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One-dimensional transient wave propagation in fluid-saturated incompressible porous media

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Summary: In this investigation, the general formalism for the field equations governing the dynamic response of fluid-saturated porous media is analyzed and employed for the study of transient wave motion. The two constituents are assumed to be incompressible. A one-dimensional analytical solution is derived by means of Laplace transform technique which, as a result of the incompressibility constraint, exhibits only one independent dilatational wave propagating in the solid and the fluid phases, respectively. The fluid-saturated porous material is supplied with characteristics similar to those occuring in viscoelastic solids. This work can provide the further understanding of the characteristics of wave propagation in porous materials and may be taken for a quantitative comparision to various numerical solutions.

Eindimensionale transiente Wellenfortpflanzung in flüssigkeitsgefüllten inkompressiblen porösen Medien

Übersicht: In dieser Arbeit wird der allgemeine Formalismus für die Feldgleichungen, die das dynamische Verhalten der fluidsaturierten Medien bestimmen, analysiert und für die Untersuchung der transienten Wellenbewegung ausgewertet. Es wird angenommen, daß beide Konstituierenden inkompressibel sind. Mit Hilfe der Laplacetransformation wird eine eindimensionale analytische Lösung abgeleitet, die als ein Resultat der Inkompressibilitätsbedingung nur eine unabhängige dilatante Wellenfortplanzung zeigt. Das fluidsaturierte poröse Material ist mit Charakteristiken versehen, die denen viskoelastischer Festkörper ähnlich sind. Diese Arbeit soll das weitere Verstehen der charakteristischen Eigenschaften der Wellenfortpflanzung in porösen Materialien erleichtern. Die Ergebnisse können zum quantitativen Vergleich mit verschiedenen numerischen Lösungen verwendet werden.

1 Introduction

The investigation of wave motion phenomena in fluid-saturated porous media is attracting more and more attention because of its significance in a great number of practical engineering problems. The general widely accepted opinion is that there are two dilatational waves and one rotational wave, which has been concluded on the basis of the theory given by Biot [1]. Biot's theory, which is based on the assumption of compressible constituents, and some of his results have been taken as standard references and the basis for much of subsequent analysis in acoustics, geophysics geomechanics and other fields up to the present. Recently, some new wave propagation theories [2–5] have been proposed and many numerical results [5–7] have been presented. However, because the coupled differential equations are generally difficult to solve exactly, it appears that numerical approaches have to be adopted to attain solutions. Meanwhile, much effort has been made to search for an exact solution so as to evaluate the availability of various numerical solution methods (e.g., finite element method and boundary element method). Garg et al. [8] examined the response of an infinitely long linear elastic fluid-saturated soil column subject to a Heaviside step function velocity boundary condition at one end, by use of Biot's theory, and derived closed-form analytical solutions only for the two extreme cases of zero and infinite drag. Recently, Simon et al. [9] also presented an analytical solution for the transient response of fluid-saturated porous elastic solids within the framework of Biot's theory; however, in the above approaches, the so-called Biot's dynamic compatibility relation of the solid and fluid materials must be satisfied. The latter paper also predicted the existence of two kinds of dilatational waves in the one-dimensional case, however, too many material parameters were involved in the procedure, where these material parameters may be physically not very clear.

In the present article, an analytical solution to analyze transient phenomena in fluid-staturated elastic porous media is presented. The fluid-saturated porous material is modelled as a two-phase system composed of an incompressible solid phase and an incompressible fluid phase, thus meeting the assumptions of many problems in engineering practice, e.g., in soil mechanics. On the basis of general porous media theories [10–13], following mixture theories extended by the concept of volume fractions, governing equations are given considering linear elastic deformation for the solid skeleton. An exact solution is obtained via Laplace transform technique for the one-dimensional problem taking into account initial and boundary conditions. Of interest is that numerical results, as a direct consequence of the incompressibility constraint, demonstrate the existence of only one independent compressible wave in the solid and the fluid phases. The included material parameters, compare, Table 1, are physically evident and can be taken from simple laboratory tests. It should furthermore be noted that the fluid-saturated porous material is endowed with characteristics similar to those occuring in viscoelastic solids. The presentation may also be used to make a critical comparision between various numerical and analytical results, as well as to provide an alternative understanding of the mechanism of wave propagation in fluid-saturated porous materials.

2 Field equations

Within the framework of modern porous media theories (the reader is referred to de Boer and Ehlers [12]), the fluid-saturated porous medium is understood as a binary mixture of superimposed but immiscible constituents φ^i with particles X^i $(i = F, S)$ denoting the fluid and the solid phases, respectively, each of which is regarded as a continuum following its own motion. This macroscopic treatment implies a model in which at any time t, each spatial position $\mathbf{x}$ of the current configuration is simultaneously occupied by particles X^i of both constituents φ^i. These particles proceed from different reference positions $\mathbf{X}_i$. Thus, each constituent φ^i is assigned its own motion function χ_i

$$\mathbf{x} = \chi_i(\mathbf{X}_i, t)\,. \tag{1}$$

The volume fractions

$$n^i = n^i(\mathbf{x}, t) \tag{2}$$

are defined as the local ratios of the constituent volumes v^i with respect to the bulk volume v. Associated with each φ^i is an effective density ϱ^{iR} and a bulk density ϱ^i. The density functions are related by

$$\varrho^i = n^i \varrho^{iR}\,. \tag{3}$$

Constituent incompressibility, as is assumed for the present binary model, implies that the effective densities are kept constant during deformation:

$$\varrho^{iR} = \text{const.} \tag{4}$$

Excluding mass and heat exchanges between the solid and the liquid phases and excluding the supply terms of moment of momentum, the concept of volume fractions and the balance equations for the constituents are given referring to de Boer and Ehlers [12] as follows

1. concept of volume fractions

$$n^S + n^F = 1\,. \tag{5}$$

2. balance of mass

$$\acute{\varrho}_i^{\,i} + \varrho^i \operatorname{div} \acute{\mathbf{x}}_i = 0\,. \tag{6}$$

3. balance of momentum

$$\operatorname{div} \mathbf{T}^i + \varrho^i(\mathbf{b}^i - \acute{\acute{\mathbf{x}}}_i) + \hat{\mathbf{p}}^i = \mathbf{0}\,, \tag{7}$$

$$\hat{\mathbf{p}}^S + \hat{\mathbf{p}}^F = \mathbf{0}\,. \tag{8}$$

4. balance of moment of momentum

$$\mathbf{T}^i = \mathbf{T}^{iT}\,. \tag{9}$$

The material time derivatives $(\ldots)'_i$ are defined by

$$(\ldots)'_i = \frac{\partial(\ldots)}{\partial t} + \operatorname{grad}(\ldots)\cdot \acute{\mathbf{x}}_i, \tag{10}$$

where $\acute{\mathbf{x}}_i$ and $\ddot{\mathbf{x}}_i$ characterize the constituent velocities and accelerations of φ^i, $\mathbf{T}^i$ the partial Cauchy stresses, $\hat{\mathbf{p}}^i$ the supply terms of momentum, and $\mathbf{b}^i$ the external body force densities. The symbol grad $(\ldots)$ means partial differentiation with respect to the spatial position $\mathbf{x}$, div $(\ldots)$ is the divergence operator corresponding to grad $(\ldots)$.

