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Antiplane shear problems of perfect and partially damaged matrix-inclusion systems

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Summary: Antiplane shear problem of an infinite medium containing a circular inclusion of different material is investigated in this article. Perfectly bonding between the matrix and the inclusion, as well as partial debonding in the form of a circumferential crack occurring at the interface of these two constituents, are considered. Using the complex variable method in conjunction with a semi-inversed technique, exact expressions for the stress, displacement and stress intensity factor in the problem are obtained for various external loading conditions, including that brought about by stress singularities. A number of numerical results of stress intensity factors for a partially damaged matrix-inclusion system are given, and a discussion is made on the general features of the antiplane shear problem of inhomogeneous matrix-inclusion systems.

Scherung von perfekten und teilweise beschädigten Matrix-Einschluß-Systemen außerhalb der Ebene

Übersicht: In dieser Abhandlung wird das Problem der Scherung außerhalb der Ebene für einen unendlichen Körper mit einem kreisförmigen Einschluß aus unterschiedlichem Material untersucht. Es wird die perfekte Verbindung zwischen der Matrix und dem Einschluß ebenso wie teilweise Ablösung in der Form eines Umkreisrisses an der Schnittfläche zwischen den beiden Bestandteilen betrachtet. Unter Verwendung der komplexen Variablenmethode in Verbindung mit einem Halbinversionsverfahren werden exakte Ausdrücke für Spannung, Verschiebung und Spannungsintensitätsfaktor für verschiedene äußere Belastungszustände, einschließlich dem Einfluß von Spannungssingularitäten, erhalten. Es werden numerische Ergebnisse für den Spannungsintensitätsfaktor für ein teilweise beschädigtes Matrix-Einschluß-System gegeben, und die allgemeine Form des Problems der Scherung außerhalb der Ebene für ein nicht homogenes Matrix-Einschluß-System wird diskutiert.

1 Introduction

In recent years, composite materials have been finding an ever growing application in various branches of engineering. Being a necessity to the damage tolerance design, the evaluation of stresses and stress intensity factors in structures of composite materials has become a significant subject in fracture mechanics, to which many references have been dedicated. England [1] studied an arc crack around a circular elastic inclusion by the method of complex variables. Using dislocation density as Green's functions, Erdogan et al. [2] solved the problem of an infinite medium containing a circular inclusion and a neighboring, arbitrary oriented crack. A more complicated situation, with a debonding around and a crack occurring from a circular rigid inclusion, was treated by Hasebe et al. [3] as a mixed boundary value problem under uniform tension. Recently, Luo and Chen discussed an interface crack in a three-phrase composite constitutive model by the use of the method of complex variables [4].

It should be pointed out that the references mentioned above are concerned with plane strain or plane stress problems. As for the problem of antiplane shear, apart from a few papers treating of straight cracks between different materials ([5] for instance), no reference seems available for circumferential interface cracks, a fact manifesting that the problem has not been treated sufficiently and conclusively.

This article deals with the antiplane shear problem of an infinite medium containing a circular inclusion of different material. Its content is now outlined as follows. Both perfect bonding and partial debonding in the form of a circumferential crack between the matrix and the inclusion are considered.

The external forces consist of uniform antiplane shear at infinity and stress singularities. Using the method of complex variables in conjunction with the semi-inversed method, exact expressions for the stresses, displacements and stress intensity factors in the problem are developed. Finally, numerical values of stress intensity factors are presented as well as a discussion on the features of the antiplane shear problem in the matrix-inclusion system.

2 Problem and basic formulation

In this article we deal with the antiplane shear problem of an infinite matrix containing a circular inclusion of different material. The region in the z plane occupied by the matrix is denoted with S_1 and that by the inclusion with S_2. The two regions are separated via the interface L, which stands for $r = 1$ (Fig. 1). It is well known that the stresses and the displacement in an antiplane shear problem are related by [6]

$$\tau_{xt_j} = G_j \frac{\partial w_j}{\partial x}, \qquad \tau_{yt_j} = G_j \frac{\partial w_j}{\partial y}, \tag{1a, b}$$

where the displacement w_j should satisfy the following equation:

$$\frac{\partial^2 w_j}{\partial x^2} + \frac{\partial^2 w_j}{\partial y^2} = 0, \tag{2}$$

where $j = 1, 2$, G_j stands for the shear modulus of the material. The subscripts 1 and 2 are used to distinguish similar quantities of the matrix and the inclusion.

To solve the problem we introduce two analytic functions $\Phi_j(z)$, $j = 1, 2$ and write [6]

$$\tau_{xt_j} - \mathrm{i}\tau_{yt_j} = \Phi_j(z) \tag{3}$$

$$w_j = \frac{\phi_j(z) + \overline{\phi_j(z)}}{2G_j}, \qquad \Phi_j(z) = \frac{d\phi_j(z)}{dz}. \tag{4a, b}$$

Also, it is ready to show that

$$\tau_{rt_j} - \mathrm{i}\tau_{\theta t_j} = e^{\mathrm{i}\theta}\Phi_j(z) \qquad \text{for} \qquad j = 1, 2. \tag{5}$$

In this way, by a basic property of an analytic function [7], equation (2) is automatically satisfied. Therefore, the unknown functions $\Phi_j(z)$ should be determined from the boundary condition of the problem at inifinity and the continuity conditions of traction and displacement at the interface of the two constituents, $r = 1$. In the case of perfect bonding between the matrix and the inclusion, the continuity conditions take the following form:

$$\tau_{rt_1} = \tau_{rt_2}, \qquad w_1 = w_2 \qquad \text{for} \qquad r = 1. \tag{6a, b}$$

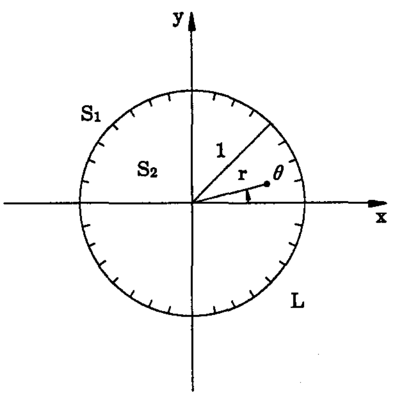


Fig. 1. The matrix and inclusion

The function $\Phi_1(z)$ is originally defined in region S_1 and $\Phi_2(z)$ in S_2 (Fig. 1). Now they are extended into their counterpart regions, S_2 and S_1, respectively, after the following formulas [8]:

$$\Phi_1(z) = \overline{\Phi_1}\left(\frac{1}{z}\right) \quad \text{for } z \text{ in } S_2, \tag{7}$$

$$\Phi_2(z) = \overline{\Phi_2}\left(\frac{1}{z}\right) \quad \text{for } z \text{ in } S_1. \tag{8}$$

