satisfying $\sup(\{\alpha_\mu : \mu < \gamma\}) = \alpha$.

With these definitions, clearly we have:

PROPOSITION 1.5. *Let Y be a Hausdorff space, W any subset of Y, and V an open subset of Y.*

(*a*) $\mathrm{Iso}(W \cap V) = \mathrm{Iso}(W) \cap V$.

(*b*) $D(W \cap V) = D(W) \cap V$.

(*c*) $D_\alpha(W \cap V) = D_\alpha(W) \cap V$.

(*d*) $\mathrm{rk}(W \cap V) \leqslant \mathrm{rk}(W)$.

(*e*) *If $X \subseteq Z \subseteq Y$, then $D_\alpha(X) \subseteq D_\alpha(Z)$.*

(*f*) *Let $\rho(Y)$ be the first ordinal ν such that $D_\nu(Y) = D_{\nu+1}(Y)$. Then* $\mathrm{rk}(Y) \leqslant \rho(Y) \leqslant \mathrm{rk}(Y) + 1$. *If $\rho(Y)$ is a successor ordinal, then* $\rho(Y) = \mathrm{rk}(Y) + 1$; *and if $\rho(Y)$ is* 0 *or limit, then* $\rho(Y) = \mathrm{rk}(Y)$.

Moreover, if Y is a non-empty compact space, then $\rho(Y)$ is a successor ordinal, and thus $\rho(Y) = \mathrm{rk}(Y) + 1$.

Finally, Y is a scattered space if and only if $D_{\mathrm{rk}(Y)+1}(Y) = \emptyset$.

In particular, if Y is a non-empty compact scattered space, then $D_{\mathrm{rk}(Y)}(Y)$ is a non-empty finite space. □

The next definition introduces the notations and notions concerning interval topologies.

DEFINITION 1.6. (*a*) Let $\langle L, < \rangle$ be a linear ordering. For $a, b \in L \cup \{-\infty, +\infty\}$, let $(a, b) = \{c \in L : a < c < b\}$, $[a, b] = \{c \in L : a \leqslant c \leqslant b\}$, $[a, b)$, etc. are defined in a similar way.

We recall that $\tau_L^<$ denotes the topology on L whose basis is $\{(a, b) : a, b \in L \cup \{-\infty, +\infty\}\}$. $\tau_L^<$ is called the *interval topology* of $<$. Y is an *interval space* if and only if there is a linear ordering $<$ on Y such that $\tau_Y = \tau_Y^<$.

For any $B \subseteq Y$, τ_B^Y denotes the topology on B, inherited from $\tau_Y = \tau_Y^<$. $\tau_B^<$ denotes the interval topology induced on B by the restriction $< \restriction B$ of $<$ on B.

(*b*) Let K, L be two linear orderings. Any strictly increasing or strictly decreasing mapping f from K into L is called an *o-embedding*.

A *to-embedding* from K into L is an o-embedding which is a t-embedding with respect to $\tau_K^<$ and $\tau_L^<$. These notions will be denoted respectively by $\leqslant^o$ and $\leqslant^{to}$.

(*c*) Let $(L, <)$ be a linear ordering, and A, B be two subsets of Y. We put $A < B$ whenever every member of A is less than every member of B. $A < a$, $A \leqslant B$, etc ... are defined in a similar manner.

We summarize some well-known and easy facts in a proposition.

PROPOSITION 1.7. *Let Y be an interval space, where the topology on Y is induced by a linear ordering $<_Y$.*

(*a*) *Let $B \subseteq Y$. Then $\tau_B^< \subseteq \tau_B^Y$. Moreover, the following are equivalent.*

(*i*) $\tau_B^< = \tau_B^Y$.

(*ii*) *For every non-empty $A \subseteq B$, if $\sup_B(A)$ exists, then so does $\sup_Y(A)$ and $\sup_Y(A) = \sup_B(A)$, and if $\inf_B(A)$ exists, then so does $\inf_Y(A)$ and $\inf_Y(A) = \inf_B(A)$.*

In particular, if B is closed, then $\tau_B^< = \tau_B^Y$.

(*b*) *A subset A of Y is compact in the relative topology if and only if for every $B \subseteq A$, $\sup_Y(B)$ and $\inf_Y(B)$ exist and belong to A.*

Hence if A is compact in the relative topology, then $\tau_A^< = \tau_A^Y$.

(*c*) *Let $\langle Y, \tau_Y^< \rangle$ and $\langle Z, \tau_Z^< \rangle$ be two interval spaces, and f be an increasing function from $\langle Y, < \rangle$ into $\langle Z, < \rangle$. Then f is continuous if and only if f preserves suprema and infima, i.e., for every non-empty subset A of Y, if $\sup_Y(A)$ exists, then $\sup_Z(f[A]) = f(\sup_Y(A))$, and if $\inf_Y(A)$ exists, then $\inf_Z(f[A]) = f(\inf_Y(A))$.*

Proof. See [Part I] (Proposition 2.6). □

Our next definition introduces notions and notations concerning ordinals.

DEFINITION 1.8. Let α be an ordinal.

(*a*) α can be uniquely expressed in the form $\alpha = \lambda + n$ where $\lambda = 0$, or limit ordinal, and $0 \leqslant n < \omega$.

Moreover α is *even* (resp. *odd*) whenever n is even (resp. odd).

Finally, we denote by $\mathrm{Succ}(\alpha)$ the set of all $\beta < \alpha$ of the form $\xi + 1$.

(*b*) α is an *indecomposable ordinal* whenever $\alpha > 0$ and for all $\beta, \gamma < \alpha$, $\beta + \gamma < \alpha$.

(*c*) A space Y is an *ordinal space* whenever there is a well-ordering $<$ on Y such that $\tau_Y = \tau_Y^<$.

We need some several well known facts about ordinals. We summarize these facts in the following lemma.

LEMMA 1.9. (*a*) *A non-zero ordinal α is compact if and only if α is a successor ordinal.*

(*b*) (1) *Every non-zero ordinal can be uniquely expressed in normal form, i.e.,*

$$\alpha = \sum_{i<n} \omega^{\alpha_i} \cdot p_i,$$

where $0 < n < \omega$, $0 < p_i < \omega$ for every $i < n$, and $\alpha_0 > \alpha_1 > \cdots > \alpha_{n-1}$.

(2) *If α is as in (b1), then $\alpha < \omega^{\alpha_0} \cdot (p_0 + 1) < \omega^{\alpha_0 + 1}$.*

(3) *If α is as in (b1), and $\beta = \Sigma_{i<n} \omega^{\beta_i} \cdot q_i$ also in such a normal form (except with no insistence that each $q_i \neq 0$), then we have $\beta < \alpha$ if and only if:*

$$\text{There is an } i \text{ such that: } \begin{cases} \forall j < i \ (\beta_j = \alpha_j \text{ and } q_j = p_j) \\ \text{and} \\ (\beta_i < \alpha_i) \text{ or } (\alpha_i = \beta_i \text{ and } q_i < p_i). \end{cases}$$

(4) *If* α *is as in* (*b*1), *we have* $D_\gamma(\alpha) = \{\beta < \alpha : \text{if } \beta = \Sigma_{i<l}\, \omega^{\delta_i} \cdot q_i$ is in normal form, then $\delta_{l-1} \geqslant \gamma\}$.

(5) *If* $\alpha \geqslant 2$ *and* $\beta < \alpha$, *then* $\beta + 1 = [0, \beta + 1) \subseteq \alpha$, *and* $\beta + 1$ *is a closed and open subset of the interval space* α.

(6) *If* α *is a limit ordinal and* $1 \leqslant n < \omega$, *then* $\alpha + n$ *is homeomorphic to* $\alpha + 1$.

(7) *If* α, β *are limit ordinals, then* $\alpha + \beta + 1$ *and* $\beta + \alpha + 1$ *are homeomorphic.*

(8) *Let* α *be an ordinal in normal form as in* (*b*1). *Then* $\mathrm{rk}(\alpha) = \alpha_0$.

Moreover, if $\alpha \geqslant \omega$ *and* α *is a successor ordinal, then* $\alpha \sim \omega^{\alpha_0} \cdot p_0 + 1$, *and* $D_{\alpha_0}(\alpha) = \{\omega^{\alpha_0} \cdot (j+1) : j < p_0\}$.

Hence every infinite compact ordinal α *is homeomorphic to* $\omega^\beta \cdot p + 1$ *for some ordinal* β *and some* $p < \omega$. *If* $\alpha \sim \omega^\beta \cdot p + 1$, *then* $\omega^\beta \cdot p + 1 \leqslant \alpha < \omega^{\beta+1}$.

(*c*) *Let* $\alpha_0, \ldots, \alpha_{n-1}$ *be compact ordinals. Then* $\Sigma_{i<n}\, \alpha_i \sim \bigsqcup_{i<n} \alpha_i$ *and if* $\bigsqcup_{i<n} \alpha_i$ *is chosen to be* $\bigcup_{i<n} \{i\} \times \alpha_i$, *then* $f(\langle i, \gamma \rangle) = \Sigma_{j<i}\, \alpha_j + \gamma$ *is a homeomorphism from* $\bigsqcup_{i<n} \alpha_i$ *onto* $\Sigma_{i<n}\, \alpha_i$.

In particular, if α *is compact, then* $\alpha \cdot n \sim \alpha \times n$.

(*d*) *Let* α *be an infinite ordinal. The following are equivalent*:

(1) α *is indecomposable.*

(2) *There is* $\beta > 0$ *such that* $\alpha = \omega^\beta$.

(3) *For every clopen* $V \subseteq \alpha + 1$, $V \sim \alpha + 1$ *if and only if* $\alpha \in V$.

(4) *For every indecomposable* $\gamma < \alpha$, *and* $n < \omega$; $\gamma \cdot n < \alpha$.

(*e*) *If* $\alpha = \omega^\theta$, *then* $\mathrm{rk}(\alpha + 1) = \theta$.

Hence if α, β *are two different indecomposable ordinals, then* $\mathrm{rk}(\alpha + 1) \neq \mathrm{rk}(\beta + 1)$.

(*f*) *If* $\alpha \geqslant \omega^\omega$, *then* $D(\alpha + 1) \sim \alpha + 1$.

(*g*) *Let* α *be an ordinal,* Y *an interval space, and* $f\colon (\alpha + 1) \sim U \subseteq Y$. *Then there is a family* $\langle W_{\beta+1} : \beta < \alpha \rangle$ *of pairwise disjoint open sets of* Y *such that for every* $\beta < \alpha$, $f(\beta + 1) \in W_{\beta+1}$.

Proof. See [Part I] (Proposition 2.8). □

We recall an easy fact on scattered linear ordering:

LEMMA 1.10. *Assume that* κ *is an infinite regular cardinal and* $\langle Y, < \rangle$ *is a scattered linear ordering of size* κ. *Then* $\kappa \leqslant^o Y$.