Considering incompressible constituents, the combination of (3), (4) and (6) yields

$$\acute{n}_i^i + n^i \operatorname{div} \acute{\mathbf{x}}_i = 0, \tag{11}$$

i.e., the balance of mass equations reduce to balance equations for the volume fractions [12]. Then, by use of (5) and (11), the relation

$$\acute{n}_S^F = \acute{n}_F^F - \operatorname{grad} n^F \cdot (\acute{\mathbf{x}}_F - \acute{\mathbf{x}}_S) \tag{12}$$

holds. Alternatively, one obtains

$$\operatorname{div}(n^S \acute{\mathbf{x}}_S + n^F \acute{\mathbf{x}}_F) = 0. \tag{13}$$

In the case of $\mathbf{b} = \mathbf{b}^S = \mathbf{b}^F$, the balance equations of momentum (7) can be rewritten in the forms

$$\operatorname{div} \mathbf{T}^S + \varrho^S(\mathbf{b} - \ddot{\mathbf{x}}_S) - \hat{\mathbf{p}}^F = \mathbf{0}, \tag{14}$$

$$\operatorname{div} \mathbf{T}^F + \varrho^F(\mathbf{b} - \ddot{\mathbf{x}}_F) + \hat{\mathbf{p}}^F = \mathbf{0}, \tag{15}$$

where, additionally, (8) has been used.

As a consequence of the incompressibility constraint (4), the stress tensors and the interaction force are additively decomposed into two terms

$$\mathbf{T}^S = -n^S p \mathbf{I} + \mathbf{T}_E^{\,S}, \tag{16}$$

$$\mathbf{T}^F = -n^F p \mathbf{I} + \mathbf{T}_E^{\,F}, \tag{17}$$

$$\hat{\mathbf{p}}^F = p \operatorname{grad} n^F + \hat{\mathbf{p}}_E^{\,F}, \tag{18}$$

where p characterizes the effective pressure of the incompressible pore fluid (de Boer and Ehlers [12]). In (16)–(18), the index $(\ldots)_E$ expresses the so called "extra quantities" for which constitutive equations must be formulated. Insertion of (16)–(18) into (14) and (15) produces

$$\operatorname{div} \mathbf{T}_E^{\,S} - n^S \operatorname{grad} p + \varrho^S(\mathbf{b} - \ddot{\mathbf{x}}_S) - \hat{\mathbf{p}}_E^{\,F} = \mathbf{0}, \tag{19}$$

$$\operatorname{div} \mathbf{T}_E^{\,F} - n^F \operatorname{grad} p + \varrho^F(\mathbf{b} - \ddot{\mathbf{x}}_F) + \hat{\mathbf{p}}_E^{\,F} = \mathbf{0}. \tag{20}$$

For further considerations, it is convenient to substitute the velocities $\acute{\mathbf{x}}_i$ and the accelerations $\ddot{\mathbf{x}}_i$ by the corresponding derivatives of the displacement vectors $\mathbf{u}_i$

$$\mathbf{x} = \mathbf{X}_i + \mathbf{u}_i, \tag{21}$$

$$\acute{\mathbf{x}}_i = \acute{\mathbf{u}}_i, \qquad \ddot{\mathbf{x}}_i = \ddot{\mathbf{u}}_i. \tag{22}$$

The investigations to follow are restricted to an isotropic, linear elastic porous solid filled with an inviscid liquid. The constitutive equations for the extra stresses and the extra momentum supply yield:

$$\mathbf{T}_E^{\,S} = 2\mu^S \mathbf{E}_S + \lambda^S(\mathbf{E}_S \cdot \mathbf{I})\,\mathbf{I}, \tag{23}$$

$$\mathbf{z}^F = \operatorname{div} \mathbf{T}_E^{\,F} \cong \mathbf{0}, \tag{24}$$

$$\hat{\mathbf{p}}_E^{\,F} = -\mathbf{S}_v(\acute{\mathbf{u}}_F - \acute{\mathbf{u}}_S), \tag{25}$$

where μ^S and λ^S are the macroscopic Lamé constants of the porous solid, and

$$\mathbf{E}_S = \frac{1}{2}(\operatorname{grad} \mathbf{u}_S + \operatorname{grad}^T \mathbf{u}_S) \tag{26}$$

is the linearized Langrangian strain tensor. The relation (24) corresponds to the fact that, in covering the fields of ground-water flow through soils or pore water flow through earth dams, masonry or concrete dams, respectively, the viscosity force $\mathbf{z}^F$ is negligible in comparison with the other terms incorporated into (20) (additionally compare, de Boer and Ehlers [12]). The volume fraction n^S is determined from (11) by integration,

$$n^S = n^S_{0S}(\det \mathbf{F}_S)^{-1} = n^S_{0S}(1 + \mathbf{E}_S \cdot \mathbf{I})^{-1}, \tag{27}$$

where, in the scope of infinitesimal deformations, all terms of higher order are neglected. Moreover, since $\mathbf{E}_S \cdot \mathbf{I} \ll 1, n^S$ may be approximated by n^S_{0S} which is the solid volume fraction in the initial configuration. In the case of isotropic permeability, the tensor $\mathbf{S}_v$, describing the coupled interaction between the solid and the fluid, is given by de Boer and Ehlers [12]

$$\mathbf{S}_v = \frac{(n^F)^2\, \gamma^{FR}}{k^F}\,\mathbf{I} = S_v \mathbf{I}, \tag{28}$$

where γ^{FR} is the effective specific weight of the fluid, and k^F is the Darcy permeability coefficient of the porous medium. Now, inserting (23)–(26) and (28) into (13), (19) and (20), we may write the field equations as follows

$$(\lambda^S + \mu^S)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}_S + \mu^S\,\mathrm{div}\,\mathrm{grad}\,\mathbf{u}_S - n^S\,\mathrm{grad}\,p + \varrho^S(\mathbf{b} - \ddot{\mathbf{u}}_S) + S_v(\dot{\mathbf{u}}_F - \dot{\mathbf{u}}_S) = \mathbf{0}, \tag{29}$$

$$-\,n^F\,\mathrm{grad}\,p + \varrho^F(\mathbf{b} - \ddot{\mathbf{u}}_F) - S_v(\dot{\mathbf{u}}_F - \dot{\mathbf{u}}_S) = \mathbf{0}, \tag{30}$$

$$\mathrm{div}\,(n^S\dot{\mathbf{u}}_S + n^F\dot{\mathbf{u}}_F) = \mathbf{0}. \tag{31}$$

In the framework of the infinitesimal theory, the superposition principle holds, i.e., the loading by body forces and by external forces can be treated separately. Furthermore, by only considering the loading by external forces, the equations of motion, (29) and (30), can be written as

$$(\lambda^S + \mu^S)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}_S + \mu^S\,\mathrm{div}\,\mathrm{grad}\,\mathbf{u}_S - n^S\,\mathrm{grad}\,p - \varrho^S\ddot{\mathbf{u}}_S + S_v(\dot{\mathbf{u}}_F - \dot{\mathbf{u}}_S) = \mathbf{0}, \tag{32}$$

$$-n^F\,\mathrm{grad}\,p - \varrho^F\ddot{\mathbf{u}}_F - S_v(\dot{\mathbf{u}}_F - \dot{\mathbf{u}}_S) = \mathbf{0}. \tag{33}$$

The set of equations (29)–(31) or, without body forces, the versions (31)–(33) are the general statement of the boundary and initial value problem for saturated elastic porous media if appropriate boundary conditions and initial conditions are given.