The regions of existence for $\Phi_1(z)$ and $\Phi_2(z)$ now cover the whole z plane except possibly the interface $r = 1$. In addition, the following properties hold for $\Phi_1(z)$ and $\Phi_2(z)$:

$$\overline{\Phi_j^{+}(t)} = \Phi_j^{-}(t), \quad \overline{\Phi_j^{-}(t)} = \Phi_j^{+}(t) \quad \text{for} \quad j = 1, 2 \tag{9a, b}$$

where t denotes a point on the interface $r = 1$, and the superscripts + and − are used to indicate that the affixed function values are approached within region S_2 and region S_1, respectively. With use of (7, 8, 9a, b), the continuity conditions (6a, b) at the interface can be written as

$$t^2\Phi_2^{+}(t) - \Phi_1^{+}(t) = t^2\Phi_1^{-}(t) - \Phi_2^{-}(t), \tag{10}$$

$$G_1t^2\Phi_2^{+}(t) + G_2\Phi_1^{+}(t) = G_2t^2\Phi_1^{-}(t) + G_1\Phi_2^{-}(t). \tag{11}$$

In (11), we have replaced equivalently (6b) with

$$\frac{dw_1}{d\theta} = \frac{dw_2}{d\theta} \quad \text{for} \quad r = 1.$$

Now the solution to the problem is reduced to the finding of two complex functions, $\Phi_1(z)$ and $\Phi_2(z)$, analytic in the whole z plane sectioned by the interface $r = 1$, and satisfying (10, 11) as well as the boundary conditions at infinity.

3 Solutions for a perfectly bonded inclusion

3.1 *Uniform shear in the x direction at infinity*

The boundary conditions for this particular case are following

$$\tau_{xt} = \tau_0, \quad \tau_{yt} = 0 \quad \text{for} \quad z \to \infty. \tag{12a, b}$$

To solve the problem it is noticed that since there is no singularity of stress or displacement in the whole z plane, $\Phi_1(z)$ and $\Phi_2(z)$ should be holomorphic in both S_1 and S_2. Besides,

$$\Phi_1(z) \to \tau_0 \quad \text{for} \quad z \to \infty, \tag{13}$$

because of the boundary conditions (12a, b). On account of the above observations, we assume

$$\text{in } S_2 \quad z^2\Phi_2(z) - \Phi_1(z) = \tau_0 z^2 + f_1(z), \tag{14}$$

$$\text{in } S_1 \quad z^2\Phi_1(z) - \Phi_2(z) = \tau_0 z^2 + f_2(z), \tag{15}$$

$$\text{in } S_2 \quad G_1z^2\Phi_2(z) + G_2\Phi_1(z) = G_2\tau_0 z^2 + f_3(z), \tag{16}$$

$$\text{in } S_1 \quad G_2z^2\Phi_1(z) + G_1\Phi_2(z) = G_2\tau_0 z^2 + f_4(z), \tag{17}$$

where $f_1(z), f_3(z)$ and $f_2(z), f_4(z)$ stand for holomorphic functions in S_2 and S_1, respectively. To satisfy (10), $f_1(z)$ and $f_2(z)$ should equal to each other on the interface $r = 1$. As a result, both of them must be a constant [7]:

$$f_1(z) = f_2(z) = \beta = \text{const}. \tag{18}$$

A similar analysis with respect to $f_3(z), f_4(z)$ and (11) leads to

$$f_3(z) = f_4(z) = \gamma = \text{const}. \tag{19}$$

Substituting (18, 19) into (14–17) and solving for the latter, we obtain:

$$\text{in } S_2 \quad \Phi_2(z) = \frac{2G_2\tau_0 z^2 + G_2\beta + \gamma}{(G_1 + G_2)\, z^2}, \quad \Phi_1(z) = \frac{(G_2 - G_1)\,\tau_0 z^2 - G_1\beta + \gamma}{G_1 + G_2}, \tag{20a, b}$$

$$\text{in } S_1 \quad \Phi_1(z) = \frac{(G_1 + G_2)\,\tau_0 z^2 + G_1\beta + \gamma}{(G_1 + G_2)\, z^2}, \quad \Phi_2(z) = \frac{-G_2\beta + \gamma}{G_1 + G_2}. \tag{21a, b}$$

Constants β and γ in (20a, b, 21a, b) can be determined from the following conditions: the function $\Phi_2(z)$ should be holomorphic in S_2, thus,

$$G_2\beta + \gamma = 0 \tag{22}$$

and due to (20b), (21a) and (7),

$$\beta = -\Phi_1(0) = -\tau_0 \tag{23}$$

Substituting (22) and (23) into (20a) and (21a), the final solution turns out to be

$$\text{in } S_2 \quad \Phi_2(z) = \frac{2G_2\tau_0}{G_1 + G_2}, \tag{24}$$

$$\text{in } S_1 \quad \Phi_1(z) = \tau_0 + \frac{G_2 - G_1}{G_1 + G_2} \cdot \frac{\tau_0}{z^2}. \tag{25}$$

Furthermore, using (4a, b), the displacements in the matrix and the inclusion can be determined from (24, 25)

$$\text{in } S_2 \quad w_2 = \frac{2\tau_0}{G_1 + G_2}\, r\cos\theta + C_0, \tag{26}$$

$$\text{in } S_1 \quad w_1 = \frac{\tau_0}{G_1}\left[r\cos\theta - \frac{G_2 - G_1}{G_2 + G_1} \cdot \frac{\cos\theta}{r}\right] + C_0 \tag{27}$$

with C_0 being a constant signifying the rigid body displacement of the matrix-inclusion system.

3.2 *Uniform shear in the y direction at infinity*

This particular case can be treated in a way similar to that described above. The final result is

$$\text{in } S_2 \quad \Phi_2(z) = -\frac{2G_2\tau_1 \mathrm{i}}{G_1 + G_2}, \tag{28}$$

$$\text{in } S_1 \quad \Phi_1(z) = -\mathrm{i}\tau_1\left[1 - \frac{G_2 - G_1}{G_1 + G_2} \cdot \frac{1}{z^2}\right], \tag{29}$$

where τ_1 denotes the uniform shear stress at infinity. And,

$$\text{in } S_2 \quad w_2 = \frac{2\tau_1}{G_1 + G_2}\, r\sin\theta + C_2, \tag{30}$$

$$\text{in } S_1 \quad w_1 = \frac{\tau_1}{G_1}\left[r\sin\theta - \frac{G_2 - G_1}{G_1 + G_2} \cdot \frac{\sin\theta}{r}\right] + C_2. \tag{31}$$

3.3 Singular solutions

First, we introduce the following basic singular solution in an antiplane shear problem for a homogeneous medium:

$$\Phi_0(z) = \frac{1}{z}. \tag{32}$$

The stress and displacement associated with this singular solution are

$$\tau_{rt_0} = \frac{1}{r}, \quad \tau_{\theta t_0} = 0 \tag{33a, b}$$

and,

$$w_0 = \frac{1}{G}\log r + C_0. \tag{34}$$

The solution indicates that there is a source of radial antiplane shear stress of unit intensity at point $z = 0$. This is another basic form of external loading under which the matrix-inclusion system can statically be deformed in antiplane shear.