Proof. Suppose that $\kappa \nleqslant^o Y$. Let E be the following equivalence relation on Y: aEb if and only if $|(a, b) \cup (b, a)| < \kappa$. Since $\kappa \nleqslant^o Y$ and κ is regular, for every $a \in Y$, $|a/E| < \kappa$. Also, since κ is regular, for every $a, b \in Y$ ($a/E < b/E$ implies $|(a, b) - a/E - b/E| = \kappa$). So $\{a/E : a \in Y\}$ is κ-dense. If $Y' \subseteq Y$ is such that, for every $a \in E$, $|(a/E) \cap Y'| = 1$, then Y' is κ-dense. □

2. Proof of Theorem 1.2

In the first part of this section, we define two properties of compact spaces F, G and H: the property $\Psi(F, G)$ and $\Theta(F, G, H)$. Let X be a scattered compact interval space. We shall show that if X has two closed subsets F, G satisfying $\Psi(F, G)$, then X is not a CO space (Theorem 2.11). Similarly, we shall show that if X has three closed subsets F, G and H satisfying $\Theta(F, G, H)$, then again X is not a CO space (Theorem 2.14).

If Y is a non-empty compact scattered space, then, by Proposition 1.5(i), $D_{\mathrm{rk}(Y)}(Y)$ is a finite non-empty set, which we denote by $\mathrm{End}(Y)$. If $\mathrm{End}(Y)$ has only one element, we write $\mathrm{End}(Y) = \{\mathrm{end}(Y)\}$. The first definition is concerned with general topological spaces:

DEFINITION 2.1. Let F and G be two compact scattered spaces.

(a) *F has a trivial invariant* if $\mathrm{End}(F)$ has only one element. In this case we put $\mathrm{Inv}(F) = (\mathrm{rk}(F), \mathrm{end}(F))$.

(b) F is *indecomposable* if it has a trivial invariant and for every $V \in \mathrm{Op}_F(\mathrm{end}(F))$ there is a $U \in \mathrm{Clopcopies}_F(F)$ such that $U \subseteq V$.

(c) F and G are *quasi-homeomorphic* if $\mathrm{Clopcopies}_F(G) \neq \emptyset \neq \mathrm{Clopcopies}_G(F)$.

(d) F and G are *similar* if $\mathrm{Clcopies}_F(G) \neq \emptyset \neq \mathrm{Clcopies}_G(F)$.

We summarize some easy facts in a lemma.

LEMMA 2.2. (a) *If F is a closed indecomposable space, has a trivial invariant, $V \in \mathrm{Op}_F(\mathrm{end}(F))$, and $U \in \mathrm{Clopcopies}_F(F)$ such that $U \subseteq V$, then $\mathrm{end}(F) \in U$ and U is indecomposable.*

(b) *Suppose $F \in \mathrm{Cl}(Y)$, Y is compact, F has a trivial invariant, $U \in \mathrm{Clopen}_F(\mathrm{end}(F))$, and $f\colon U \sim V \subseteq Y$, then V has a trivial invariant* $(\mathrm{rk}(F), f(\mathrm{end}(F)))$.

(c) *If F and G are quasi-homeomorphic, then they are similar.*

Proof. (a) Obviously $\mathrm{rk}(F) = \mathrm{rk}(U)$ and $\mathrm{End}(U)$ has only one element. Now by Proposition 1.5(c),

$$\mathrm{End}(U) = D_{\mathrm{rk}(U)}(U) = D_{\mathrm{rk}(F)}(F) \cap U.$$

Hence $\mathrm{end}(F) \in U$. Hence clearly U is indecomposable.

(b) Again we have $D_{\mathrm{rk}(F)}(U) = D_{\mathrm{rk}(F)}(F) \cap U$ by Proposition 1.5(c), and the result follows.

(c) Obvious. □

The next definition introduces the relation Ψ.

DEFINITION 2.3. For any two scattered compact spaces F, G, we put $\Psi(F, G) = \bigwedge_{i=1}^{5} \Psi_i(F, G)$, where:

$\Psi_1(F, G)$ is: F, G have trivial invariants;

$\Psi_2(F, G)$ is: $\mathrm{rk}(F) = \mathrm{rk}(G)$;

$\Psi_3(F, G)$ is: F, G are indecomposable;

$\Psi_4(F, G)$ is: F, G are similar;

$\Psi_5(F, G)$ is: F, G are not quasi-homeomorphic.

EXAMPLE 2.4. Let $F = (\omega_1 + 1 + \omega^*) \cdot \omega + 1$ and $G = (\omega_1 + 1 + \omega^* + \omega_1 + 1) \cdot \omega + 1$. Then F and G have property Ψ. To see this, let:

$$\mathbf{F} \stackrel{\text{def}}{=} \omega_1 + 1 + \omega^* + \omega_1 + 1 + \omega^* \sim (\omega_1 + 1 + \omega^*) \sqcup (\omega_1 + 1 + \omega^*).$$

$$\mathbf{G} \stackrel{\text{def}}{=} \omega_1 + 1 + \omega^* + \omega_1 + 1 \sim (\omega_1 + 1 + \omega^*) \sqcup (\omega_1 + 1).$$

It is easy to check that $\text{rk}(\mathbf{F}) = \text{rk}(\mathbf{G}) = \omega_1$ as an easy consequence of Lemma 1.9($b4$) and 1.9(e). With obvious notations, $F = \mathbf{F} \cdot \omega + 1$ and $G = \mathbf{G} \cdot \omega + 1$. Hence $D_{\omega_1}(F) \sim D_{\omega_1}(G) \sim \omega + 1$, and thus $\text{rk}(F) = \text{rk}(G) = \omega_1 + 1$ and $|\text{End}(F)| = |\text{End}(G)| = 1$, that show $\Psi_1(F, G)$ and $\Psi_2(F, G)$ hold. Clearly $\Psi_3(F, G)$ holds. Let us prove $\Psi_4(F, G)$, i.e., F and G are similar. If in $\mathbf{F}$, we delete the second ω^*, we get a closed subspace homeomorphic to $\mathbf{G}$ and this shows that $\text{Clcopies}_F(G) \neq \emptyset$. If in $\mathbf{G}$, we delete the right part after ω^*, we get a closed subspace homeomorphic to $\omega_1 + 1 + \omega^*$, and this shows that $\text{Clcopies}_G(F) \neq \emptyset$. Finally, for $\Psi_5(F, G)$, note that in every $W \in \text{Op}_G(\text{end}(G))$, there is a point x such that x is a limit of a ω_1-sequence, but x is not a limit of a ω-sequence, while x has a neighborhood having no other element of this sort. If U is a clopen subset of F homeomorphic to G, then U have $\text{end}(F)$ as a member, and hence must contain a terminal segment of F. Thus U contains a clopen subset V of F which is still a terminal segment of F. V contains no element of this sort indicated. But the image of V contains a terminal segment of G, which does not contain such an element. This shows that $\text{Clopcopies}_F(G) = \emptyset$. The argument for $\text{Clopcopies}_G(F) = \emptyset$ is essentially the same. □

LEMMA 2.5. *Let X be a compact scattered interval space, let $H_0, H_1 \in \text{Cl}(X)$, and assume that:*

($H1$) *H_i has a trivial invariant for $i = 0, 1$.*

($H2$) *For any $W_i \in \text{Clop}(H_i)$ such that $\text{end}(H_i) \in W_i$, $i = 0, 1$, we have $W_0 \not\sim W_1$.*

Let $U_i, V_i \in \text{Clop}(X)$ for $i = 0, 1$. Assume that $\text{end}(H_i) \in U_i$ and $H_i \cap U_i \sim V_i$ for $i = 0, 1$. Then V_i has a trivial invariant $(\text{rk}(H_i), \text{end}(V_i))$, $i = 0, 1$, and $\text{end}(V_0) \neq \text{end}(V_1)$.

Proof. By Lemma 2.2(b), $U_i \cap H_i$ has a trivial invariant $(\text{rk}(H_i), \text{end}(H_i))$. Hence V_i has a trivial invariant $(\text{rk}(H_i), \text{end}(V_i))$.

For a contradiction, suppose $\text{end}(V_0) = \text{end}(V_1) = b$. Let $f_i\colon H_i \cap U_i \sim V_i$. Then $f_i(\text{end}(H_i)) = \text{end}(V_i)$. Let $W = V_0 \cap V_1$. Then by Lemma 2.2(b), W also has trivial invariant $(\text{rk}(H_i), b)$. Now $f_i^{-1}[W]$ is a clopen subset of H_i and $f_0^{-1}[W] \stackrel{\text{def}}{=} W_0 \sim W_1 \stackrel{\text{def}}{=} f_1^{-1}[W]$, contradicting ($H2$). □

The following general result will be frequently used.

PROPOSITION 2.6. *If X is a compact scattered interval space, then X is Boolean (that means that X is compact and 0-dimensional).*

Proof. By Theorem 15.4 of the Handbook on Boolean algebras (Koppelberg [1989]), it suffices to show that the jumps are dense, and this is clear since otherwise one would get a dense-in-itself interval. □

LEMMA 2.7. *Let X be a CO-space, $H_0 \in \mathrm{Clop}(X)$, and $H_1 \in \mathrm{Cl}(X)$. Assume that:*

(H1) $H_1 \subseteq H_0$.

(H2) H_i has a trivial invariant for $i = 0, 1$, $\mathrm{rk}(H_1) \leqslant \mathrm{rk}(H_0)$, and $\mathrm{end}(H_0) = \mathrm{end}(H_1) \stackrel{\mathrm{def}}{=} a$.

(H3) If $U_i, V_i \in \mathrm{Clop}(X)$ satisfy $V_i \sim U_i \cap H_i$ for $i = 0, 1$, and $a \in U_0 \cap U_1$, then $\mathrm{end}(V_0) \neq \mathrm{end}(V_1)$.

Then for each integer $m > 0$ there is a system $\langle\langle V_i^m, W_i^m\rangle : i < m\rangle$ such that:

(i) $W_i^m \in \mathrm{Clop}_{H_0}(a)$ and $V_i^m \in \mathrm{Clop}(X)$ for $i < m$.

(ii) $\langle V_i^m : i < m\rangle$ is a pairwise disjoint family.

(iii) $a \notin V_i^m$ for $i < m$.

(iv) $V_i^m \sim H_1 \cap W_i^m$ for $i < m$.

(v) If H_1 is indecomposable, then $V_i^m \sim H_1$ for $i < m$.

Proof. Let $\alpha = \mathrm{rk}(H_1)$ and notice that (*i*) and (*iv*) imply:

(*vi*) $\mathrm{rk}(V_i^m) = \alpha$ *and V_i^m has a trivial invariant, for each $i < m$.*

We do the construction by recursion on m. First suppose that $m = 1$. Let $W_0^1 = H_0$ and let V_0^1 be a clopen set such that $V_0^1 \sim H_1$. Then (*i*), (*ii*), (*iv*), and (*v*) are clear. To prove (*iii*), apply (*H*3) with $U_0 = U_1 = H_0$, $V_0 = H_0$, $V_1 = V_0^1$; we get $a = \mathrm{end}(V_0) \neq \mathrm{end}(V_1) = \mathrm{end}(V_0^1)$, as desired.