3 One-dimensional transient wave propagation solution

The balance equations described above are usually solved by numerical methods. However, in the case of one-dimensional small strain, an analytical solution is possible, in which the small variation of the volume fraction is approximately neglected. Now, the focus is put on a one-dimensional infinitely long column, Fig. 1, separated from a half space consisting of a liquid-saturated porous elastic skeleton material. The motion of both the solid and the fluid materials is constrained to take place in the vertical direction; loading as a function of time, $\sigma(z = 0, t) = f(t)$ by a permeable punch with ideal permeability, is applied to the half space boundary. The z axis is taken as the normal on the boundary and thus, for

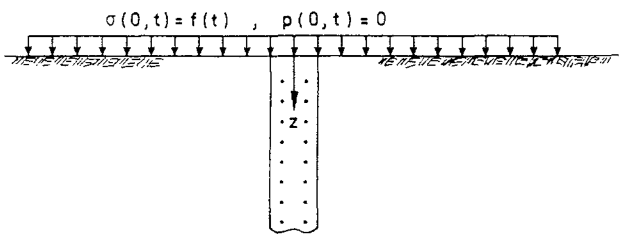


Fig. 1. Geometry of the investigated problem

the one-dimensional problem, the governing equations (31)–(33) are directly simplified as

$$(\lambda^S + 2\mu^S)\, u_{,zz} - n^S p_{,z} - \varrho^S u_{,tt} + S_v(v_{,t} - u_{,t}) = 0, \tag{34}$$

$$-n^F p_{,z} - \varrho^F v_{,tt} - S_v(v_{,t} - u_{,t}) = 0, \tag{35}$$

$$n^S u_{,tz} + n^F v_{,tz} = 0, \tag{36}$$

where the vertical values of $\mathbf{u}_S$ and $\mathbf{u}_F$ are substituted by u and v. In (36), homogeneous pore distribution is assumed. Furthermore, $u_{,z}$ or $u_{,zz}$, respectively, mean the first or the second differentiation of u with respect to the spatial coordinate z; the partial time derivatives are denoted by $u_{,t}$ and $u_{,tt}$, etc.

For the present investigations, the loading function at the free boundary is given by

$$\sigma(0, t) = f(t), \qquad p(0, t) = 0 \tag{37), (38}$$

(Fig. 1), where $f(t)$ is an arbitrary function of time which describes surface loading onto the skeleton material. The boundary condition (38) furthermore implies a free liquid surface assuming the half space boundary to show an adequate permeability. The initial conditions are

$$u(z, 0) = 0, \qquad v(z, 0) = 0, \tag{39), (40}$$

$$u_{,t}(z, 0) = 0, \qquad v_{,t}(z, 0) = 0. \tag{41), (42}$$

Taking the Laplace transforms of (34)–(36), with initial conditions (39)–(42), and using matrix notation, one obtains

$$\mathbf{A}\mathbf{m}_{,zz} + \mathbf{B}\mathbf{m}_{,z} + \mathbf{C}\mathbf{m} = \mathbf{0}, \tag{43}$$

where $\mathbf{m}^T = (L(u), L(v), L(p))$. The functions $L(u)$, $L(v)$, $L(p)$ are the Laplace transforms of the solid displacement, the fluid displacement and the pore pressure, respectively

$$L(u) = \int_0^\infty e^{-rt} u \, dt, \tag{44}$$

$$L(v) = \int_0^\infty e^{-rt} v \, dt, \tag{45}$$

$$L(p) = \int_0^\infty e^{-rt} p \, dt, \tag{46}$$

where r is the Laplace transform parameter. In (43), the corresponding matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are defined by

$$\mathbf{A} = \begin{bmatrix} \lambda^S + 2\mu^S & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{47}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & -n^S \\ 0 & 0 & -n^F \\ -n^S r & -n^F r & 0 \end{bmatrix}, \tag{48}$$

$$\mathbf{C} = \begin{bmatrix} -\varrho^S r^2 - S_v r & S_v r & 0 \\ S_v r & -\varrho^F r^2 - S_v r & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{49}$$

Assuming $\mathbf{m}$ to be solved via

$$\mathbf{m} = \mathbf{m}_0 e^{\alpha z}, \tag{50}$$

where $\mathbf{m}_0$ and α are functions of the transform parameter r, one obtains the eigenvalue problem by substitution of (50) into (43)

$$(\alpha^2 \mathbf{A} + \alpha \mathbf{B} + \mathbf{C})\, \mathbf{m}_0 e^{\alpha z} = \mathbf{0}. \tag{51}$$

By use of (47) – (49), the corresponding characteristic equation is

$$(n^F)^2 (\lambda^S + 2\mu^S)\,\alpha^4 - [(n^S)^2 \varrho^F + (n^F)^2 \varrho^S]\, r^2\alpha^2 - S_v r\alpha^2 = 0\,. \tag{52}$$

Eqn. (52) may be solved to yield two pairs of roots

$$\alpha_{1,2} = \pm\sqrt{ar^2 + br}\,, \qquad \alpha_{3,4} = 0\,, \tag{53), (54}$$

where

$$a = \frac{(n^S)^2 \varrho^F + (n^F)^2 \varrho^S}{(\lambda^S + 2\mu^S)(n^F)^2}\,, \qquad b = \frac{S_v}{(\lambda^S + 2\mu^S)(n^F)^2}\,. \tag{55), (56}$$

The eigenvectors associated with $\alpha_{1,2}$ and $\alpha_{3,4}$, respectively, are

$$\mathbf{q}_{1,2}^T = \left[-\frac{n^F}{n^S},\, 1,\, \mp\frac{n^S\varrho^F r^2 + S_v r}{n^S n^F\sqrt{ar^2 + br}}\right], \tag{57}$$

$$\mathbf{q}_{3,4}^T = [0,\, 0,\, 1]\,. \tag{58}$$

Thus, the transformed solution for the transient response of the porous medium is

$$\mathbf{m} = C_1\mathbf{q}_1 e^{\alpha_1 z} + C_2\mathbf{q}_2 e^{\alpha_1 z} + C_3\mathbf{q}_3 + C_4 z\mathbf{q}_4\,. \tag{59}$$

In order to insure boundedness at infinity, the coefficients C_1 and C_4 are

$$C_1 = 0\,, \qquad C_4 = 0\,. \tag{60), (61}$$

Then, the solution has the form

$$\mathbf{m} = C_2\mathbf{q}_2 e^{\alpha_2 z} + C_3\mathbf{q}_3\,, \tag{62}$$

in which C_2 and C_3 may be determined from the transformed boundary condition

$$C_2 = \frac{n^S L[f(t)]}{n^F(\lambda^S + 2\mu^S)\sqrt{ar^2 + br}}\,, \qquad C_3 = \frac{-(S_v + n^S\varrho^F r)\, rL[f(t)]}{n^{F2}(\lambda^S + 2\mu^S)(ar^2 + br)}\,. \tag{63), (64}$$