Now consider the case, when a source of intensity A (A: a real number) is located at point $z = a$ in the matrix. Obviously, in this case $\Phi_2(z)$ is holomorphic in both S_2 and S_1, whereas $\Phi_1(z)$ has a simple pole at $z = a$ so that

$$\text{in } S_1 \quad \Phi_1(z) = \frac{A}{z-a} + \text{a holomorphic function}, \tag{35}$$

$$\text{in } S_2 \quad \Phi_1(z) = \frac{Az}{1-\bar{a}z} + \text{a holomorphic function}. \tag{36}$$

On ground of the above considerations, it is put forward that

$$\text{in } S_2 \quad z^2\Phi_2(z) - \Phi_1(z) = a_0z^2 + a_1z + a_2 + \frac{Az^2}{z-a} + \frac{Az}{\bar{a}z-1} + \alpha_1(z), \tag{37}$$

$$\text{in } S_1 \quad z^2\Phi_1(z) - \Phi_2(z) = a_0z^2 + a_1z + a_2 + \frac{Az^2}{z-a} + \frac{Az}{\bar{a}z-1} + \alpha_2(z). \tag{38}$$

Similarly,

$$\text{in } S_2 \quad G_1z^2\Phi_2(z) + G_2\Phi_1(z) = G_2\left(a_0z^2 + a_1z + \frac{Az^2}{z-a} + \frac{mAz}{\bar{a}z-1}\right) + a_3 + \alpha_3(z), \tag{39}$$

$$\text{in } S_1 \quad G_2z^2\Phi_1(z) + G_1\Phi_2(z) = G_2\left(a_0z^2 + a_1z + \frac{Az^2}{z-a} + \frac{mAz}{\bar{a}z-1}\right) + a_3 + \alpha_4(z), \tag{40}$$

where $\alpha_i(z)$, $i = 1, 2, 3, 4$ are holomorphic functions in S_2 or S_1 and m is a constant to be fixed. Following an argument as that applied to $f_i(z)$, $i = 1, 2, 3, 4$ in (14–17), we have

$$\alpha_1(z) = \alpha_2(z) = \delta = a \text{ const.}, \quad \alpha_3(z) = \alpha_4(z) = \eta = a \text{ const.} \tag{41a, b}$$

Since these constants can be regarded as having been included in the constants a_2 and a_3 in (37–38) and (39–40), respectively, they can be taken as zero. Solving for (37–40) thus simplified, we obtain

$$\text{in } S_2 \quad \Phi_2(z) = \frac{1}{G_1+G_2}\cdot\frac{1}{z^2}\left\{2G_2\left(a_0z^2 + a_1z + \frac{Az^2}{z-a}\right) + G_2a_2 + a_3 + \frac{(G_2+m)Az}{\bar{a}z-1}\right\}. \tag{42}$$

Since $\Phi_2(z)$ is holomorphic at $z = 0$, the following equations must be true:

$$a_1 = 0, \quad a_3 + G_2a_2 = 0, \quad G_2 + m = 0. \tag{43a, b, c}$$

Substituting the above equations into (42) results:

$$\text{in } S_2 \quad \Phi_2(z) = \frac{2G_2}{G_1 + G_2} \cdot \frac{1}{z^2} \left\{ a_0 z^2 + \frac{Az^2}{z - a} \right\}. \tag{44}$$

On the other hand, for the matrix we have

$$\text{in } S_1 \quad \Phi_1(z) = \frac{1}{(G_1 + G_2) z^2} \left\{ (G_1 + G_2) \left(a_0 z^2 + \frac{Az^2}{z - a} \right) + a_2 G_1 + a_3 + \frac{(G_1 - G_2) Az}{\bar{a}z - 1} \right\}. \tag{45}$$

Using the boundary condition at finity

$$\Phi_1(z) \to 0 \quad \text{for} \quad z \to \infty \tag{46}$$

and noticing that (37) implies

$$a_2 = -\Phi_1(0), \tag{47}$$

all the constants in (44, 45) can be fixed as

$$a_0 = a_1 = a_2 = a_3 = 0, \qquad m = -G_2. \tag{48 a, b}$$

Combining the above results, the final expressions for $\Phi_1(z)$ and $\Phi_2(z)$ are as follows:

$$\text{in } S_2 \quad \Phi_2(z) = \frac{2AG_2}{G_1 + G_2} \cdot \frac{1}{z - a}, \tag{49}$$

$$\text{in } S_1 \quad \Phi_1(z) = \frac{A}{z - a} + \frac{G_1 - G_2}{G_1 + G_2} \cdot \frac{A}{z(\bar{a}z - 1)}, \tag{50}$$

$$\text{in } S_2 \quad w_2 = \frac{A}{G_1 + G_2} \log\left[R^2 + r^2 - 2Rr\cos(\phi - \theta)\right] + C, \tag{51}$$

$$\text{in } S_1 \quad w_1 = \frac{A}{G_1 + G_2} \log\left[R^2 + r^2 - 2Rr\cos(\phi - \theta)\right] - \frac{A(G_1 - G_2)}{G_1(G_1 + G_2)} \ln r + C, \tag{52}$$

where R and ϕ stand for the modulus and the argument of the complex number a ($a = Re^{i\phi}$), respectively.