Suppose that V_i^m, W_i^m have been constructed for all $i < m$ so that (*i*)–(*v*) hold. For $i < m$ let $W_i^{m+1} \stackrel{\mathrm{def}}{=} W_i^m$. Now if H_1 is not indecomposable, set $W_m^{m+1} \stackrel{\mathrm{def}}{=} H_0 - \bigcup_{i<m} V_i^m$. Suppose H_1 is indecomposable. Choose $S \in \mathrm{Clopcopies}_{H_1}(H_1)$ so that $S \subseteq H_1 - \bigcup_{i<m} V_i^m$. Thus $S \in \mathrm{Clop}(H_1)$ and $S \sim H_1$; and by Lemma 2.2(*a*), $a \in S$. Say $S = T \cap H_1$ with $T \in \mathrm{Clop}(H_0)$. Set $W_m^{m+1} \stackrel{\mathrm{def}}{=} T - \bigcup_{i<m} V_i^m$. So $W_m^{m+1} \cap H_1 = S$. Note that in either case $W_m^{m+1} \in \mathrm{Clop}_{H_0}(a)$. Now by the CO property, pick $U \in \mathrm{Clop}(X)$ and f with

$$f: (W_m^{m+1} \cap H_1) \cup \bigcup_{i<m} V_i^m \sim U.$$

Put $V_i^{m+1} \stackrel{\mathrm{def}}{=} f[V_i^m]$ for $i < m$ and $V_m^{m+1} \stackrel{\mathrm{def}}{=} f[W_m^{m+1} \cap H_1]$. We claim that (*i*)–(*v*) hold for $m + 1$. In fact, (*i*), (*ii*), and (*iv*) are obvious. For (*v*), we have $V_m^{m+1} \sim W_m^{m+1} \cap H_1 = S \sim H_1$. Finally we prove (*iii*), which amounts to showing that $a \notin U$. Note by (*i*) and Proposition 1.5(*c*) that $\mathrm{rk}(H_1 \cap W_i^{m+1}) = \mathrm{rk}(H_1)$ for all $i < m$, and hence by (*iv*) $\mathrm{rk}(U) = \mathrm{rk}(H_1)$. Moreover, $\mathrm{End}(U) = \{f(a)\} \cup \{f(\mathrm{end}(V_i^m)) : i < m\}$. Now if $a \in U$, then by Proposition 1.5(*c*), $a \in \mathrm{End}(U)$.

CLAIM 1. $a \neq f(a)$.

Proof. In fact, $\mathrm{end}(V_m^{m+1}) = f(a)$ since $f^{-1}: V_m^{m+1} \sim W_m^{m+1} \cap H_1$; and $W_m^{m+1} \sim W_m^{m+1} \cap H_0$, so, by (*H*3), $f(a) = \mathrm{end}(V_m^{m+1}) \neq \mathrm{end}(W_m^{m+1}) = a$. □

CLAIM 2. $a \neq f(\mathrm{end}(V_i^m))$ *for all* $i < m$.

Proof. In fact, $V_i^{m+1} \sim V_i^m \sim H_1 \cap W_i^m$ and $W_m^{m+1} = W_m^{m+1} \cap H_0$, so by ($H3$), $f(\mathrm{end}(V_i^m)) = \mathrm{end}(V_i^{m+1}) \neq \mathrm{end}(W_m^{m+1}) = a$. □

This finishes the proof of Lemma 2.7. □

The next definition and proposition are introduced to show that a CO space does not contain closed subsets satisfying Ψ.

DEFINITION 2.8. (a) Let γ be a limit ordinal, $S \subseteq \gamma$, and D be a complete chain. Denote by d_0 the first element of D, and put

$$Z'_\gamma(S, D) = \{(\alpha, d) \in \gamma \times D : \alpha \in S \text{ and } d \in D, \text{ or } \alpha \notin S \text{ and } d = d_0\}.$$

The order on $Z'_\gamma(S, D)$ is inherited from the lexicographic ordering on $\gamma \times D$. Note that $Z'_\gamma(S, D)$ is closed under bounded suprema, so adding (γ, d_0) to it makes it a complete chain denoted by $Z_\gamma(S, D)$.

(b) $\Sigma_\gamma^+(D) \overset{\text{def}}{=} Z_\gamma(\mathrm{Succ}(\gamma), D)$, where γ and D are as in (a) and $\mathrm{Succ}(\gamma)$ is the set of all successor ordinals less than γ.

COMMENT. $A \sim B$ does not imply that $Z_\gamma(S, A) \sim Z_\gamma(S, B)$. To see this, let $A \overset{\text{def}}{=} 1 + \omega^* \sim B \overset{\text{def}}{=} 2 + \omega^*$. We claim that $Z_{\omega_2}(\omega_2, A) \not\sim Z_{\omega_2}(\omega_2, B) \sim \omega_2 + 1$. First, note that in $Z_{\omega_2}(\omega_2, A)$, there is a point which has both ω_1- and ω-sequences converging to it, but there is no such point in $\omega_2 + 1$. To show that $Z_{\omega_2}(\omega_2, B) \sim \omega_2 + 1$, consider $2 + \omega^*$ as $ab\omega^*$; then $Z_{\omega_2}(\omega_2, B)$ can be considered as

$$ab\omega^* b\omega^* b\omega^* \cdots ab\omega^* \cdots \infty,$$

with a's only at 0 and at limit places; and the first $ab\omega^*$ and each $b\omega^*$ can be mapped to $\omega + 1$ continuously.

LEMMA 2.9. (a) *If D is a scattered compact interval space, γ is a limit ordinal, and $S \subseteq \gamma$, then $Z_\gamma(S, D)$ is a scattered compact interval space. Moreover, the mappings $\pi : Z_\gamma(S, D) \to \gamma + 1$ and $\sigma : \gamma + 1 \to Z_\gamma(S, D)$ defined respectively by $\pi(\langle \alpha, d \rangle) = \alpha$ and $\sigma(\alpha) = \langle \alpha, d_0 \rangle$ are each continuous and increasing.*

(b) *Let A be a scattered compact interval space, γ be a non-zero indecomposable ordinal, and set $a_0 = \min(A)$. Then*:

(1) *If B is an interval space homeomorphic to A, then $\Sigma_\gamma^+(B) \sim \Sigma_\gamma^+(A)$.*

(2) *$\Sigma_\gamma^+(A)$ has a trivial invariant, and*

$$\mathrm{Inv}(\Sigma_\gamma^+(A)) = (\mathrm{rk}(A) + \mathrm{rk}(\gamma + 1), \langle \gamma, a_0 \rangle, \langle \gamma, a_0 \rangle).$$

Moreover,

- *For every limit $\delta \leqslant \gamma$ and $V \in \mathrm{Clop}_{\Sigma_\gamma^+(A)}(\langle \delta, a_0 \rangle)$, $\mathrm{rk}(V) > \mathrm{rk}(A)$.*
- $\mathrm{rk}(Z_\gamma(\gamma, A)) = \mathrm{rk}(A) + \mathrm{rk}(\gamma + 1)$.

(3) *For each* $\alpha \in \mathrm{Succ}(\gamma)$, $\{\alpha\} \times A$ *is clopen in* $\Sigma_\gamma^+(A)$ *and is homeomorphic to* A.

(4) $t \in \Sigma_\gamma^+(A) - \bigcup \{\{\alpha\} \times A : \alpha \in \mathrm{Succ}(\gamma)\}$ *if and only if* $t = \langle \zeta, a_0 \rangle$ *for some limit ordinal* $\zeta \leqslant \gamma$.

(5) *If* $t = \langle \zeta, a_0 \rangle$ *for some limit* $\zeta < \gamma$, *then* t *has a successor in* $\Sigma_\gamma^+(A)$, *namely* $\langle \zeta + 1, a_0 \rangle$.

(6) $\langle 0, a_0 \rangle$ *is isolated in* $\Sigma_\gamma^+(A)$, *and* $\Sigma_\gamma^+(A) - \{\langle 0, a_0 \rangle\} \sim \Sigma_\gamma^+(A)$.

(7) $\Sigma_\gamma^+(A)$ *is indecomposable.*

(*c*) *Let* A *be a complete scattered chain,* $\langle \gamma_\mu : \mu < \rho \rangle$ *be a sequence of limit ordinals with* ρ *a limit ordinal too,* X *be a complete scattered chain, and* $\langle F_\mu : \mu < \rho \rangle$ *be a family of closed subspaces of* X. *Put* $F^- \stackrel{\mathrm{def}}{=} \bigcup_{\mu < \rho} F_\mu$, $F \stackrel{\mathrm{def}}{=} \mathrm{cl}_X(F^-)$, *and* $\gamma \stackrel{\mathrm{def}}{=} \Sigma_{\mu < \rho} \gamma_\mu$. *Suppose also that*:

(1) *If* $\mu < \nu < \rho$, *then* $F_\mu < F_\nu$,

(2) *For* $\mu < \rho$ *and* $u \in F_\mu$, $u \notin \mathrm{cl}_X(\bigcup \{F_\nu : \nu < \rho, \nu \neq \mu\})$,

(3) *For* $\mu < \rho$, $F_\mu \sim \Sigma_{\gamma_\mu}^+(A)$.

Then $F \sim \Sigma_\gamma^+(A)$.

Proof. (*a*) is clear.

(*b*). (*b*1). Let $a_0 = \min(A)$, $b_0 = \min(B)$ and $b_1 = \max(B)$. Suppose that $g: A \sim B$. Define $f: \Sigma_\gamma^+(A) \to \Sigma_\gamma^+(B)$ as follows: $f((\varepsilon, a_0)) = (\varepsilon, b_0)$ for ε limit, and $f((\varepsilon, a)) = (\varepsilon, g(a))$ for ε non-limit. Clearly f is one-to-one and onto. To show that f is continuous, suppose that U is open in $\Sigma_\gamma^+(B)$ and $(\varepsilon, a) \in f^{-1}[U]$.

Case 1: ε is successor or 0 (i.e., non-limit).
Now $(\varepsilon - 1, b_1) < (\varepsilon, g(a)) < (\varepsilon + 1, b_0)$. So without loss of generality U has the form $\{\varepsilon\} \times U'$, with U' open in B. Say V is open in A such that $g[V] \subseteq U'$. Then $f[\{\varepsilon\} \times V] \subseteq U$, as desired.

Case 2: ε is limit.
Thus $(\varepsilon, a_0) \in f^{-1}[U]$. So without loss of generality U has the form $((\delta, a), (\varepsilon + 1, a_0))$. Then $(\varepsilon, a_0) \in ((\delta + 1, a_0), (\varepsilon + 1, a_0))$ and $f[((\delta + 1, a_0), (\varepsilon + 1, a_0))] \subseteq U$, as desired.

(*b*3), (*b*4) and (*b*5) are obvious.

(*b*2). Let us show the first part. It is easily proved by induction that $D_\alpha(\Sigma_\gamma^+(A)) = \Sigma_\gamma^+ D_\alpha(A)$ for $\alpha < \mathrm{rk}(A)$, and $D_\alpha(\Sigma_\gamma^+(A)) = D_{1+\beta}(\gamma + 1) \times \{d_0\}$ if $\alpha = \mathrm{rk}(A) + \beta$. The desired result follows. The proof for $Z_\gamma(\gamma, A)$ is similar.