By use of the convolution integral and transform formultion (Abramowitz and Stegun [14]), the inverse transformation of Eqn. (59) produces the exact solution for the transient response problem in the fluid saturated porous medium

$$u(z, t) = -\frac{1}{\sqrt{a}\,(\lambda^S + 2\mu^S)}\int_0^t f(t - \tau)\, e^{-\frac{b}{2a}\tau}\, I_0\left(\frac{b\sqrt{\tau^2 - az^2}}{2a}\right) U(\tau - \sqrt{a}\, z)\,\mathrm{d}\tau\,, \tag{65}$$

$$v(z, t) = \frac{n^S}{n^F\sqrt{a}\,(\lambda^S + 2\mu^S)}\int_0^t f(t - \tau)\, e^{-\frac{b}{2a}\tau}\, I_0\left(\frac{b\sqrt{\tau^2 - az^2}}{2a}\right) U(\tau - \sqrt{a}\, z)\,\mathrm{d}\tau\,, \tag{66}$$

$$p(z, t) = \frac{1}{(n^F)^2(\lambda^S + 2\mu^S)}\,[n^S\varrho^F L_{,tt}(z, t) + S_v L_{,t}(z, t)]\,, \tag{67}$$

where

$$L(z, t) = \int_0^t Q(t - \tau)\, G(z, \tau)\,\mathrm{d}\tau\,, \tag{68}$$

$$Q(t) = \frac{1}{\sqrt{a}} \int_0^t f(t-\tau)\, e^{-\frac{b}{2a}\tau} I_0\left(\frac{b}{2a}\tau\right) d\tau, \tag{69}$$

$$G(z,t) = \frac{1}{\sqrt{a}} e^{-\frac{b}{2a}t} I_0\left(\frac{b\sqrt{t^2 - az^2}}{2a}\right) U(t - \sqrt{a}\, z) - \frac{1}{\sqrt{a}} e^{-\frac{b}{2a}t} I_0\left(\frac{b}{2a} t\right). \tag{70}$$

Moreover, it is not difficult to determine the one-dimensional extra stress $\sigma_E{}^S$

$$\sigma_E{}^S = \frac{b}{2\sqrt{a}} \int_0^t f(t-\tau)\, e^{-\frac{b}{2a}\tau} I_1\left(\frac{b\sqrt{\tau^2 - az^2}}{2a}\right) \frac{z}{\sqrt{\tau^2 - az^2}} U(\tau - \sqrt{a}\, z)\, d\tau + f(\tau - \sqrt{a}\, z)\, e^{-\frac{b}{2\sqrt{a}} z}. \tag{71}$$

In the above equations, $I_0(z)$ and $I_1(z)$ are the modified Bessel functions of zero and one order, respectively, and $U(t)$ is the unit step function (Heaviside function). A computer program may be provided to evaluate u, v, p and $\sigma_E{}^S$ for any form of the surface loading function numerically.

General properties of the analytical solution

In the preceding section, a one-dimensional analytical solution via Laplace transform technique was presented for an incompressible linear elastic skeleton material saturated by a single incompressible pore liquid.

The resulting expressions for the solid and the liquid displacements, Eqns. (62) and (63), and for the liquid pressure and the solid extra stresses, Eqns. (64) and (68), exhibit a strong history dependence comparable to that in the theory of viscoelasticity. In particular, the different response functions do not only depend on time, but furthermore depend on the previous loading history. This point can easily be understood from the squeezing out of water (when, e.g., the material is subject to external loads), combined with effects of internal friction included into the momentum supply term $\hat{\mathbf{p}}^F$ (de Boer and Ehlers [12]). Thus, a saturated porous skeleton material is provided with certain features similar to those appearing in viscoelastic solids.

The wave motion in the porous medium may be expressed by the solid and the fluid displacements or the solid extra stresses, respectively, but it cannot be expressed by the pore pressure which, of course, is nothing else than the Lagrangian multiplier corresponding to the incompressibility constraint of the binary medium. The incompressibility condition of the model furthermore produces the ratio of the solid and the liquid displacements to yield

$$u(z,t)/v(z,t) = -n^F/n^S. \tag{72}$$

i.e., as a matter of fact, there is only one disturbance propagating in the medium. The unit step function (Heaviside function) included into the formulae (65)–(71) regulates the relation between the disturbed spatial position and the necessary propagation time. Thus, the propagation velocity c_0 included into the argument of the unit step function U yields

$$c_0 = \frac{z}{t} = \frac{1}{\sqrt{a}} = \sqrt{\frac{(n^F)^2 (\lambda^S + 2\mu^S)}{(n^F)^2 \varrho^S + (n^S)^2 \varrho^F}}. \tag{73}$$

If the pore liquid is absent or if gas is contained in the pores of the matrix, the term ϱ^F is zero or can be neglected in comparison with ϱ^S. Then, since in this case n^F means porosity, one obtains the propagation velocity c' of the dilatational wave in incompressible empty porous solids as

$$c' = \sqrt{(\lambda^S + 2\mu^S)/\varrho^S}, \tag{74}$$

where the corresponding volume changes are only due to changes in porosity. The above expression can be compared with the well-known result of classical elasticity theories.

Finally, if only the solid constituent is present and if furthermore $n^F \to 0$, which corresponds to a non-porous incompressible solid material, then, the propagation velocity c'' of te dilatational wave is zero

$$c'' = 0. \tag{75}$$

This is a direct consequence of the incompressibility constraint.

4 An illustrative example of a one-dimensional soil column subject to three different surface loadings

In this section, a number of numerical results for the exact solution in a one-dimensional water-saturated soil column is presented. The loading function at the surface is $\sigma(0, t) = f(t)$, where $f(t)$ is chosen to be a sine function, a step function or an impulse function, respectively. The physical properties of the soil are assumed as shown in Table 1.

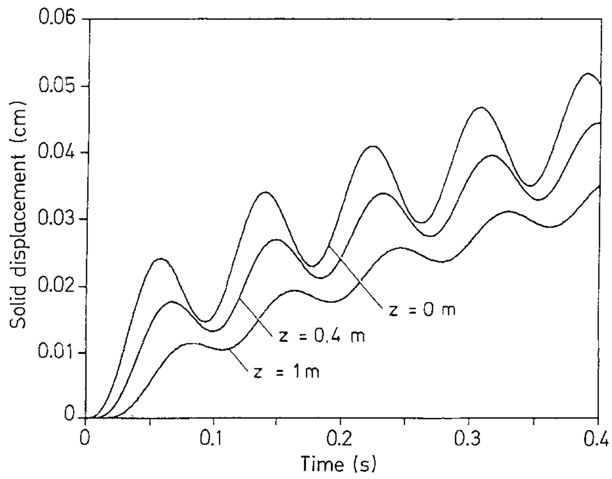
Table 1. Material properties

$n^S = 0.67$	$n^F = 0.33$
$\varrho^S = 1.34\ \mathrm{Mg/m^3}$	$\varrho^F = 0.33\ \mathrm{Mg/m^3}$
$E = 30\ \mathrm{MN/m^2}$	$\nu = 0.20$
$\lambda^S = 5.5833\ \mathrm{MN/m^2}$	$\mu^S = 8.3750\ \mathrm{MN/m^2}$
$qk^F = 0.01\ \mathrm{m/s}$	$\gamma^{FR} = 10.00\ \mathrm{kN/m^3}$

In contrast to the various papers on Biot's approach, the following section concerns an illustration of the characteristics of one-dimensional transient wave motion for the incompressible binary model under discussion. In particular, the solid and the liquid displacements, the solid extra stresses and the pore pressure are given with respect to time and with respect to different spatial positions within the framework of three loading forms, sinusoidal, step loading and impulsive loading.