In case the singularity lies within the inclusion, that is, when $|a| < 1$, the problem can be treated similarly. The result is:

$$\text{in } S_2 \quad \Phi_2(z) = \frac{A}{z - a} + \frac{A\bar{a}(G_2 - G_1)}{G_1 + G_2} \cdot \frac{1}{\bar{a}z - 1}, \tag{53}$$

$$\text{in } S_1 \quad \Phi_1(z) = \frac{A(G_2 - G_1)}{G_1 + G_2} \cdot \frac{1}{z} + \frac{2G_1 A}{G_1 + G_2} \cdot \frac{1}{z - a}, \tag{54}$$

$$\text{in } S_2 \quad w_2 = \frac{A}{G_2(G_1 + G_2)} \left[(G_2 - G_1)\log r + G_2 \log\left\{R^2 + r^2 - 2Rr\cos(\phi - \theta)\right\}\right] + C, \tag{55}$$

$$\text{in } S_1 \quad w_1 = \frac{A}{G_1(G_1 + G_2)} \left[(G_2 - G_1)\log r + G_1 \log\left\{R^2 + r^2 - 2Rr\cos(\phi - \theta)\right\}\right] + C. \tag{56}$$

Another interesting case occurs when the singularity is located precisely on the interface, *i.e.*, $a = a_0$, $|a_0| = 1$. Solution for this case can be obtained by letting $a \to a_0$ either in (49–52) or in (53–56). Both approaches give the same result. It is,

$$\text{in } S_2 \quad \Phi_2(z) = \frac{2AG_2}{G_1 + G_2} \cdot \frac{1}{z - a_0}, \tag{57}$$

$$\text{in } S_1 \quad \Phi_1(z) = \frac{A}{z - a_0} + \frac{G_1 - G_2}{G_1 + G_2} \cdot \frac{A}{z(\bar{a}_0 z - 1)}, \tag{58}$$

and

$$\text{in } S_2 \quad w_2 = \frac{A}{G_1 + G_2} \log\left[1 + r^2 - 2r\cos(\phi - \theta)\right] + C, \tag{59}$$

$$\text{in } S_1 \quad w_1 = \frac{A}{G_1}\left\{\frac{G_2 - G_1}{G_2 + G_1}\log r + \frac{G_1}{G_1 + G_2}\log\left[1 + r^2 - 2r\cos(\phi - \theta)\right]\right\} + C. \tag{60}$$

4 Solutions for a partially debonded inclusion

Suppose there is a partial debonding on the interface between the matrix and the inclusion in the form of a circumferential crack on $r = 1$ for $|\theta| \leqq \theta_0$, Fig. 2, the following antiplane shear problems for this partially damaged system will be considered.

4.1 *Uniform shear in the x direction at infinity*

In this case, the boundary conditions at infinity (12 a, b) still applies. The continuity conditions at the interface $r = 1$, *i.e.*, (6 a, b) remain valid for $|\theta| \geqq \theta_0$. For $|\theta| < \theta_0$ they should be replaced by traction free condition on the crack surfaces,

$$|\theta| < \theta_0, \quad \tau_{rt} = 0 \quad \text{for} \quad r = 1. \tag{61}$$

To solve the problem, we start from the observation that as can be seen from (61), continuity of stress at the interface is still valid over the whole interface $r = 1$. Therefore, the function $\tau_{rt_2} - \tau_{rt_1}$ is continuous in both S_2 and S_1, and vanish at $r = 1$. As a result, $z^2\Phi_2(z) - \Phi_1(z)$ and $z^2\Phi_1(z) - \Phi_2(z)$ are holomorphic respectively in S_2 and S_1, taking a same value at $r = 1$. In view of this, both of them should be equal to a polynomial in the z plane. Since $\Phi_1(z)$ and $\Phi_2(z)$ are regular at infinity, the order of the polynomial must not exceed 2. Combining the above points, we obtain,

$$\text{in } S_2 \quad z^2\Phi_2(z) - \Phi_1(z) = a_0 + a_1 z + a_2 z^2, \tag{62}$$

$$\text{in } S_1 \quad z^2\Phi_1(z) - \Phi_2(z) = a_0 + a_1 z + a_2 z^2. \tag{63}$$

By a similar consideration and noticing that w_1 and w_2 are discontinuous across the crack surfaces, we write

$$\text{in } S_2 \quad G_1 z^2\Phi_2(z) + G_2\Phi_1(z) = b_0 + b_1 z + b_2 z^2 + f(z), \tag{64}$$

$$\text{in } S_1 \quad G_2 z^2\Phi_1(z) + G_1\Phi_2(z) = b_0 + b_1 z + b_2 z^2 + f(z), \tag{65}$$

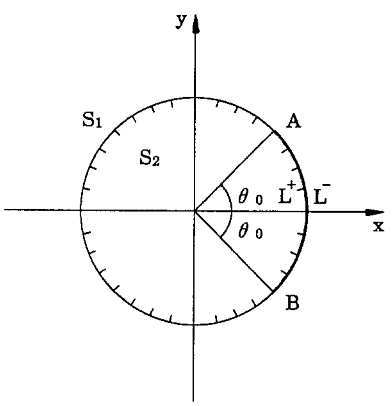


Fig. 2. The partially damaged matrix-inclusion system

where

$$f(z) = \frac{c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n}{X(z)} \tag{66}$$

and $X(z)$ stands for a single-valued branch of function $\sqrt{(z - e^{i\theta_0})(z - e^{-i\theta_0})}$ under the condition $\lim_{z\to\infty} X(z)/z \to 1$.

Note that $f(z)$ takes values of opposite sign on the crack surfaces such that

$$f^+(t) = -f^-(t) \qquad t = e^{i\theta} \tag{67}$$

and therefore is used to take account of the jump of displacement across the crack surfaces.

Substituting (66) into (64, 65) and subsequently solving (62–65), the following expressions for $\Phi_1(z)$ and $\Phi_2(z)$ are obtained:

$$\text{in } S_2 \quad \Phi_2(z) = \frac{1}{(G_1 + G_2)\, z^2} \times \left\{ b_0 + a_0 G_2 + (b_1 + a_1 G_2)\, z + (b_2 + a_2 G_2)\, z^2 + \frac{c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n}{X(z)} \right\}, \tag{68}$$

$$\text{in } S_2 \quad \Phi_1(z) = \frac{1}{G_1 + G_2} \times \left\{ b_0 - a_0 G_1 + (b_1 - a_1 G_1)\, z + (b_2 - a_2 G_1)\, z^2 + \frac{c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n}{X(z)} \right\}, \tag{69}$$

$$\text{in } S_1 \quad \Phi_1(z) = \frac{1}{(G_1 + G_2)\, z^2} \times \left\{ b_0 + a_0 G_1 + (b_1 + a_1 G_1)\, z + (b_2 + a_2 G_1)\, z^2 + \frac{c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n}{X(z)} \right\}, \tag{70}$$

$$\text{in } S_1 \quad \Phi_2(z) = \frac{1}{G_1 + G_2} \times \left\{ b_0 - a_0 G_2 + (b_1 - a_1 G_2)\, z + (b_2 - a_2 G_2)\, z^2 + \frac{c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n}{X(z)} \right\}. \tag{71}$$