Now, let δ and V as in the second assertion of (*b*2). Let $\delta = \Sigma_{i \leqslant n} \omega^{\delta_i} \cdot p_i$ be its normal form. Let $\delta^- = \Sigma_{i < n} \omega^{\delta_i} \cdot p_i + \omega^{\delta_n} \cdot (p_n - 1)$ and $\gamma \in (\delta^-, \delta)$ be such that $\langle \delta, a_0 \rangle \in W \stackrel{\mathrm{def}}{=} (\langle \gamma, \max(A) \rangle, \langle \delta + 1, a_0 \rangle) \subseteq V$. Note that $\langle \gamma, \max(A) \rangle$ has an immediate successor, namely $\langle \gamma + 1, a_0 \rangle$; and $\langle \delta + 1, a_0 \rangle$ has an immediate predecessor, namely $\langle \delta, a_0 \rangle$. Hence $W \in \mathrm{Clop}_{\Sigma_\gamma^+(A)}(\langle \delta, a_0 \rangle)$. Let $f: \Sigma_{\omega^{\delta_n}}^+(A) \to W$ defined by $f(\langle \varepsilon, a_0 \rangle) = \langle \gamma + 1 + \varepsilon, a_0 \rangle$ for $\varepsilon < \omega^{\delta_n}$ limit, and $f(\langle \varepsilon, a \rangle) = \langle \gamma + 1 + \varepsilon, a \rangle$ otherwise. It is easy to check that f is an order-isomorphism from $\Sigma_{\omega^{\delta_n}}^+(A)$ onto W, and thus $f: \Sigma_{\omega^{\delta_n}}^+(A) \sim W$. Hence, $\mathrm{rk}(W) = \mathrm{rk}(A) + \mathrm{rk}(\omega^{\delta_n} + 1) = \mathrm{rk}(A) + \delta_n > \mathrm{rk}(A)$. Clearly $\mathrm{rk}(V) \geqslant \mathrm{rk}(W) > \mathrm{rk}(A)$, that finishes the proof of this part.

(*b*6) Obviously $\langle 0, a_0 \rangle$ is isolated in $\Sigma_\gamma^+(A)$. The second assertion of (*b*6) is an instance of the following general fact:

CLAIM. *If X is a compact interval space with infinitely many isolated points, and if x is any isolated point, then $X \sim X - \{x\}$.*

Proof. There is either an increasing ω-sequence of isolated points or a decreasing one; say without loss of generality an increasing one. So without loss of generality $x < x_0 < x_1 < \cdots < x_n < \cdots$ or $x_0 < x_1 < \cdots < x_n < \cdots < x$ all isolated. Let $c = \sup_{\iota < \omega} x_\iota$. Note that $c \neq x$. We define $f: X \to X - \{x\}$ as follows: $f(x) = x_0$, $f(x_\iota) = x_{\iota+1}$, $f(y) = y$ otherwise. Clearly f maps X one-one onto $X - \{x\}$, so it suffices to show that f is continuous. So suppose U is open in $X - \{x\}$ (hence also in X), any $y \in f^{-1}[U]$. We want to find an open set V in X such that $y \in V \subseteq f^{-1}[U]$. First suppose that $x < x_0 < x_1 < \cdots < x_n < \cdots$.

Case 1: $y < x$.
Then $y \in (-\infty, x) \cap U \subseteq f^{-1}[U]$.

Case 2: $x < y < x_0$, or $x_i < y < x_{i+1}$ for some i.
Similar to Case 1.

Case 3: $y = x$.
Then $y \in \{x\} \subseteq f^{-1}[U]$.

Case 4: $y = x_\iota$ for some i.
Similar to Case 3.

Case 5: $y = c$.
Say $c = f(c) \in (u, v) \subseteq U$, $x < u$. Then $c \in (u, v) \subseteq f^{-1}1[U]$ since if $x_\iota \in (u, v)$, then also $x_{\iota+1} \in (u, v)$.

Case 6: $c < y$.
Then $y \in (c, \infty) \cap U \subseteq f^{-1}[U]$.

The case where $x_0 < x_1 < \cdots < x_n < \cdots < x$ is similar. □

($b7$) For each $\beta \in \mathrm{Succ}(\gamma)$ let $V_\beta = \{\langle \beta, a_0 \rangle\} \cup [\langle \beta + 1, a_0 \rangle, \langle \gamma, a_0 \rangle]$. Then if U is an open neighborhood of $\mathrm{end}(\Sigma_\gamma^+(A))$, which is $\langle \gamma, a_0 \rangle$ by ($b2$), there is a $\beta \in \mathrm{Succ}(\gamma)$ such that $V_\beta \subseteq U$. Clearly each V_β is clopen, so it suffices to show that each one is homeomorphic to $\Sigma_\gamma^+(A)$. A homeomorphism is given by $f(\langle 0, a_0 \rangle) = \langle \beta, a_0 \rangle$, $f(\langle \alpha, d \rangle) = \langle \beta + \alpha, d \rangle$ for $\alpha \neq 0$.

(c) For $\tau \leqslant \rho$ limit let $c_\tau \stackrel{\mathrm{def}}{=} \sup_X(\bigcup_{\nu < \tau} F_\nu)$ and $\lambda_\tau \stackrel{\mathrm{def}}{=} \Sigma_{\nu < \tau} \gamma_\nu$. By ($c2$), $c_\tau \notin F^-$ for $\tau \leqslant \rho$ and c_τ is isolated in $\{c_\tau\} \cup F_\tau$, for every $\tau < \rho$. Now note that both $[\lambda_\tau + 1, \lambda_{\tau+1})$ are order-isomorphic to γ_τ. For $\mu < \rho$ let, by ($c3$),

$$f_\mu: \Sigma_\gamma^+(A) \cap ([\lambda_\mu + 1, \lambda_{\mu+1}] \times A) \sim F_\mu.$$

For μ limit $\leqslant \rho$ let $f_\mu^*(\langle \lambda_\mu, a_0 \rangle) = c_\mu$. Set

$$f = \bigcup \{f_\mu : \mu < \rho\} \cup \bigcup \{f_\mu^* : \mu \leqslant \rho \text{ limit}\}.$$

Clearly f is one-to-one and onto. To show that it is continuous, let $x \in f^{-1}[U]$ with U open in F.

Case 1: $x = (\alpha, a)$, with $\lambda_\mu + 1 \leqslant \alpha \leqslant \lambda_{\mu+1}$.
Then $U \cap F_\mu$ is open in F_μ, and $x \in f_\mu^{-1}[U \cap F_\mu]$. Hence there is a V open in $(\Sigma_\gamma^+(A)) \cap ([\lambda_\mu + 1, \lambda_{\mu+1}] \times A)$ such that $x \in V \subseteq f_\mu^{-1}[U \cap F_\mu] \subseteq f^{-1}[U]$. Since

$(\Sigma_\gamma^+(A)) \cap ([\lambda_\mu + 1, \lambda_{\mu+1}] \times A)$ is clopen in $\Sigma_\gamma^+(A)$, V is also open in $\Sigma_\gamma^+(A)$, as desired.

Case 2: $x = (\lambda_\mu, a_0)$ with $\mu \leqslant \rho$ limit.

Thus $c_\mu \in U$. Without loss of generality $U = (d, e)$ with d the first element of some F_ν, $\nu < \mu$, and e the first element of F_μ if $\mu < \rho$, or $e = \infty$ if $\mu = \rho$. Then for $\mu < \rho$ we have

$$f^{-1}[U] = (\Sigma_\gamma^+(A)) \cap ([\lambda_\nu + 1, \lambda_\mu] \times A),$$

which is open, and for $\mu = \rho$ we have:

$$f^{-1}[U] = ((\lambda_\nu, a_0), \infty),$$

which is also open. This finishes the proof of Lemma 2.9. □

Now, we are ready to prove:

LEMMA 2.10. *Let γ be an indecomposable ordinal, A and B scattered compact interval spaces. If $\Psi(A, B)$ holds, then so does $\Psi(\Sigma_\gamma^+(A), \Sigma_\gamma^+(B))$.*

Proof. Set $a_0 = \min(A)$, $b_0 = \min(B)$, $A' = \Sigma_\gamma^+(A)$, $B' = \Sigma_\gamma^+(B)$. By Lemma 2.9(*b*), A' and B' have the same rank $\mathrm{rk}(A) + \mathrm{rk}(\gamma + 1)$ and are indecomposable, and both have trivial invariants.

Thus $\Psi_1(A', B')$, $\Psi_2(A', B')$, and $\Psi_3(A', B')$ hold.

For $\Psi_4(A', B')$, choose $U \in \mathrm{Cl}(A)$ such that $U \sim B$. Hence by Lemma 2.9(*b*1), $\Sigma_\gamma^+(U) \sim \Sigma_\gamma^+(B)$. Clearly $\Sigma_\gamma^+(U)$ is a closed subset of $\Sigma_\gamma^+(A)$, so this proves that $\mathrm{Clcopies}_{A'}(B') \neq \emptyset$. Similarly $\mathrm{Clcopies}_{B'}(A') \neq \emptyset$.

Next, we show that $\Psi_5(A', B')$ holds. Suppose it does not. Pick $U \in \mathrm{Clop}(B')$ with $f: A' \sim U$. It is clear that $\{1\} \times A \in \mathrm{Clop}(A')$ and thus $f[\{1\} \times A] \in \mathrm{Clop}(B')$. Say $f(1, \mathrm{end}(A)) = \langle \delta, b \rangle$. If δ is limit, then $b = b_0$. $f[\{1\} \times A] \in \mathrm{Clop}(B')$, and thus, by Lemma 2.9(*b*), $\mathrm{rk}(f[\{1\} \times A]) > \mathrm{rk}(B)$. But $\mathrm{rk}(A) = \mathrm{rk}(B)$ and $\{1\} \times A \sim A$, contradiction. Thus δ is a successor. Let $V = [(\delta, b_0), (\delta + 1, b_0))$, where b_0 is the first element of B. So, V is clopen in B'. By the indecomposability of A pick a clopen $W \subseteq f^{-1}[V \cap U] \cap (\{1\} \times A)$ so that $W \sim A$. thus $A \sim f[W]$, which is a clopen subset of V, and $V \sim B$. This proves that $\mathrm{Clopcopies}_A(B) \neq \emptyset$. Similarly, $\mathrm{Clopcopies}_B(A) \neq \emptyset$, and this contradicts $\Psi_5(A, B)$. □

THEOREM 2.11. *If X is a compact scattered interval* CO-*space, then there are no $F, G \in \mathrm{Cl}(X)$ such that $\Psi(F, G)$ holds.*

Proof. Suppose such F, G exist. Consider the following statement, for indecomposable γ.

$\bullet(\gamma)$: *There are $F_\gamma, G_\gamma \in \mathrm{Cl}(X)$ such that $F_\gamma \sim \Sigma_\gamma^+(F)$ and $G_\gamma \sim \Sigma_\gamma^+(G)$.*

Assume that $\bullet(\gamma)$ is true for each indecomposable ordinal γ. Then by Lemma 2.9(*b*2), $\mathrm{rk}(\gamma + 1) \leqslant \mathrm{rk}(F_\gamma)$ for all γ, and this is impossible since $\mathrm{rk}(\gamma + 1)$ can be arbitrarily large.