4.1 Responses to sinusoidal loading

In the case of sinusodial loading, the responses of the solid and the liquid displacements versus time and versus depth measured from the free surface are shown in Fig. 2–5. From the comments on (72), it is clear that there exists only one independent dilatational wave propagating through the medium, i.e.,



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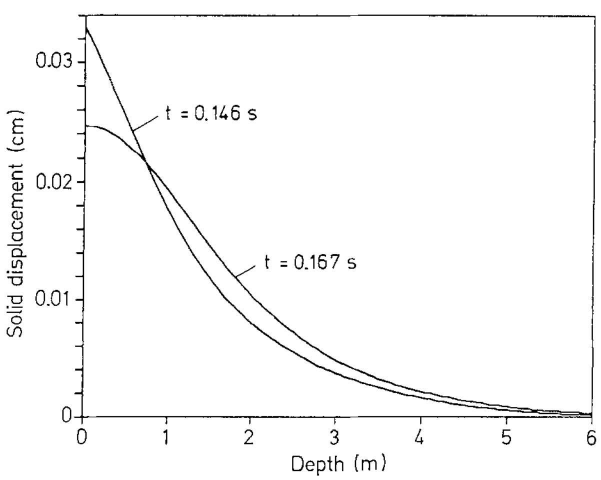

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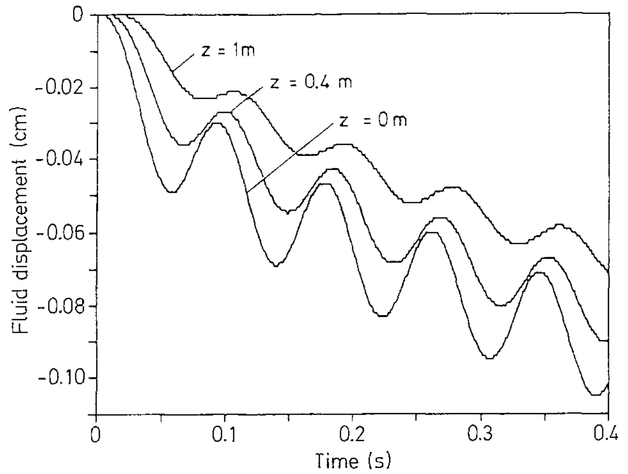
Fig. 2 and **3. 2** Response of solid displacement vs. time to sine loading; **3** Response of solid displacement vs. depth to sine loading



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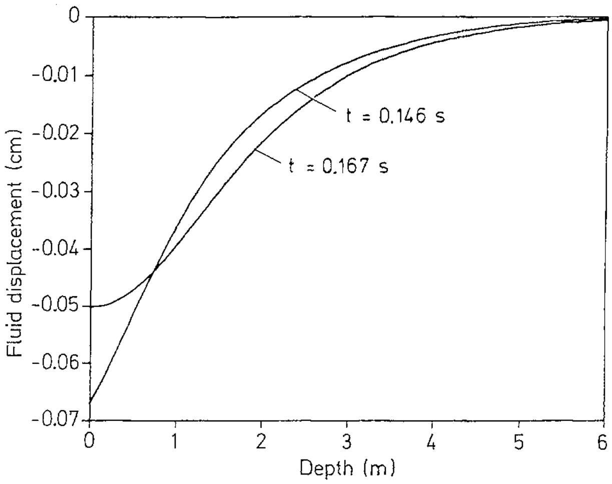

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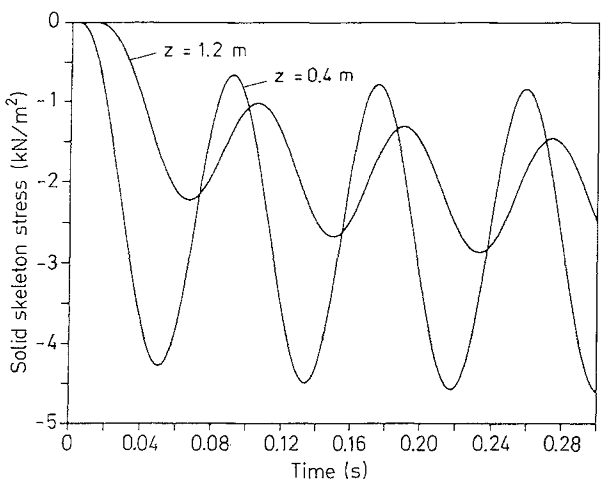
Fig. 4 and 5. 4 Response of fluid displacement vs. time to sine loading; 5. Response of fluid displacement vs. depth to sine loading



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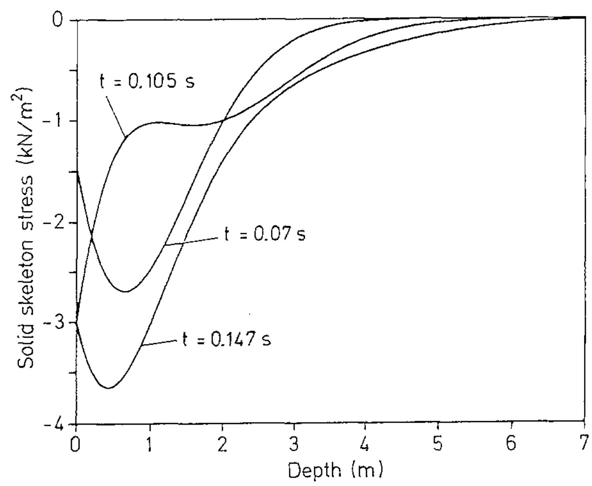

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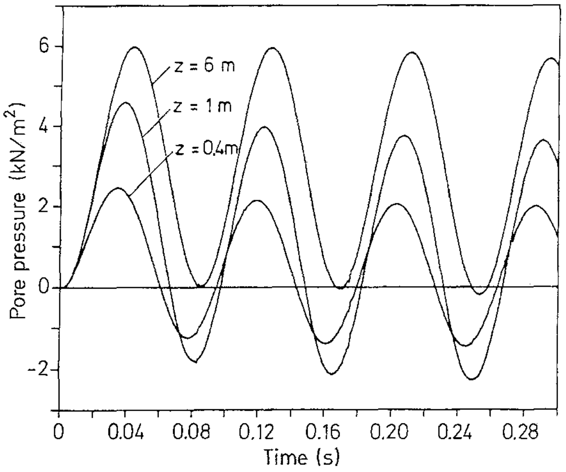
Fig. 6 and 7. 6 Response of solid skeleton effective stress vs. time to sine loading; 7 Response of solid skeleton effective stress vs. depth to sine loading

the solid displacement can be given as a function of the fluid displacement and vice versa. It is furthermore concluded from (72), in the case of the geometrical linear theory with approximately constant volume fractions, that the sum of the solid and the liquid volume fluxes vanishes, namely, as a direct consequence of the incompressibility constraint

$$n^S \acute{u}_S + n^F \acute{v}_F = n^S \acute{u}_S - n^F (n^S/n^F)\, \acute{u}_S = 0 \,. \tag{76}$$

The effective stress functions of the solid skeleton and the pore water pressure variations versus time and versus depth are shown in Fig. 5–9. It is not difficult to understand that the solid extra stress waves are very sensitive to the external loading close to the surface and tend to vanish in a certain distance from the loading face. This effect can be interpreted as a result of viscous damping caused by internal friction from the interaction mechanism between the skeleton and pore liquid materials.