Applying (8), $\Phi_2(z)$ in S_1 can also be obtained in the following form:

$$\text{in } S_1 \quad \Phi_2(z) = \frac{1}{G_1 + G_2} \times \left\{ (\overline{b_0} + \overline{a_0} G_2)\, z^2 + (\overline{b_1} + \overline{a_1} G_2)\, z + (\overline{b_2} + \overline{a_2} G_2) - \frac{\overline{c_0} z^3 + \overline{c_1} z^2 + \overline{c_2} z + \overline{c_3} + \overline{c_4}/z + \cdots + \overline{c_n}/z^{n-3}}{X(z)} \right\}. \tag{72}$$

In deriving the above formula, the following relation has been used:

$$\bar{X}(1/z) = -zX(z). \tag{73}$$

By comparing similar terms in (71) and (72), we obtain,

$$c_3 = -\overline{c_0}, \qquad c_2 = -\overline{c_1}, \qquad c_4 = c_5 = \cdots = c_n = 0, \tag{74a, b, c}$$

$$b_0 - a_0 G_2 = \overline{b_2} + \overline{a_2} G_2, \qquad b_1 - a_1 G_2 = \overline{b_1} + \overline{a_1} G_2, \tag{75a, b}$$

$$b_2 - a_2 G_2 = \overline{b_0} + \overline{a_0} G_2. \tag{76}$$

Similarly, by applying (7) to $\Phi_1(z)$ in S_2 and following a procedure similar to that described above, we obtain,

$$b_0 - a_0 G_1 = \overline{b_2} + \overline{a_2} G_1, \tag{77}$$

$$b_1 - a_1 G_1 = \overline{b_1} + \overline{a_1} G_1, \qquad b_2 - a_2 G_1 = \overline{b_0} + \overline{a_0} G_1. \tag{78a, b}$$

From (75a, b), (76), (77) and (78a, b), the following additional result is obtained:

$$a_0 = -\overline{a_2}, \qquad a_1 = \mathrm{i} a_R, \tag{79a, b}$$

$$b_0 = \overline{b_2}, \qquad b_1 = b_R, \tag{80a, b}$$

where a_R and b_R are real numbers. The remaining constants contained in the expressions for $\Phi_1(z)$ and $\Phi_2(z)$ should be determined by the following conditions.

Since for $z \to \infty$, $\Phi_1(z) \to \tau_0$, therefore,

$$b_2 + a_2 G_1 + c_3 = \tau_0 (G_1 + G_2). \tag{81}$$

Since for $z \to 0$, $\Phi_2(z)$ should be holomorphic, so that

$$b_2 - a_2 G_2 + c_3 = 0, \tag{82}$$

$$b_1 + \mathrm{i} a_R G_2 + \overline{c_2} + \overline{c_3} \cos \theta_0 = 0. \tag{83}$$

Since on the crack surfaces, any traction should be absent, we have,

$$\text{on } L^+ \quad t^2 \Phi_2^{\,+}(t) + \Phi_2^{\,-}(t) = 0. \tag{84}$$

The above equation, (84), turns out to be equivalent to the following conditions:

$$c_3 = G_2 \tau_0, \qquad c_2 = -G_2 \tau_0 \cos \theta_0 + \mathrm{i} c_{2R}, \tag{85a, b}$$

where c_{2R} is a real number.

The term $a_1 = \mathrm{i} a_R$ in the expressions for $\Phi_1(z)$ and $\Phi_2(z)$ corresponds to the following stress field:

$$\tau_{rt} = 0, \qquad \tau_{\theta t} = a_R / r. \tag{86a, b}$$

A similar explanation applies to the term of Im $[c_2] = c_{2R}$ (see (85b)). Since a stress field of the type of (86a, b) lacks physical reality and is inconsistent with the nature of this problem, we should take

$$a_R = c_{2R} = 0. \tag{87}$$

Using (74a, b, c)–(76), (77)–(83), (85a, b) and (87), all unknown constants contained in the expressions for $\Phi_1(z)$ and $\Phi_2(z)$ can uniquely be fixed. Substitution of these fixed values into (68) and (70) yields

$$\text{in } S_2 \quad \Phi_2(z) = \frac{G_2 \tau_0}{G_1 + G_2} \left\{ 1 - \frac{1}{z^2} + \frac{1}{X(z)} \left[z - \frac{1}{z^2} - \cos \theta_0 \left(1 - \frac{1}{z} \right) \right] \right\}, \tag{88}$$

$$\text{in } S_1 \quad \Phi_1(z) = \frac{\tau_0}{G_1 + G_2} \left\{ G_1 \left(1 - \frac{1}{z^2} \right) + \frac{G_2}{X(z)} \left[z - \frac{1}{z^2} - \cos \theta_0 \left(1 - \frac{1}{z} \right) \right] \right\}. \tag{89}$$

A parameter of engineering importance in this problem is the mode *III* stress intensity factors at crack tips, A and B. The factor at tip A can be evaluated from the following formula:

$$K_A = \lim_{\substack{r \to 1 \\ \theta \to \theta_0^+}} \sqrt{2\pi r (\theta - \theta_0)}\, \tau_{rt}. \tag{90}$$

Noticing (5), (88) and the following result:

$$\lim_{\substack{r \to 1 \\ \theta \to \theta_0^+}} \frac{\sqrt{2\pi(\theta - \theta_0) r}}{X(z)} = -\frac{i \sqrt{\pi}\, e^{-i\theta_0/2}}{\sqrt{\sin \theta_0}} \tag{91}$$

the stress intensity factor at tip A can be worked out as

$$K_A = \frac{2G_2\tau_0}{G_1 + G_2} \cdot \frac{\sqrt{\pi}}{\sqrt{\sin\theta_0}} \left(\sin\frac{3\theta_0}{2} - \cos\theta_0 \sin\frac{\theta_0}{2}\right). \tag{92}$$

The stress intensity factor at tip B can be found similarly and it turns out that

$$K_B = K_A. \tag{93}$$

4.2 *Uniform shear in the y direction at infinity*

In this case, the external loading is effected by a uniform antiplane shear stress $\tau_{yt} = \tau_1$ at infinity. The basic functions in the problem, $\Phi_1(z)$ and $\Phi_2(z)$, can be determined via a way similar to that for the uniform shear in the x direction. The final result is as follows:

$$\text{in } S_2 \quad \Phi_2(z) = -\frac{iG_2\tau_1}{G_1 + G_2}\left\{1 + \frac{1}{z^2} + \frac{1}{X(z)}\left[-\left(z + \frac{1}{z^2}\right) + \cos\theta_0\left(1 + \frac{1}{z}\right)\right]\right\}, \tag{94}$$

$$\text{in } S_1 \quad \Phi_1(z) = -\frac{i\tau_1}{G_1 + G_2}\left\{G_1\left(1 + \frac{1}{z^2}\right) + \frac{G_2}{X(z)}\left[z + \frac{1}{z^2} - \cos\theta_0\left(1 + \frac{1}{z}\right)\right]\right\}. \tag{95}$$