Now we reach a contradiction by showing that $\bullet(\gamma)$ holds for every indecomposable γ, by induction on γ. We recall that $\omega^0 = 1$, and, by Lemma 1.9, that every indecomposable ordinal γ has the form ω^σ. We set $\Sigma_0^+(F) = F$ and $\Sigma_0^+(G) = G$. The case $\gamma = \omega^0 = 1$ is trivial.

Case 1: $\gamma = \theta \cdot \omega$, θ indecomposable.
By the induction hypothesis pick $F_\theta, G_\theta \in \mathrm{Cl}(X)$ so that $F_\theta \sim \Sigma_\theta^+(F)$ and $G_\theta \sim \Sigma_\theta^+(G)$. Thus by Lemma 2.10, $\Psi(F_\theta, G_\theta)$ holds. Now we claim that the hypotheses of Lemma 2.5 hold for F_θ, G_θ. In fact, $(H1)$ follows from $\Psi_1(F_\theta, G_\theta)$. For $(H2)$, suppose that $W_0 \in \mathrm{Clop}(F_\theta)$, $W_1 \in \mathrm{Clop}(G_\theta)$, $\mathrm{end}(F_\theta) \in W_0$, $\mathrm{end}(G_\theta) \in W_1$, and $W_0 \sim W_1$. Since G_θ is indecomposable by $\Psi_3(F_\theta, G_\theta)$, let V be a subset of G_θ clopen in G_θ such that $V \sim G_\theta$ and $V \subseteq W_1$. Then $V \sim U$ for some clopen subset U of W_0. Thus $\mathrm{Clopcopies}_{F_\theta}(G_\theta) \neq \emptyset$. Similarly $\mathrm{Clopcopies}_{G_\theta}(F_\theta) \neq \emptyset$. This contradicts $\Psi_5(F_\theta, G_\theta)$. So $(H2)$ in Lemma 2.5 holds.

Now by the CO property $F_\theta \sim F'_\theta$ for some clopen subset F'_θ of X. And since $\Psi_4(F_\theta, G_\theta)$ holds, we have $G_\theta \sim G'_\theta$ for some closed subset G'_θ of F'_θ. Put $\varphi \stackrel{\text{def}}{=} \mathrm{rk}(F_\theta) = \mathrm{rk}(G_\theta)$. Now $\mathrm{end}(F'_\theta) = \mathrm{end}(G'_\theta)$ since $D_\alpha(G'_\theta) \subseteq D_\alpha(F'_\theta)$ for all α; set $a_\theta \stackrel{\text{def}}{=} \mathrm{end}(F'_\theta)$. Of course the hypotheses of Lemma 2.5 still hold for F'_θ, G'_θ. This now implies that the hypotheses of Lemma 2.7 hold. Thus for each integer $m \geqslant 1$ there are clopen subsets $V_0^m, \ldots, V_{m-1}^m$ of X satisfying:

(1) $V_i^m \sim G_\theta$ for $i < m$;

(2) If $i < j < m$, then $\mathrm{end}(V_i^m) \neq \mathrm{end}(V_j^m)$.

Put $T \stackrel{\text{def}}{=} \{\mathrm{end}(V_i^m) : i < m < \omega\}$. Hence T is infinite, and we may assume it contains a strictly increasing sequence $\langle \mathrm{end}(V_{i_k}^{m_k}) : k < \omega \rangle$. For each $k < \omega$ let $c_k = \mathrm{end}(V_{i_k}^{m_k})$. Since F_θ and G_θ are similar, and $V_{i_k}^{m_k} \sim G_\theta$, it follows that F_θ and $V_{i_k}^{m_k}$ are similar. Since $c_k \in (c_{k-1}, c_{k+1}) \cap V_{i_k}^{m_k}$ and $V_{i_k}^{m_k}$ is indecomposable, there is a clopen subset U of $V_{i_k}^{m_k}$ so that $U \subseteq (c_{k-1}, c_{k+1}) \cap V_{i_k}^{m_k}$ and $U \sim V_{i_k}^{m_k}$. So F_θ and U are similar; hence there is a closed subset H_k of U such that $H_k \sim F_\theta$. Thus $H_k \cap H_l = \emptyset$ for distinct odd k and l. Put $H = \bigcup \{H_k : k < \omega, k \text{ odd}\}$, $F_\gamma = \mathrm{cl}_X(H)$, and $c^* = \sup_X(H)$. Then $F_\gamma = H \cup \{c^*\}$, $\mathrm{rk}(F_\gamma) = \mathrm{rk}(F_\theta) + 1$, and $\mathrm{end}(F_\gamma) = c^*$. Now by Lemma 2.9$(c)$ we get $F_\gamma \sim \Sigma_\gamma^+(F)$, if $\theta \geqslant \omega$. If $\theta = 1$, it is clear that $F_\gamma \sim \Sigma_\gamma^+(F)$. G_γ is constructed similarly.

Case 2: $\gamma = \omega^\theta$, θ limit.
Put $\delta = \mathrm{cf}(\gamma) = \mathrm{cf}(\theta)$, and let $\langle \gamma_\nu : \nu < \delta \rangle$ be a strictly increasing sequence of indecomposable ordinals cofinal in γ. Hence for $\nu < \delta$, $(\gamma_\nu, \gamma_{\nu+1})$ is order-isomorphic to $\gamma_{\nu+1}$. Now by the induction hypothesis and the CO property of X, for each $\nu < \delta$ let $F_\nu \in \mathrm{Clop}(X)$ and $G_\nu \in \mathrm{Cl}(F_\nu)$ be such that $F_\nu \sim \Sigma_{\gamma_\nu+1}^+(F)$ and $G_\nu \sim \Sigma_{\gamma_\nu+1}^+(G)$. By Lemma 2.10, $\Psi(F_\nu, G_\nu)$ holds. Put $c_\nu = \mathrm{end}(F_\nu) = \mathrm{end}(G_\nu)$. By Lemma 2.9$(b2)$, $\mu < \nu$ implies $\mathrm{rk}(F_\mu) < \mathrm{rk}(F_\nu)$. Hence $c_\mu \neq c_\nu$ by Proposition 1.5(c). Put $\mathbf{X} = \{c_\nu : \nu < \delta\}$. Since $\mathbf{X}$ is scattered and δ is regular, it follows by Lemma 1.10 that without loss of generality $c_\mu < c_\nu$ for $\mu < \nu < \delta$. Fix $\nu \in \mathrm{Succ}(\delta)$. Since F_ν is indecomposable because of $\Psi(F_\nu, G_\nu)$, we can choose a clopen subset H_ν of F_ν such that $H_\nu \subseteq (c_{\nu-1}, c_{\nu+1}) \cap F_\nu$ and $H_\nu \sim F_\nu$. Thus $H_\mu \cap H_\nu = \emptyset$ for odd $\mu < \nu$ in $\mathrm{Succ}(\delta)$. Next, let $H = \bigcup \{H_\nu : \nu \in \mathrm{Succ}(\delta) \text{ and } \nu \text{ odd}\}$ and $F_\gamma = \mathrm{cl}_X(H)$. For each

limit ordinal $\rho \leqslant \delta$ put $c_\rho^* = \sup_X(\{c_\nu : \nu < \rho\})$. Hence $F_\gamma = H \cup \{c_\rho^* : \rho \leqslant \gamma, \rho \text{ limit}\}$. By Lemma 2.9(*c*) $F_\gamma \sim \Sigma_\gamma^+(F)$. G_γ is constructed similarly, and this completes the proof of Theorem 2.11. □

The next definition introduces the relation Θ, between three closed subspaces of a space X.

DEFINITION 2.12. Let X be a compact scattered interval space. Suppose $H_i \in \mathrm{Cl}(X)$ for $i = 0, 1, 2$. We define $\Theta(H_0, H_1, H_2)$ as the conjunction of the following:

$\Theta_1(H_0, H_1, H_2)$ is: $H_2 \subseteq H_1 \subseteq H_0$,

$\Theta_2(H_0, H_1, H_2)$ is: H_i has a trivial invariant for $i = 0, 1, 2$, and $\mathrm{rk}(H_2) = \mathrm{rk}(H_1) \leqslant \mathrm{rk}(H_0)$.
Moreover, $\mathrm{end}(H_2) = \mathrm{end}(H_1) = \mathrm{end}(H_0) \stackrel{\mathrm{def}}{=} a$.

$\Theta_3(H_0, H_1, H_2)$ is: H_1 satisfies either $\Theta_{3.1}(H_0, H_1, H_2)$ or $\Theta_{3.2}(H_0, H_1, H_2)$ where:

- $\Theta_{3.1}(H_0, H_1, H_2)$: H_1 is indecomposable.
- $\Theta_{3.2}(H_0, H_1, H_2)$: Every intersection of countably many neighborhoods of a in H_1 is a neighborhood of a in H_1.

$\Theta_4(H_0, H_1, H_2)$ is: For any clopen neighborhoods U and V of a in H_0, the spaces $U \cap H_0$ and $V \cap H_1$ are not homeomorphic.

EXAMPLE 2.13. Let $H_0 = \omega_1^2 + 1 + \omega^*$, $H_1 = \{\omega_1 \cdot \alpha : \alpha < \omega_1\} + 1 + \omega^*$ and $H_3 = \{\omega_1 \cdot \alpha : \alpha < \omega_1\} + 1$. It is easy to check that $\Theta(H_0, H_1, H_2)$ holds (H_1 is indecomposable). With the notations of Example 2.4, notice that $\mathbf{F} \sim H_1 + H_1$ and $\mathbf{G} \sim H_1 + H_2$, that is the basic example of the proof of Theorem 2.14.

THEOREM 2.14. *Let X be a compact scattered interval space. If there are some closed subsets H_0, H_1, H_2 of X such that $\Theta(H_0, H_1, H_2)$ holds, then X is not a CO-space.*

Proof. By contradiction: suppose X is a CO-space. Then we may assume that H_0 is clopen in X. By $\Theta_2(H_0, H_1, H_2)$ and $\Theta_5(H_0, H_1, H_2)$, the hypotheses of Lemma 2.5 are met for H_0, H_1. Hence the hypotheses of Lemma 2.7 hold. So we choose $\langle\langle V_i^m, W_i^m\rangle : i < m\rangle$ as in the conclusion of 2.7. Hence $T \stackrel{\mathrm{def}}{=} \{\mathrm{end}(V_i^m) : i < m < \omega\}$ is infinite. Without loss of generality, T has a strictly increasing sequence $\langle \mathrm{end}(V_{i_k}^{m_k}) : k < \omega\rangle$; set $c_k \stackrel{\mathrm{def}}{=} \mathrm{end}(V_{i_k}^{m_k})$. For k odd, let $f_k : H_1 \cap W_{i_k}^{m_k} \sim V_{i_k}^{m_k}$. Still for k odd, $(c_{k-1}, c_{k+1}) \cap V_{i_k}^{m_k}$ is an open neighborhood of c_k, so since X is a Boolean space let V_k be a clopen subset of X such that $c_k \in V_k \subseteq (c_{k-1}, c_{k+1}) \cap V_{i_k}^{m_k}$.