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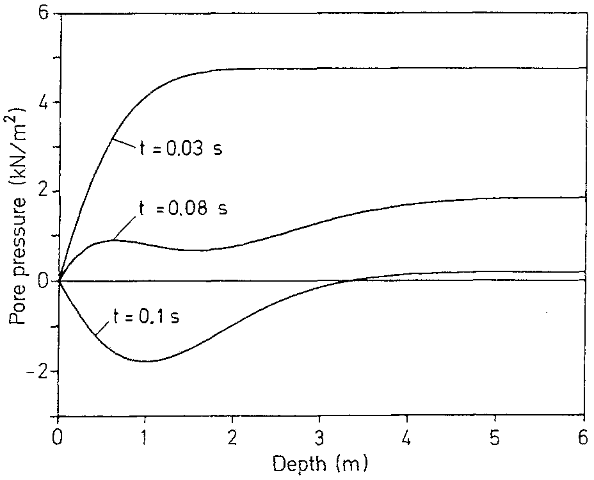

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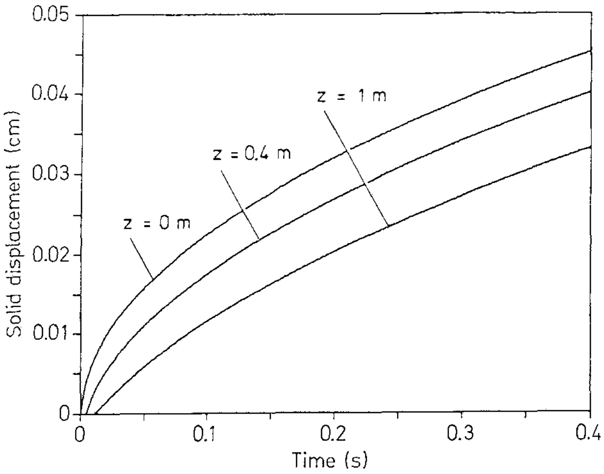
Fig. 8 and **9. 8** Response of pore pressure vs. time to sine loading; **9** Response of pore pressure vs. depth to sine loading



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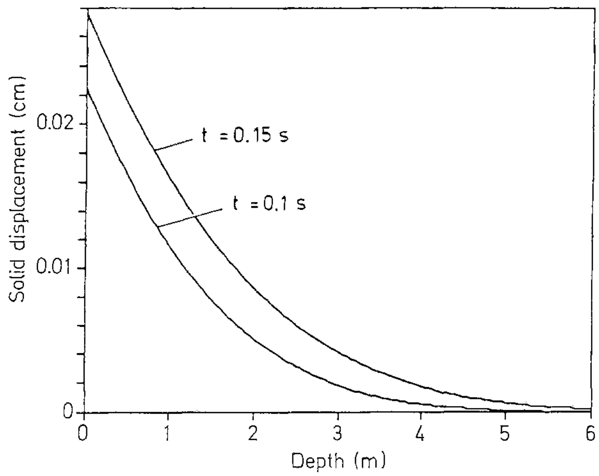

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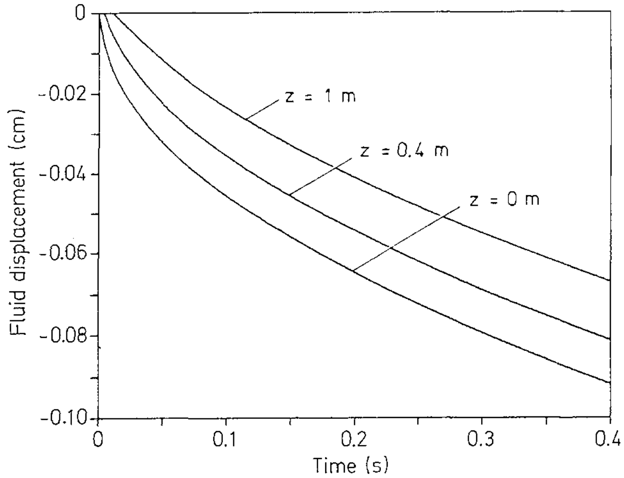
Fig. 10 and **11.** Response of solid displacement vs. time to step loading; **11** Response of solid displacement vs. depth to step loading

Furthermore, it is worth paying attention to the result that the pore water pressure can show negative values (pore water suction) in the vicinity of the loading surface. This result is due to the recovery of the elastic skeleton matrix close to the surface during sinusodial loading, where the pore liquid does not squeeze out but is absorbed into the pores accompanied by liquid suction.

4.2 Responses to step loading

The responses of the medium due to step loading can be utilized to analyze a consolidation process with a free pore water surface Figs. 10–17. In particular, Figs. 10–13 show the solid and the liquid displacements changing with respect to time and depth. With growing time, the solid moves downwards and the liquid is squeezed out from the pore volume. This process again exhibits viscoelastic properties as a result of internal friction. During the consolidation process, the solid extra stresses increase with time at a given depth, Fig. 14 but decrease with the distance from the



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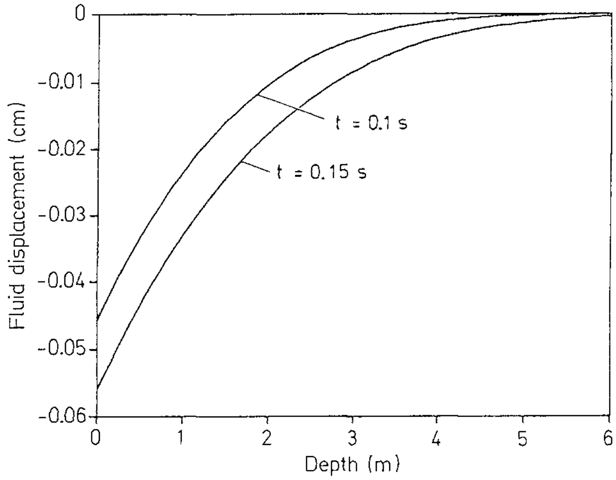