The stress intensity factors at crack tips A and B are,

$$K_A = -K_B = \frac{2G_2\tau_1}{G_1 + G_2}\sqrt{\frac{\pi}{\sin\theta_0}}\left(\cos\frac{3\theta_0}{2} - \cos\theta_0\cos\frac{\theta_0}{2}\right). \tag{96}$$

4.3 *Shear tractions on the crack surfaces*

In the preceding two subsections, we solve antiplane shear problems of the matrix-inclusion system with a partially debonded interface. The solution approach used is the semi-inversed method. With some physical reasoning, the basic form of the solution for a particular problem is firstly proposed, and the unknown coefficients contained therein are then fixed with various conditions the solution should satisfy. In what follows, this solution method will also be applied.

Let us consider the general case in which all antiplane shear stresses vanish at infinity, whereas on the crack surfaces the following external antiplane shear traction is applied:

$$\tau_{rt} = \cos n\theta \qquad n = 0, 1, 2, \ldots \tag{97}$$

For this general case we first propose

$$\text{in } S_2 \quad z^2\Phi_2(z) - \Phi_1(z) = 0, \tag{98}$$

$$\text{in } S_2 \quad G_1 z^2\Phi_2(z) + G_2\Phi_1(z) = \frac{b_{1-n}}{z^{n-1}} + b_{n+1}z^{n+1} + \frac{1}{X(z)} \times\left[\frac{c_{1-n}}{z^{n-1}} + \frac{c_{2-n}}{z^{n-2}} + \cdots + \frac{c_{-1}}{z} + c_0 + c_1 z + \cdots + c_{n+1}z^{n+1} + c_{n+2}z^{n+2}\right], \tag{99}$$

$$\text{in } S_1 \quad z^2\Phi_1(z) - \Phi_2(z) = 0, \tag{100}$$

$$\text{in } S_1 \quad G_2 z^2\Phi_1(z) + G_1\Phi_2(z) = \frac{b_{1-n}}{z^{n-1}} + b_{n+1}z^{n+1} + \frac{1}{X(z)} \times\left[\frac{c_{1-n}}{z^{n-1}} + \frac{c_{2-n}}{z^{n-2}} + \cdots + \frac{c_{-1}}{z} + c_0 + c_1 z + \cdots + c_{n+1}z^{n+1} + c_{n+2}z^{n+2}\right]. \tag{101}$$

By solving for (98 – 101), it is obtained that

$$\text{in } S_2 \quad \Phi_2(z) = \frac{1}{G_1 + G_2} \cdot \frac{1}{z^2} \left\{ \frac{b_{1-n}}{z^{n-1}} + b_{n+1} z^{n+1} + \frac{1}{X(z)} \times \left[\frac{c_{1-n}}{z^{n-1}} + \frac{c_{2-n}}{z^{n-2}} + \cdots + \frac{c_{-1}}{z} + c_0 + c_1 z + \cdots + c_{n+1} z^{n+1} + c_{n+2} z^{n+2} \right] \right\}, \quad (102)$$

$$\text{in } S_2 \quad \Phi_1(z) = z^2 \Phi_2(z), \quad (103)$$

$$\text{in } S_1 \quad \Phi_1(z) = \frac{1}{G_1 + G_2} \cdot \frac{1}{z^2} \left\{ \frac{b_{1-n}}{z^{n-1}} + b_{n+1} z^{n+1} + \frac{1}{X(z)} \times \left[\frac{c_{1-n}}{z^{n-1}} + \frac{c_{2-n}}{z^{n-2}} + \cdots + \frac{c_{-1}}{z} + c_0 + c_1 z + \cdots + c_{n+1} z^{n+1} + c_{n+2} z^{n+2} \right] \right\}, \quad (104)$$

$$\text{in } S_1 \quad \Phi_2(z) = z^2 \Phi_1(z). \quad (105)$$

To determine the unknown constants in the above expressions, we write the expression for $\Phi_2(z)$ in S_1 directly from (102) by employing (8). This gives,

$$\text{in } S_1 \quad \Phi_2(z) = \frac{z^2}{G_1 + G_2} \left\{ \overline{b_{1-n}} z^{n-1} + \frac{\overline{b_{n+1}}}{z^{n+1}} - \frac{z}{X(z)} \times \left[\overline{c_{1-n}} z^{n-1} + \overline{c_{2-n}} z^{n-2} + \cdots + \overline{c_{-1}} z + \overline{c_0} + \frac{\overline{c_1}}{z} + \cdots + \frac{\overline{c_{n+1}}}{z^{n+1}} + \frac{\overline{c_{n+2}}}{z^{n+2}} \right] \right\}. \quad (106)$$

Since (105) and (106) should be identical, comparing the coefficients for similar terms in these two equations results:

$$b_{n+1} = \overline{b_{1-n}} \quad (107)$$

$$\overline{c_{1-n}} = -c_{n+2}, \quad \overline{c_{2-n}} = -c_{n+1}, \quad \overline{c_{3-n}} = -c_n, \quad \overline{c_{-1}} = -c_4, \quad \overline{c_0} = -c_3, \quad \overline{c_1} = -c_2 \quad (108\text{a, b, c, d, e, f})$$

Similarly, an alternative expression for $\Phi_1(z)$ in S_2 can be written directly from (104) by employing (7). But this does not give new result.