Case 1: $\Theta_{3.1}$ holds.

Then by indecomposability of $V_{i_k}^{m_k}$ (by Lemma 2.7(*v*)), we may assume additionally that $V_k \sim H_1$. Pick $g_k : H_1 \sim V_k$ and put $G_1 = H_1, G_2 = H_2$.

Case 2: $\Theta_{3.2}$ holds.
Now for each odd k, $f_k^{-1}[V_k]$ is a clopen neighborhood of a in H_1, so $W \overset{\text{def}}{=} \bigcap \{f_k^{-1}[V_k] : k \text{ odd}\}$ is also a neighborhood of a in H_1. Let $U \in \mathrm{Clop}_X(a)$ such that $U \cap H_1 \subseteq W$. Set $G_1 = U \cap H_1$ and $G_2 = U \cap H_2$. By Proposition 1.5(c), $\mathrm{inv}(G_1) = \mathrm{inv}(G_2) = (\mathrm{rk}(H_2), a)$. Put $g_k = f_k$.

So we have constructed G_1 and G_2 in either case so that $G_1 \in \mathrm{Clop}(H_1)$, $G_2 \in \mathrm{Clop}(H_2)$, $a \in G_1 \cap G_2$, and $\mathrm{rk}(G_1) = \mathrm{rk}(G_2) = \mathrm{rk}(H_2)$; and we have certain mappings g_k.

Next put:

$$W(G_1) = \bigcup \{g_{2k+1}[G_1] : k < \omega\},$$

$$W(G_1, G_2) = \bigcup \{g_{4k+1}[G_1] : k < \omega\} \cup \bigcup \{g_{4k+3}[G_2] : k < \omega\},$$

$$c = \sup(\{c_k : k < \omega\}),$$

$$F = \mathrm{cl}_X(W(G_1)),$$

$$G = \mathrm{cl}_X(W(G_1, G_2)).$$

Thus $F = W(G_1) \cup \{c\}$ and $G = W(G_1, G_2) \cup \{c\}$. Note that $g_k[G_1] \in \mathrm{Clop}(G)$ for each odd k, $g_k[G_1] \in \mathrm{Clop}(G)$ for k of the form $4l + 1$, and $g_k[G_2] \in \mathrm{Clop}(G)$ for k of the form $4l + 3$.

CLAIM. $\Psi(F, G)$ *holds* (*and hence Theorem* 2.11 *is contradicted*).

Proof. $\Psi_1(F, G)$ and $\Psi_2(F, G)$ are clear.

That $\Psi_3(F, G)$ holds follows from $g_k[G_1] \sim G_1 \sim g_l[G_1]$ for odd k, l, and similarly for G_2.

Next, $\mathrm{Clcopies}_F(G) \neq \emptyset$ since $\bigcup \{g_{4k+1}[G_1] : k < \omega\} \cup \{c\}$ is a closed subset of F. And G_1 closed implies that $g_{4k+3}[G_1]$ is closed in $g_{4k+3}[G_2]$ and hence $\mathrm{Clcopies}_G(F) \neq \emptyset$. Thus $\Psi_4(F, G)$ holds.

To check $\Psi_5(F, G)$, suppose that $\mathrm{Clopcopies}_F(G) \neq \emptyset$. Let $U \in \mathrm{Clop}(F)$ with $U \sim G$. Now $g_3[G_2]$ is clopen in G, so there is a $V \in \mathrm{Clop}(F)$ such that $g_3[G_2] \sim V$. Say

$$s : g_3[G_2] \sim V.$$

Case 1: $\Theta_{3.1}$ holds.
Thus $G_1 = H_1$, $G_2 = H_2$. Now $s(g_3(a)) \in V$, so clearly there is a k such that $g_{2k+1}(a) = s(g_3(a))$. Thus $g_{2k+1}^{-1}[V \cap g_{2k+1}[G_1]]$ is an open neighborhood of a in H_1, so by the indecomposability of H_1 there is an $S \in \mathrm{Clopcopies}_{H_1}(H_1)$ such that

$$a \in S \subseteq g_{2k+1}^{-1}[V \cap g_{2k+1}[G_1]]$$

$$= g_{2k+1}^{-1}[V] \cap G_1.$$

Thus $S \in \mathrm{Clop}(H_1)$, and

$$S \sim g_{2k+1}[S] \sim g_3^{-1}[s^{-1}[g_{2k+1}[S]]],$$

and $g_3^{-1}[s^{-1}[g_{2k+1}[S]]]$ is a clopen set in H_2. This contradicts $\Theta_4(H_0, H_1, H_2)$.

Case 2: $\Theta_{3\,2}$ holds.

Then $a \in G_2$, so again there is a k such that $g_{2k+1}(a) = s(g_3(a))$. Then

$$s \stackrel{\text{def}}{=} g_{2k+1}^{-1}[V \cap g_{2k+1}[G_1]] = g_{2k+1}^{-1}[V] \cap G_1$$

is clopen in H_1, and the same computation as in Case 1 gives a contradiction.

Thus $\mathrm{Clopcopies}_F(G) = \emptyset$, and the proof is complete. □

Now, we will apply Theorems 2.11 and 2.14 to prove Theorem 1.2. The following definition introduces trivial generalizations and some notation on stationary sets in a chain.

DEFINITION 2.15. Let L be a chain.

(*a*) L is *relatively complete* if every non-empty bounded subset of L has a supremum and infimum in L. A non-empty subset D of L is *strongly relatively complete* if:

(1) D is relatively complete as a subchain of L.

(2) If $A \subseteq D$ is bounded, then $\sup_L(A)$ and $\inf_L(A)$ exist and equal respectively $\sup_D(A)$ and $\inf_D(A)$.

(*b*) Let κ be regular and W be a well-ordered chain of type κ. A subset C of W is *closed and unbounded* (*club*) if $\mathrm{cl}_W(C) = C$ and C is cofinal in W. A subset $S \subseteq W$ is *stationary* if it meets all clubs in W. A *non-stationary* set in W is a subset which is not stationary. The set of clubs of W will be denoted by $\mathrm{club}(W)$.

(*c*) $N \subseteq L$ is *hereditarily non-stationary* in L if for any uncountable regular cardinal κ and any $W \subseteq L$ which is a well ordered subchain of type κ such that W is strongly relatively complete, $W \cap N$ is non-stationary in W. $\mathrm{Hernonst}(L)$ will denote the set of hereditarily non-stationary subsets of L.

(*d*) For any $x \in L$, $\tau_L(x) \stackrel{\text{def}}{=} \langle \mathrm{cf}^l_L(x), \mathrm{cf}^r_L(x) \rangle$ is the *character* of x in L. For x the greatest element of L (respectively the smallest element of L) we let $\mathrm{cf}^r_L(x) = 0$ (respectively $\mathrm{cf}^l_L(x) = 0$).

(*e*) L is said to be *countably two-sided* if for every $x \in L$, if $\tau_Y(x) = \langle \xi, \eta \rangle$ with $\xi, \eta \geqslant \omega$, then $\xi = \eta = \omega$. We denote by $\mathrm{Countwosid}(L)$ the set of all $x \in L$ such that $\tau_L(x) = \langle \omega, \omega \rangle$.

Before giving a characterization of ordinal spaces in the class of countably two-sided interval spaces, we will establish some simplifying assumptions.

LEMMA 2.16. *Let Y be a scattered compact interval space, with* $\mathrm{End}(Y)$ *a singleton* $\{\mathrm{end}(Y)\}$.

(*a*) *If* $\tau_Y(\mathrm{end}(Y)) = \langle \omega, \omega \rangle$, *then there is a countable basis* $\langle U_n : n \in \omega \rangle$ *of clopen neighborhood of* end(Y) *such that* $Y = U_0 \supseteq \cdots \supseteq U_n \supseteq \cdots$.

(*b*) *Assume that* Y *is countably two-sided. Then*:

(1) *There is a countably two-sided chain* $\mathbf{Y}$ *such that*: $\mathbf{Y}$ *is homeomorphic to* Y, end($\mathbf{Y}$) *is the greatest element of* $\mathbf{Y}$ *and* end(Y) = end($\mathbf{Y}$).

(2) *Assume that* $\mathrm{rk}(Y) = \beta$ *and* $\tau_Y(\mathrm{end}(Y)) = \langle \delta, 0 \rangle$. *Then* $\delta = \mathrm{cf}(\beta)$ *if* β *is limit, and* $\delta = \omega$ *otherwise.*

Proof. (*a*) Obvious.

(*b*1) If $\tau_Y(\mathrm{end}(Y)) = \langle \omega, 1 \rangle$, then the portion of Y to the right of end(Y) can be inserted between two consecutive elements coming before end(Y). For $\tau_Y(\mathrm{end}(Y)) = \langle 1, \omega \rangle$ or $\langle \omega, 0 \rangle$ one can reverse the order. Finally, for $\tau_Y(\mathrm{end}(Y)) = \langle \omega, \omega \rangle$, one can interlace successive intervals formed by isolated points converging upwards to end(Y) with those formed by isolated points converging downwards to end(Y).

For (*b*2), we distinguish different cases:

Case 1: $\mathrm{rk}(Y) = \beta$ limit.
Let $\langle a_\nu : \nu < \delta \rangle$ be a strictly increasing sequence cofinal in $Y - \mathrm{End}(Y)$, with a_0 the first element of Y. Put $Y_\nu = [a_0, a_\nu]$ for each $\nu < \delta$. Hence $\mathrm{rk}(Y_\mu) \leqslant \mathrm{rk}(Y_\nu)$ for $\mu < \nu < \delta$. Clearly $D_\alpha(Y) = \bigcup_{\nu < \delta} D_\alpha(Y_\nu) \cup \{\mathrm{end}(Y)\}$, so $\beta = \sup\{\mathrm{rk}(Y_\nu) : \nu < \delta\}$. So $\delta = \mathrm{cf}(\beta)$.

Case 2: $\mathrm{rk}(Y) = \gamma + 1$ for some γ.
Then $D_\gamma(Y)$ is order-isomorphic to $\omega + 1$, so $\delta = \omega$. □

LEMMA 2.17. *If* Y *is a countably two-sided scattered compact interval space, then*:

(*a*) *If* Countwosid(Y) $\in$ Hernonst(Y), *then* $Y \sim \alpha + 1$ *for some ordinal* α.

(*b*) *If* Countwosid(Y) $\notin$ Hernonst(Y), *then there are a regular cardinal* $\kappa > \omega$ *and a stationary subset* T *of* κ *such that* $Z_\kappa(T, 1 + \omega^*)$ *is homeomorphic to a closed subspace of* Y.

Proof. We proceed by induction on rk(Y). Without loss of generality, End(Y) is a singleton {end(Y)}. By Lemma 2.16(*b*1) we may assume that end(Y) is the greatest element of Y.

(*a*): Countwosid(Y) $\in$ Hernonst(Y).