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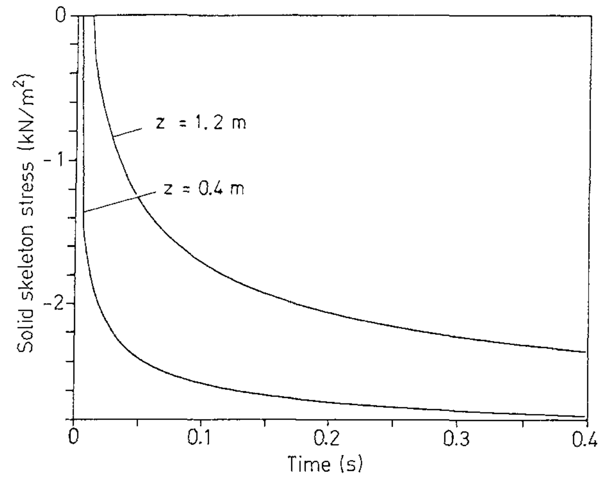
Fig. 12 and 13. 12 Response of fluid displacement vs. time to step loading; **13** Response of fluid displacement vs. depth to step loading



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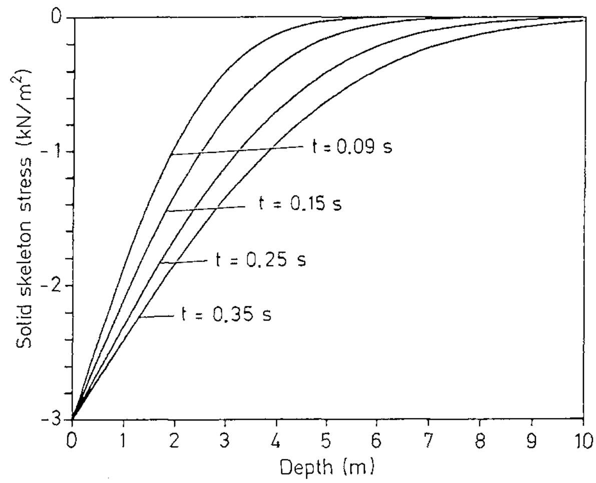

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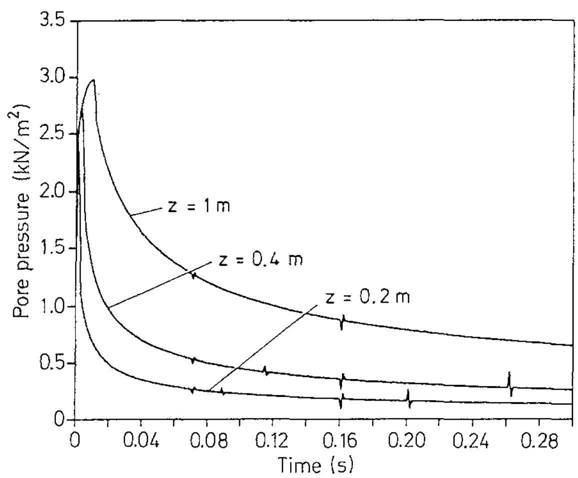
Fig. 14 and 15. 14 Response of solid skeleton effective stress vs. time to step loading; **15** Response of solid skeleton effective stress vs. depth to step loading



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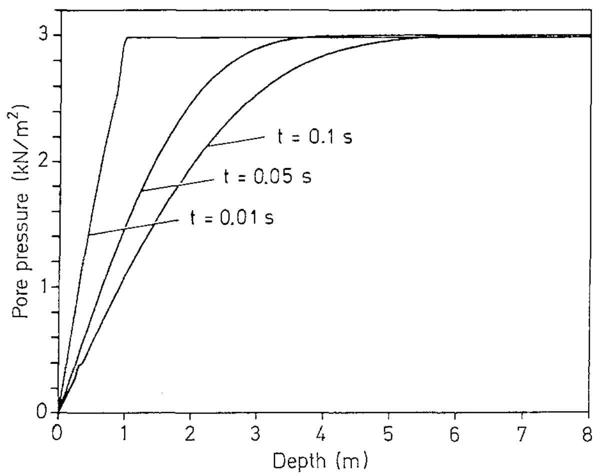

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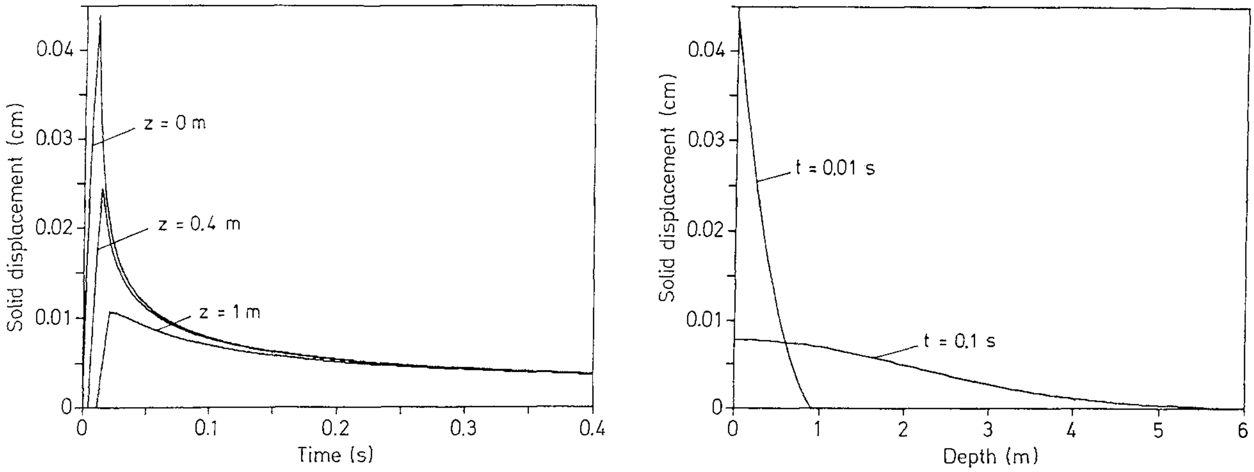
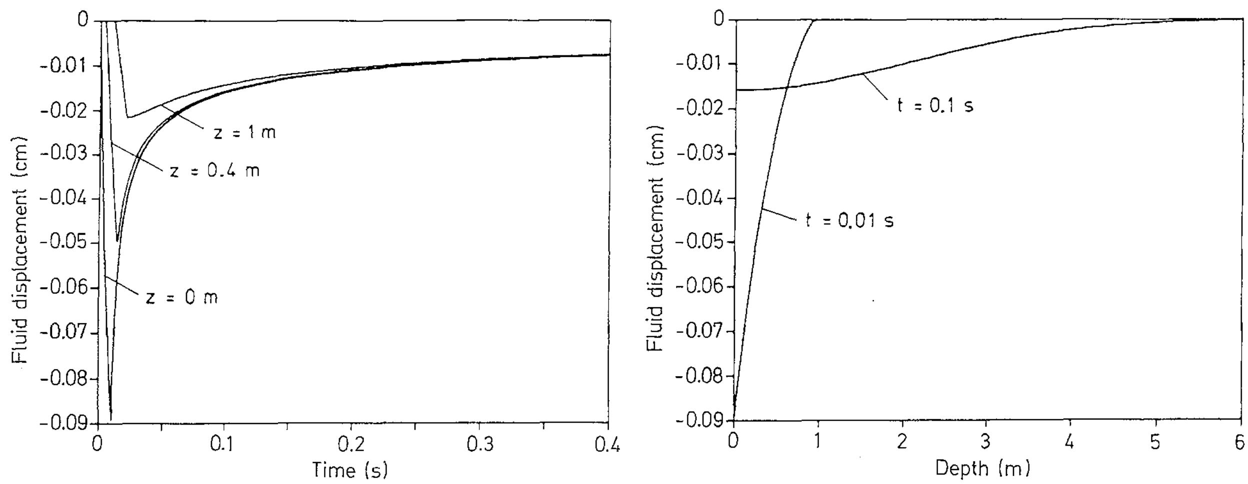
Fig. 16 and 17. 16 Response of pore pressure vs. time to step loading; **17** Response of pore pressure vs. depth to step loading