To utilizing the boundary condition at infinity, that is,

$$\Phi_1(z) \to 0 \quad \text{for} \quad z \to \infty \quad (109)$$

we first notice that the following expansion hold for $1/X(z)$ in $|z| > 1$:

$$\frac{1}{X(z)} = \frac{1}{z} \sum_{n=1}^{\infty} D_n \left(\frac{1}{z} \right)^{n-1}, \quad (110)$$

where

$$D_n = \sum_{K=1}^{n} u_K u_{n+1-K} \cos(n + 1 - 2K) \theta_0, \quad (111)$$

$$u_1 = 1, \quad u_n = \frac{2n-3}{2n-2} u_{n-1}, \quad n = 2, 3, 4, \ldots. \quad (112\text{a, b})$$

Using the expansion (110), it is easy to show that the condition (109) will be met provided the following equations hold:

$$b_{n+1} + c_{n+2} D_1 = 0, \quad (113)$$

$$c_{n+2} D_2 + c_{n+1} D_1 = 0, \quad (114)$$

$$c_{n+2}D_3 + c_{n+1}D_2 + c_nD_1 = 0, \tag{115}$$

$\dots \dots$

$$c_{n+2}D_{n-1} + c_{n+1}D_{n-2} + c_nD_{n-3} + \cdots + c_5D_2 + c_4D_1 = 0, \tag{116}$$

$$c_{n+2}D_n + c_{n+1}D_{n-1} + c_nD_{n-2} + \cdots + c_4D_2 + c_3D_1 = 0. \tag{117}$$

On the other hand, by using the following expansion for $1/X(z)$ holding in $|z| < 1$,

$$\frac{1}{X(z)} = -\sum_{n=1}^{\infty} D_n z^{n-1} \tag{118}$$

the regular condition of $\Phi_2(z)$ at $z = 0$, that is, $\Phi_2(z)$ should be holomorphic at the origin, can be treated readily. This condition, in addition to once more giving (113–117), leads to the following equation the unknown coefficients should satisfy:

$$c_{1-n}D_{n+1} + c_{2-n}D_n + c_{3-n}D_{n-1} + \cdots + c_{-1}D_3 + c_0D_2 + c_1D_1 = 0. \tag{119}$$

There are $b_{1-n}, b_{n+1}; c_{1-n}, c_{2-n}, \dots, c_{-1}, c_0, c_1, \dots, c_{n+1}, c_{n+2}$: in total $2n + 4$ unknown coefficients in (102–105), and we have had (107), (108 a, b, c, d, e, f), (113–117) and (119) in total $2n + 3$ equations. Another equation needed is afforded by the boundary condition on the crack surfaces, (97). Noticing $X^+(t) = -X^-(t)$, this condition finally reduces to

$$\tau_{rt} = \frac{2}{G_1 + G_2}\left\{\frac{b_{1-n}}{t^n} + \overline{b_{1-n}}t^n\right\} = \cos n\theta. \tag{120}$$

This equation is satisfied by taking b_{1-n} a real number determined by

$$b_{1-n} = \frac{G_1 + G_2}{4}. \tag{121}$$

With b_{1-n} thus fixed, $b_{n+1}, c_{n+2}, c_{n+1}, \dots, c_2, c_1, c_0, c_{-1}, c_{-2}, \dots, c_{1-n}$ can successively be determined from (107), (113–117), (119) and (108 a, b, c, d, e, f). Substituting these values into (102) and (104), we obtain the explicit expressions for $\Phi_2(z)$ and $\Phi_1(z)$, and by this way completely solve the problem.

The stress intensity factors at the crack tips are

$$K_A = K_B = \sqrt{\frac{\pi}{\sin\theta_0}}\left[-2c_{1-n}\sin\left(n + \frac{1}{2}\right)\theta_0 - 2c_{2-n}\sin\left(n - \frac{1}{2}\right)\theta_0 - \cdots - 2c_1\sin\frac{\theta_0}{2}\right]. \tag{122}$$

The solution for any particular case can easily be obtained from the general solution. For instance, for $n = 0$ in (97) we have

$$\text{in } S_2 \quad \Phi_2(z) = \frac{1}{2z^2}\left[z + \frac{z - z^2}{X(z)}\right], \tag{123}$$

$$\text{in } S_1 \quad \Phi_1(z) = \frac{1}{2z^2}\left[z + \frac{z - z^2}{X(z)}\right], \tag{124}$$

$$K_A = K_B = -\sqrt{\frac{\pi}{\sin\theta_0}}\sin\frac{\theta_0}{2}. \tag{125}$$

It is easy to see that by taking b_{1-n} an imaginary number determined by

$$b_{1-n} = \mathrm{i}\,\frac{G_1 + G_2}{4}. \tag{126}$$

and fix other unknown coefficients in (102) and (104) in accordance with this value of b_{1-n}, we obtain an exact and explicit solution for the case when on the crack surfaces the following external antiplane shear traction is exclusively applied:

$$\tau_{rt} = \sin n\theta . \tag{127}$$

In this case the stress intensity factors are

$$K_A = -K_B = \sqrt{\frac{\pi}{\sin \theta_0}} \left[2c_{1-n}^0 \cos\left(n + \frac{1}{2}\right)\theta_0 + 2c_{2-n}^0 \cos\left(n - \frac{1}{2}\right)\theta_0 + \cdots + 2c_1{}^0 \cos\frac{\theta_0}{2} \right], \tag{128}$$

where $c_K{}^0 = c_K/\mathrm{i}$, $K = 1 - n, 2 - n, \ldots, 0, 1$.

Finally, since any form of external antiplane traction on the crack surfaces, $f(\theta)$, can be expressed by a Fourier series consisted of cosine and sine terms, the general result can be utilized to solve the problem when on the crack surfaces there is an arbitrary distribution of τ_{rt}.

4.4 *Singular loading*

By using the result developed in the preceding subsection, the problem in which the partially damaged matrix-inclusion system is effected by a singularity of the type of (32) can be treated simply by the method of superposition.

Suppose there is a singularity, *i.e.*, a source of intensity A located at point $z = a$ on the x axis in the matrix. The solution to this problem for a perfect matrix-inclusion system has been given by (49, 50), and stress τ_{rt} at the interface is expressed by

$$\tau_{rt} = -\frac{2AG_2}{G_1 + G_2}\left(\frac{\cos\theta}{a} + \frac{\cos 2\theta}{a^2} + \cdots + \frac{\cos n\theta}{a^n} + \cdots\right) \tag{129}$$

This distribution of τ_{rt} should be absent on the crack surfaces of the partially damaged matrix-inclusion system. Therefore, to eliminate τ_{rt} on the crack surfaces we should apply a distribution of τ_{rt} on them, which is equal in magnitude but opposite in sign to that given by (129). This is a problem when the partially damaged matrix-inclusion system is exclusively loaded on the crack surfaces, and the solution is given by (102) and (104). The final solution to this singular loading problem is obtained by adding a solution of type (49, 50) to another one of type (102, 104). Other cases of singular loading can be dealt with similarly.

5 Numerical results and discussion

The antiplane shear problem of matrix-inclusion systems has been solved in the preceding sections rigorously. Using various exact expressions developed therein, antiplane shear stresses and stress intensity factors for the systems can easily be obtained. In this section, numerical values of stress intensity factors at the crack tips of a partially damaged matrix-inclusion system (Fig. 2) are given.