Case 1: rk(Y) is a successor ordinal $\gamma + 1$, or $\mathrm{cf}(\mathrm{rk}(Y)) = \omega$.
Pick $\langle U_n : n < \omega \rangle$ as in Lemma 2.16(*a*). Then Countwosid(Y) $\cap (U_n - U_{n+1}) \in$ Hernonst($U_n - U_{n+1}$) for each $n < \omega$. Since $\mathrm{rk}(U_n - U_{n+1}) < \mathrm{rk}(Y)$, it follows that $U_n - U_{n+1}$ is homeomorphic to some ordinal $\alpha_n + 1$. Hence $Y \sim \Sigma_{n < \omega} (\alpha_n + 1) + 1$.

Case 2: $\mathrm{cf}(\mathrm{rk}(Y)) > \omega$.
Let $\gamma = \mathrm{cf}(\mathrm{rk}(Y))$. By Lemma 2.16(*b*2), let $\langle y'_\alpha : \alpha < \gamma \rangle$ be a strictly increasing continuous sequence with supremum end(Y). Let $W = \{y'_\alpha : \alpha < \gamma\}$. Let C be a club in W disjoint from Countwosid(Y). Obviously if $y \in C$, then $\tau_Y(y) \neq \langle \omega, \omega \rangle$, and thus $\mathrm{cf}^l_Y(y) \in \{0, 1\}$ or $\mathrm{cf}^r_Y(y) \in \{0, 1\}$.

CLAIM 1. *There is a strictly increasing sequence* $\langle y_\nu : \nu < \gamma \rangle$ *in* Y *cofinal in* $Y - \{\mathrm{end}(Y)\}$ *such that*:

(1) $\{[y_\nu, y_{\nu+1}) : \nu < \gamma\}$ *is a partition of* $Y - \{\text{end } Y\}$.

(2) $[y_\nu, y_{\nu+1})$ *is a clopen subset of* Y *for each* $\nu < \gamma$.

Proof. To see this, consider two possibilities:

Possibility 1: $\{y \in C : \text{cf}^l_Y(y) = 1\}$ is cofinal in C.
Each such y has an immediate predecessor y^-. Let $\langle y_\nu : \nu < \gamma \rangle$ enumerate in increasing order all such y's. Now $[y_\nu, y_{\nu+1}) = (y_\nu^-, y_{\nu+1}) = [y_\nu, y_{\nu+1}^-]$, so such intervals are clopen. Replacing y_0 by the first element of Y, we obtain (1), (2).

Possibility 2: $\{y \in C : \text{cf}^r_Y(y) = 1\}$ is cofinal in C.
So each element of this set has an immediate successor. Enumerating this set as $\langle y_\nu : \nu < \gamma \rangle$, what we really want is $\langle y_\nu^+ : \nu < \gamma \rangle$, except for replacing y_0^+ by the first element of Y. □

Now, having (1) and (2), we have $\text{rk}([y_\nu, y_{\nu+1})) < \text{rk}(Y)$ and so $[y_\nu, y_{\nu+1}] \sim \mu_\nu + 1$ for some ordinal μ_ν. Hence $Y \sim (\Sigma_{\nu < \gamma} (\mu_\nu + 1)) + 1$, as desired.

(*b*): $\text{Countwosid}(Y) \notin \text{Hernonst}(Y)$.

Case 1: $\text{rk}(Y)$ is a successor ordinal, or $\text{cf}(\text{rk}(Y)) = \omega$.
Pick $\langle U_n : n \in \omega \rangle$ as in Lemma 2.16(*a*).

CLAIM 2. *There is an n such that* $\text{Countwosid}(Y) \cap (U_n - U_{n+1}) \notin \text{Hernonst}(Y)$.

Proof. Indeed, since $\text{Countwosid}(Y) \notin \text{Hernonst}(Y)$, there is an uncountable regular cardinal κ and a subchain W of Y order-isomorphic to κ so that W is strongly relatively complete and $W \cap \text{Countwosid}(Y)$ is stationary in W. Note that $W \cap (U_n - U_{n+1})$ is a club in W if it has cardinality κ. Since one of these sets does have power κ, the claim follows. □

Taking such an n, we have $\text{rk}(U_n - U_{n+1}) < \text{rk}(Y)$, so from the induction hypothesis there are a regular uncountable cardinal κ and a stationary subset T of κ such that $Z_k(T, 1 + \omega^*)$ is homeomorphic to a closed subset of $U_n - U_{n+1}$, hence of Y, as desired.

Case 2: $\text{cf}(\text{rk}(Y)) > \omega$.
By the assumption of (*b*), let D be a strongly relatively complete subset of Y such that:

(1) D is order-isomorphic to an uncountable regular cardinal.

(2) $S \overset{\text{def}}{=} \text{Countwosid}(Y) \cap D$ is stationary in D.

Let $\langle y_\xi : \xi < \delta \rangle$ be the canonical increasing enumeration of D. For $y_\xi \notin S$ put $F_\xi = \{y_\xi\}$, and for $y_\xi \in S \cap D$ choose a strictly decreasing sequence $\langle y_{\xi n} : n < \omega \rangle$ in $(y_\xi, y_{\xi+1})$ converging to y_ξ and set $F_\xi = \{y_\xi\} \cup \{y_{\xi n} : n \in \omega\}$. Finally, we put $F_\delta = \{\text{end}(Y)\}$. Thus $F \overset{\text{def}}{=} \bigcup_{\xi \leq \delta} F_\xi$ is closed. Clearly $F \sim Z_\delta(S, 1 + \omega^*)$, as desired.

This finishes the proof of Lemma 2.17. □

COROLLARY 2.18. *Let X be a compact scattered* CO *interval space, and Y a closed and countably two-sided subset of X. Then*:

(*a*) Countwosid(Y) $\in$ Hernonst(Y).

(*b*) $Y \sim \alpha + 1$ *for some ordinal* α.

Proof. (*a*) By contradiction. By Lemma 2.17(*b*) there exist a closed set $H_1 \subseteq Y$ and a homeomorphism $f\colon Z_\kappa(T_1, 1+\omega^*) \sim H_1$ for some stationary subset T_1 of a regular uncountable cardinal κ. Let $T_3 \subseteq T_2 \subseteq T_1$ be sets such that T_3, $T_2 - T_3$, and $T_1 - T_2$ are stationary in κ. Let $H_i = f[Z_\kappa(T_i, 1+\omega^*)]$ for $i = 2, 3$. Clearly $Z_\kappa(T_i, 1+\omega^*)$ is closed in $Z_\kappa(T_{i-1}, 1+\omega^*)$ for $i = 2, 3$, so H_i is a closed subset of H_{i-1} for $i = 2, 3$.

CLAIM. $\Theta(H_1, H_2, H_3)$ *holds.*

Proof. Indeed, $\Theta_1(H_1, H_2, H_3)$, $\Theta_2(H_1, H_2, H_3)$, and $\Theta_{3.2}(H_1, H_2, H_3)$ are clear. From Lemma 2.1 of Bonnet and Si-Kaddour [1987], it follows that $\Theta_4(H_1, H_2, H_3)$ and $\Theta_5(H_1, H_2, H_3)$ hold. This proves the claim. □

The claim contradicts Theorem 2.14, so (*a*) is proved.

(*b*) Follows from (*a*) and Lemma 2.17. □

The next definition is used to conclude the proof of Theorem 1.2.

DEFINITION 2.19. Let X be a chain. We denote by Bad(X) the set of all $x \in X$ such that:

(1) $\mathrm{cf}^l_X(x)$, $\mathrm{cf}^r_X(x) \geqslant \omega$;

(2) $\mathrm{cf}^l_X(x) \geqslant \omega_1$ or $\mathrm{cf}^r_X(x) \geqslant \omega_1$.

PROPOSITION 2.20. *If X is a* CO *compact scattered interval space, then* Bad(X) *is a finite set.*

Proof. Assume that Bad(X) is infinite. There is no loss in generality that there is a strictly increasing sequence $\langle a_n : n < \omega \rangle$ of elements of Bad(X) such that each a_n is isolated in Bad(X). So for each $n < \omega$ there is a clopen interval U^n such that $U^n \cap \mathrm{Bad}(X) = \{a_n\}$. With no loss in generality, the U^n's are pairwise disjoint, and $U^n < U^m$ for $n < m$. Let

$$a = \sup_X \{a_n : n < \omega\}.$$

Now by Corollary 2.18, $(-\infty, a_n] \cap U^n \sim \xi_n + 1$ and $[a_n, \infty) \cap U^n \sim 1 + \zeta_n^*$ for some ordinals ξ_n, ζ_n. Since $a_n \in \mathrm{Bad}(X)$, we have $\xi_n, \zeta_n \geqslant \omega$ and $\mathrm{cf}(\xi_n) \geqslant \omega_1$ or $\mathrm{cf}(\zeta_n) \geqslant \omega_1$. Clearly $U^n \sim \xi_n + 1 + \zeta_n^*$. So there is a closed subset F_n of U^n such that $F_n \sim \mathrm{cf}(\xi_n) + 1 + (\mathrm{cf}(\zeta_n))^*$. Without no loss of generality, $\omega_1 \leqslant \mathrm{cf}(\xi_n) \geqslant \mathrm{cf}(\zeta_n)$ for all n. By Ramsey's theorem, we can suppose that each sequence $\langle \mathrm{cf}(\xi_n) : n < \omega \rangle$, $\langle \mathrm{cf}(\zeta_n) : n < \omega \rangle$ is either constant or strictly increasing.

Case 1: $\mathrm{cf}(\xi_n) = \xi$, $\mathrm{cf}(\zeta_n) = \zeta$ for all n.

Hence $\xi \geqslant \zeta$. Let $H = \mathrm{cl}_X(\bigcup_{n<\omega} F_n)$. Clearly $H \sim (\xi + 1 + \zeta^*) \cdot \omega + 1$. Now let G be the closed subspace of $H' \overset{\mathrm{def}}{=} (\xi + 1 + \zeta^*) \cdot \omega + 1$ obtained by eliminating the even ζ^*'s, i.e.,

$$H' = \xi + 1 + \zeta^* + \xi + 1 + \zeta^* + \xi + 1 + \zeta^* + \xi + 1 + \zeta^* + \cdots$$

is transformed to

$$G = \xi + 1 + \zeta^* + \xi + 1 + \xi + 1 + \zeta^* + \xi + 1 + \cdots$$

So $G = (\xi + 1 + \zeta^* + \xi + 1) \cdot \omega + 1$. We claim that $\Psi(H', G)$ holds, contradicting Theorem 2.11 (the reader can be helped by Example 2.13). All parts of $\Psi(H', G)$ are easy except $\Psi_5(H', G)$. We claim, in fact, that $\mathrm{Clopcopies}_{H'}(G) = \emptyset$. For, let U be a clopen subset of H' and suppose $h: U \sim G$. If we take the middle element b in a part $\xi + 1 + \xi$ of G we have the following situation: b has a clopen neighborhood U'', namely $(-\infty, b]$, such that if V and W are closed subsets of U'' such that $V \cup W = U''$, $b \in V \cap W$, and b is not isolated in V and in W, then $V \cap W - \{b\} \neq \emptyset$ (since $V - \{b\}$ and $W - \{b\}$ are clubs in ξ, and ξ is regular and uncountable). Hence the preimage c of b under $U \sim G$ has such a neighborhood U' too. Since there is a ξ-sequence of distinct elements converging to b, the same is true for c, so c is the "middle" element in a part $\xi + 1 + \zeta^*$ of H' (since $\zeta \leqslant \xi$). Let $V' = (-\infty, c] \cap U'$, $W' = [c, \infty) \cap U'$: this contradicts the above property.