18 19

Fig. 18 and 19. 18 Response of solid displacement vs. time to impulse loading; 19 Response of solid displacement vs. depth to impulse loading



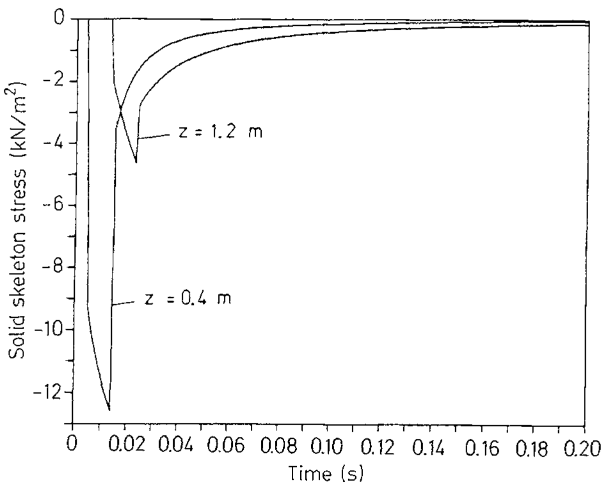
20 21

Fig. 20 and 21. 20 Response of fluid displacement vs. time to impulse loading; 21 Response of fluid displacement vs. depth to impulse loading

loading surface at a given time, Fig. 15. At any depth, the pore pressure decreases up to zero, Fig. 18, when previously, the pressure increasing process has taken as a function of depth and time, Fig. 17.

4.3 *Responses to impulse loading*

The responses of the medium due to impulse loading can be taken from Figs. 18–25. As a matter of fact, when an impact is applied to the loading surface, the displacements of the solid and the liquid phases reach their maximum values within a very short time, and then very quickly decrease to some smaller values, Figs. 18–21. The wave propagating process clearly appears in the solid extra stress and the pore pressure curves given with time at different depths and with depth at different times, Figs. 22–25. Note the sharpness of the pore pressure functions at different



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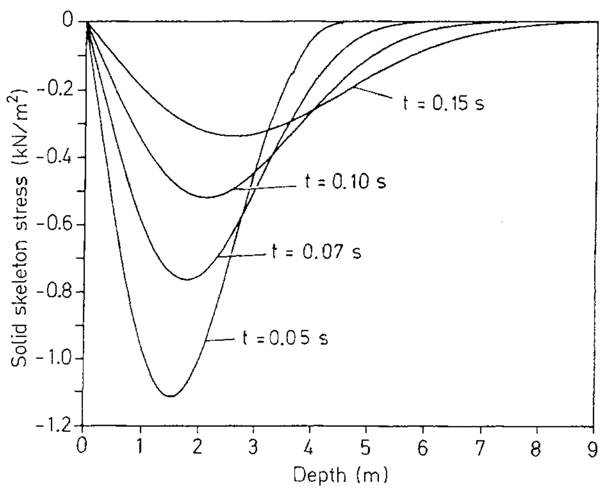

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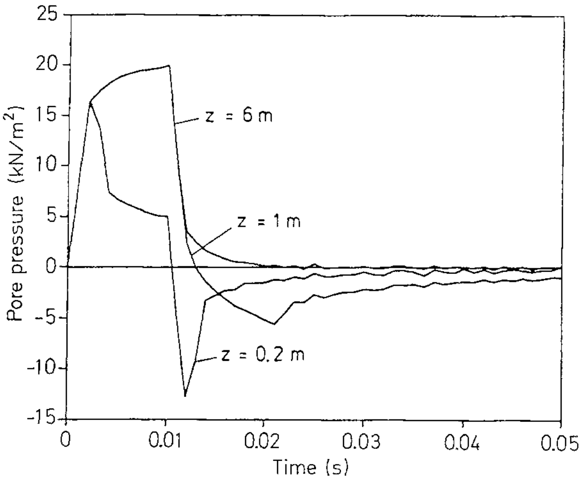
Fig. 22 and **23**. **22** Response of solid skeleton effective stress vs. time to impulse loading; **23** Response of solid skeleton effective stress vs. depth to impulse loading



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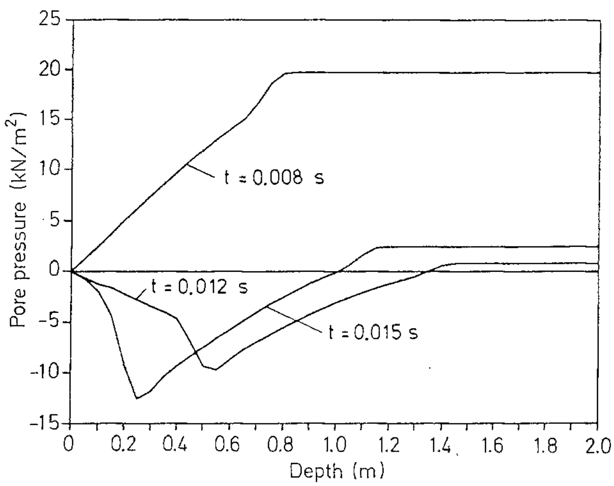

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Fig. 24 and **25**. **24** Response of pore pressure vs. time to impulse loading; **25** Response of pore pressure vs. depth to impulse loading

depths at time $t = 0.01$ s, compare, Fig. 27, which corresponds to the impulsive loading. The variation in pore pressure from positive to negative values close to the surface again exhibits the elastic recovery of the solid skeleton combined with a water absorption process.

5 Concluding remarks

An exact solution for a transient analysis of a one-dimensional column of a liquid-saturated elastic porous skeleton was presented in this paper. The saturated porous medium was modelled as a two-phase system with two incompressible constituents, where the general field equations were directly adopted according to the work of de Boer and Ehlers [12]. The exact solution was obtained by taking the Laplace transform of the governing equations with the initial and boundary conditions. The transient response of the medium was demonstrated with respect to three boundary loading functions. As a result of the incompressibility constraint, only one independent dilatational wave in the two phases was obtained, while, consequently, the disturbance propating velocity included into the unit step function was the same for both the solid and the liquid constituents. Apparently, the solution holds for two incompressible constituents within the framework of the geometrically linear theory approximately neglecting the variations in volume fractions during the deformation process. Nevertheless, the solution exhibits all the features of wave motion. Furthermore, the assumption of two incompressible constituents does not only meet the properties appearing in many branches of engineering practice (e.g., soil mechanics, etc.), but it also avoids, on the basis of an exact mechanical approach, the introduction of many complicated material parameters as must be considered in the theory of Biot and his subsequent disciples. Moreover, the present solutions allow for the evaluation of the accuracy of some numerical procedures for transient problems in such media.

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