First we deal with the case when there is a singularity of intensity A_0 located at the center of the inclusion. In this case, the following stress τ_{rt} would develop for a similar perfect matrix-inclusion system ((53, 54) and (5)):

$$\tau_{rt} = A_0 = \text{const.} \qquad \text{for} \qquad r = 1, \tag{130}$$

which is independent of material constants. The stress intensity factors at the crack tips of the partially damaged matrix-inclusion system should be sought under the condition that on the crack surfaces the following antiplane shear stress is applied:

$$\tau_{rt} = -A_0 \qquad \text{for} \qquad r = 1. \tag{131}$$

Numerical values of the stress intensity factors have been obtained from (125) and is presented in Fig. 3 for various geometric and material parameters.

Nextly we consider the case in which there is a singularity of intensity A_0 located precisely on the interface $r = 1$. Using (57, 58) and (5) it is found that the following stress τ_{rt} would develop on the interface $r = 1$ for a similar perfect matrix-inclusion system:

$$\tau_{rt} = \frac{G_2 A_0}{G_1 + G_2}. \tag{132}$$

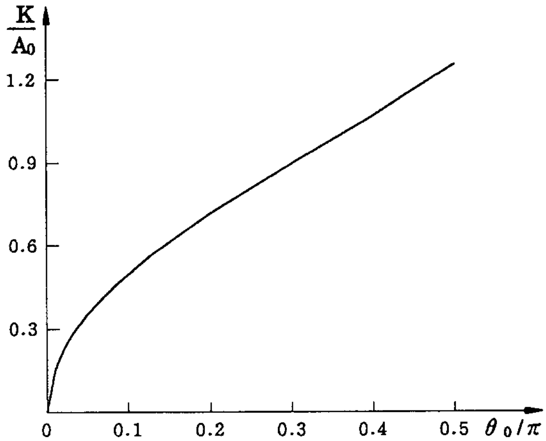
Equation (132) shows a wonderful fact, that the stress τ_{rt} developed on $r = 1$ is always a constant, irrelevant to the location of the singularity as long as it lies on the interface $r = 1$. Consequently, the stress intensity factors in the partially damaged matrix-inclusion system are independent of the location of the singularity also. These stress intensity factors have been worked out from (125) and is presented in Fig. 4.

In case the singularity is located in the matrix at a point $x = a$ on the real axis, the stress τ_{rt} developed in a similar perfect matrix-inclusion system would be

$$\tau_{rt} = -\frac{2A_0 G_2}{G_1 + G_2} \sum_{n=1}^{\infty} \frac{\cos n\theta}{a^n} \quad \text{for} \quad r = 1 \tag{133}$$

And, when the singularity lies in the matrix at a point $y = a$ on the imaginary axis, the stress τ_{rt} described above would take the following form:

$$\tau_{rt} = \frac{2A_0 G_2}{G_1 + G_2} \left[\sum_{n=1,3,5}^{\infty} (\mathrm{i})^{n+1} \frac{\sin n\theta}{a^n} + \sum_{n=2,4,6}^{\infty} (-1)\, \mathrm{i}^n \frac{\cos n\theta}{a^n} \right] \quad \text{for} \quad r = 1 \tag{134}$$



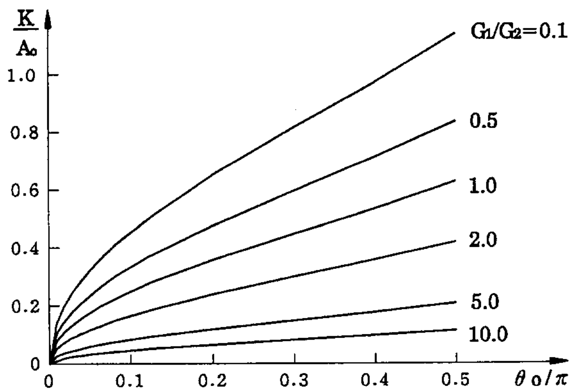


Fig. 3 and **4. 3** Numerical values of stress intensity factors ($K_A = K_B$); **4.** Numerical values of stress intensity factors ($K_A = K_B$)

Table 1. Numerical values of stress intensity factors ($K = K_A/A_0 = K_B/A_0$, $a = 2.0$)

θ	$\pi/24$	$\pi/20$	$\pi/16$	$\pi/12$	$\pi/8$
G_1/G_2			K		
0.1	0.5541	0.6020	0.6629	0.7408	0.8283
0.5	0.4064	0.4415	0.4861	0.5433	0.6075
1.0	0.3048	0.3311	0.3646	0.4075	0.4559
2.0	0.2032	0.2207	0.2431	0.2716	0.3037
5.0	0.1016	0.1104	0.1215	0.1358	0.1519
10.0	0.0554	0.0602	0.0663	0.0741	0.0828

Table 2. Numerical values of stress intensity factors ($a = 2.0$)

θ	$\pi/24$	$\pi/20$	$\pi/16$	$\pi/12$	$\pi/8$
G_1/G_2			K_A/A_0		
0.1	−0.2677	−0.2722	−0.2832	−0.3085	−0.3725
0.5	−0.1963	−0.1996	−0.2077	−0.2262	−0.2731
1.0	−0.1472	−0.1497	−0.1558	−0.1690	−0.2049
2.0	−0.0981	−0.0998	−0.1038	−0.1131	−0.1366
5.0	−0.0491	−0.0499	−0.0519	−0.0565	−0.0683
10.0	−0.0267	−0.0272	−0.0283	−0.0308	−0.0372
θ	$\pi/24$	$\pi/20$	$\pi/16$	$\pi/12$	$\pi/8$
G_1/G_2			K_B/A_0		
0.1	0.0374	0.0371	0.0398	0.0502	0.0866
0.5	0.0274	0.0272	0.0292	0.0368	0.0635
1.0	0.0205	0.0204	0.0219	0.0276	0.0476
2.0	0.0137	0.0136	0.0146	0.0184	0.0318
5.0	0.0068	0.0068	0.0073	0.0092	0.0160
10.0	0.0037	0.0037	0.0040	0.0050	0.0087

Employing (122) and (128), numerical values of stress intensity factors in the above two cases have been obtained and are presented in Table 1 (for the singularity lying on the x axis) and Table 2, (for the singularity lying on the y axis) respectively.

It is seen from both the numerical data and the theoretical analysis, that the stress intensity factors are identical for a homogeneous medium and a matrix-inclusion system, provided external loads are applied on the crack surfaces. When external loads in the form of singularities are applied, stress intensity factors for the homogeneous medium and the matrix-inclusion system are different from each other only in a constant factor $G_2/(G_1 + G_2)$. On this ground, the special features of a matrix-inclusion system in antiplane shear are less remarkable than those in plane problems.

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