Case 2: Not Case 1, i.e., $\langle \mathrm{cf}(\xi_n) : n < \omega \rangle$ or $\langle \mathrm{cf}(\zeta_n) : n < \omega \rangle$ is strictly increasing. For each infinite subset A of ω, put

$$F(A) = \mathrm{cl}(\bigcup \{F_n : n \in A\}) = \{a\} \cup \bigcup \{F_n : n \in A\}.$$

Pick $U_A \in \mathrm{Clop}(X)$ and $f_A : F(A) \sim U_A$.

CLAIM. *Suppose that A and B are infinite subsets of ω such that $A \cap B$ is finite. Then the following holds*:

(*) *For every $V \in \mathrm{Clop}_{F(A)}(a)$ and $W \in \mathrm{Clop}_{F(B)}(a)$ we have $V \not\sim W$.*

Proof. In fact, assume otherwise and choose $n \in A - B$ such that $F_n \subseteq V$. Then the "middle" element in F_n must map to a "middle" element in $F(B)$, by the argument in Case 1. This gives rise to the following situation: We have an element b mapping to an element c, where there exist ξ_n- and ζ_n- sequences converging to b, while every sequence of regular type converging to c has type ξ_m or ζ_m, where $n \neq m$. This is clearly impossible. So (*) holds. □

Because (*) holds, we have $f_A(a) \neq f_B(a)$.

Now let $\langle A_\alpha : \alpha < \omega_1 \rangle$ be a sequence of infinite subsets of ω such that $|A_\alpha \cap A_\beta|$ is finite, for $\alpha \neq \beta$. Since $\langle f_{A_\alpha}(a) : \alpha < \omega_1 \rangle$ is one-one, we may suppose that it is strictly increasing (by Lemma 1.10). For $\alpha < \omega_1$ let $F_\alpha = F(A_\alpha)$, $U_\alpha = U_{A_\alpha}$, $f_\alpha = f_{A_\alpha}$, $b_\alpha = f_\alpha(a)$. By omitting some b_α's if necessary we may assume that there are pairwise disjoint clopen sets V_α, $\alpha < \omega_1$, such that $V_\alpha \in \mathrm{Clop}_{U_\alpha}(a_\alpha)$. Now fix $\alpha < \omega_1$. Since $f_\alpha : F_\alpha \sim U_\alpha$, there is an $n \in A_\alpha$, say $h(\alpha)$ such that $f_\alpha[F_{h(\alpha)}] \subseteq V_\alpha$. Because $h: \omega_1 \to \omega$, there is $q \in \omega$ such that $\Gamma \overset{\mathrm{def}}{=} h^{-1}(q)$ is infinite. Then for $\alpha, \beta \in \Gamma$, $f_\alpha[F_q] \sim F_q \sim f_\beta[F_q]$ and $f_\alpha[F_q] \cap f_\beta[F_q] = \emptyset$. For $\alpha \in \Gamma$, let $c_\alpha = \mathrm{end}(f_\alpha[F_q])$. There is no loss in assuming that $\langle c_\alpha : \alpha \in \Gamma \rangle$ is strictly increasing. Let $c = \sup(\{c_\alpha : \alpha \in \Gamma\})$ and

$$H' = \mathrm{cl}_X\Big(\bigcup_{\alpha\in\Gamma} f_\alpha[F_q]\Big) = \Big(\bigcup_{\alpha\in\Gamma} f_\alpha[F_q]\Big)\cup\{c\}.$$

Then H' puts us back in Case 1, so we have a contradiction. □

PROPOSITION 2.21. *Let X be a compact scattered chain such that $|\mathrm{Bad}(X)| = 1$. Then the following are equivalent*:

(*i*) *X is a CO-space.*

(*ii*) *$X \sim \alpha + 1 + \beta + 1 + \gamma^*$ for some infinite regular cardinals β, γ with $\beta \geqslant \gamma$, $\beta \geqslant \omega_1$, and $\alpha \geqslant \beta \cdot \omega$.*

Proof. (*i*) implies (*ii*). Put $\mathrm{Bad}(X) = \{a\}$. Hence there are b, c such that:

(1) $b < a < c$.

(2) $U \stackrel{\mathrm{def}}{=} [b, c] \in \mathrm{Clop}(X)$ and $D_{\mathrm{rk}(U)}(U) = \{a\}$.

Thus $\mathrm{Bad}([b, a]) = \emptyset$. Hence, by the definition, $[b, a]$ is countably two-sided; so by Corollary 2.18(*b*) it is homeomorphic to an infinite successor ordinal. Similarly for $[a, c]$ and $X - U$. So $X \sim \delta + 1 + \gamma^*$ for some ordinals δ, γ, with δ, γ infinite. And since a is bad we have $\mathrm{cf}(\delta) \geqslant \omega_1$ or $\mathrm{cf}(\gamma) \geqslant \omega_1$; without loss of generality $\mathrm{cf}(\gamma) \geqslant \omega_1$ and $\mathrm{cf}(\gamma) \geqslant \mathrm{cf}(\delta) \geqslant \omega$.

Now by Lemma 1.9(*b*), $\gamma + 1 \sim \varepsilon \cdot p + 1$ for some indecomposable ordinal ε and some $p \in \omega (p \geqslant 1)$. Hence $\gamma + 1 \sim (\varepsilon + 1)(p - 1) + \varepsilon + 1$ and so $1 + \gamma^* \sim 1 + \varepsilon^* + (\varepsilon + 1) \cdot (p - 1)$. The clopen part $(\varepsilon + 1) \cdot (p - 1)$ can be moved to the front of $\delta + 1 + \gamma^*$. So, without loss of generality γ is indecomposable. Arguing similarly with δ, we may assume that:

$$X \sim \alpha + 1 + \beta + 1 + \gamma^*$$

where α is some ordinal and β and γ are indecomposable, with $\mathrm{cf}(\beta) \geqslant \omega_1$ and $\mathrm{cf}(\beta) \geqslant \mathrm{cf}(\gamma) \geqslant \omega$.

Now we claim

CLAIM 1. *$\gamma + 1$ is homeomorphic to $\lambda + 1$ for some regular cardinal λ.*

Proof. By contradiction. Write $\beta + 1 + \gamma^* = G_0 \cup \{b\} \cup G_1$, where $G_0 < b < G_1$, $G_0 \sim \beta$, and $G_1 \sim \gamma^*$. Let

$H_1 = G_0 \cup \{b\} \cup G_1$,

$H_2 = G_0 \cup \{b\} \cup G_1'$ where G_1' is a coinitial subset of G_1 of order type $(\mathrm{cf}(\gamma))^*$, and

$H_3 = G_0 \cup \{b\}$.

We claim that $\Theta(H_1, H_2, H_3)$; this will contradict Theorem 2.14.

$\Theta_1(H_1, H_2, H_3)$, $\Theta_2(H_1, H_2, H_3)$, and $\Theta_{3.1}(H_1, H_2, H_3)$ are clear.

To prove $\Theta_4(H_1, H_2, H_3)$, let U be clopen in H_1, $b \in U$ and let V be clopen in H_2, $b \in V$, and suppose that $f\colon U \sim V$. Because $\mathrm{Inv}(U) = \mathrm{Inv}(V)$, $b = \mathrm{end}(U) = \mathrm{end}(V)$, and $f\colon U \sim V$, we have $f(b) = b$. Let $S = (-\infty, b) \cap U$, $T = (b, \infty) \cap U$. Let $K \subseteq (-\infty, b) \cap V$ be closed, cofinal, and of order type $\mathrm{cf}(\beta)$. If $K \cap f[S]$ and $K \cap f[T]$ are both club in K, then so is $K \cap f[S] \cap f[T]$, hence this set is non-empty. But $S \cap T = \{b\}$, contradiction. So say with no loss of generality $K \cap f[T]$ is not club in K. Then $(f[T]-\{b\})\cap G_0$ is not cofinal in G_0. Hence by passing to a clopen subset $W\ni b$ of V, we may assume that $b\notin W\cap f[T]$. Then $b\notin W\cap \mathrm{cl}_{H_2}(f[T])\supseteq W\cap f[\mathrm{cl}_{H_1}(T)]\ni b=f(b)$; contradiction.

Similarly, $\Theta_5(H_1, H_2, H_3)$ holds; that finishes the proof of Claim 1. □

Similarly,

CLAIM 2. *$\beta+1$ is homeomorphic to $\kappa+1$ for some regular cardinal κ.* □

Thus $X\sim\alpha+1+\kappa+1+\lambda^*$, with $\kappa\geqslant\lambda, \kappa, \lambda\geqslant\omega$ regular cardinals, $\kappa\geqslant\omega_1$. If $\alpha<\kappa\cdot\omega$, then $X\sim\kappa\cdot n+1+\lambda^*$ for some positive integer n. We claim that:

CLAIM 3. *The closed set $\kappa\cdot n+1$ is not homeomorphic to a clopen set of $\kappa\cdot n+1+\lambda^*\sim X$.*

Proof. By contradiction. Write $X=G_0\cup\{a\}\cup G_1$, with $G_0<a<G_1$, $G_0\sim\kappa\cdot n$ and $G_1\sim\lambda^*$. Let $F=G_0\cup\{a\}$, and let $f\colon F\sim U\subseteq X$ where U is clopen in X. Clearly, $a\in \mathrm{rk}(F)=\mathrm{rk}(U)=\mathrm{rk}(X)$, $\mathrm{End}(F)=\mathrm{End}(U)=\mathrm{End}(X)\ni a$ and $U\sim X$. Let $b\in\mathrm{End}(F)$ be such that $f(b)=a$. This gives rise to the following situation: there is a clopen neighborhood U' of $b\in F$ such that $\mathrm{End}(U')=\{b\}$; if V and W are closed subsets of U' such that $V\cup W=U'$, $b\in V\cap W$, and b is not isolated in V and in W, then $V\cap W-\{b\}\neq\emptyset$ (since $V-\{b\}$ and $W-\{b\}$ are clubs in κ, and κ is regular and uncountable). Hence a has such a neighborhood U'' in X too. But a is the "middle" element in a part $\kappa+1+\lambda^*$. Let $V''=(-\infty,a]\cap U''$, $W''=[a,\infty)\cap U''$: this contradicts the above property. □

This finishes the proof of (*i*) implies (*ii*).

(*ii*) implies (*i*) follows from the fact that every closed subspace of $\beta+1+\gamma^*$ is either homeomorphic to clopen subspace of the ordinal space $\beta+1$ or to $\beta+1+\gamma^*$, and the fact that every compact ordinal space is a CO space.

This finishes the proof of Proposition 2.21. □

Proof of Theorem 1.2. It follows from Proposition 2.20 and Proposition 2.21. □

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